

The Banach Algebra of Bounded Φ -Variation Functions on Compact Subsets of $\mathbb{C}$

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Abstract: A concept of generalized bounded Φ -variation for complex-valued functions is introduced. We prove that the space of functions of this kind is a Banach algebra with respect to pointwise multiplication. Previously, its properties were shown and the Minkowski functional was used to provide this class of functions with a norm that makes it a Banach space. In addition, some maximal ideals are described.

Keywords: Φ -Variation, Banach algebra, Bounded variation, Ideal.

1 Introduction

Fourier's work showed that a wide class of functions can be represented by trigonometric series (see [1]). However, the problem of finding precise conditions on the functions which would possess a convergent Fourier series was and is still the subject of the researches conducted by many mathematicians.

Jordan gave a sufficient condition in terms of what a bounded variation function is (see [2]). Since then, this class of functions has gained great interest as a function space, and also multiple applications have been found in areas, such as Fourier series in several variables, geometric measure theory, calculus of variations, generalized solutions for nonlinear partial differential equations and mathematical physics (see [3]).

More recently, [4] have established an improved version of the definition of variation for functions previously presented in [5], where the authors gave the definition of the variation of complex values functions defined in a compact subset of $\mathbb{C}$.

2 Notation and basic definitions

This section addresses, the notation and definitions that are specific of bounded variation. Given these, we state our results.

Throughout this paper, K denotes a compact subset of the complex plane $\mathbb{C}$. In addition, if $z, z' \in \mathbb{C}$, then $[z, z']$ denotes the line segment, joining z and z' .

If $\xi = [z_0, z_1, \dots, z_n]$ is an ordered finite list of elements of $\mathbb{C}$ (not necessarily different) where $n \geq 1$, Doust and Al-shakarchi [4] define crossing segment as follows:

Definition 1(See [4]). Let $\xi = [z_0, z_1, \dots, z_n]$ as before and suppose that ℓ is a line on the plane. The line segment $[z_i, z_{i+1}]$ is a crossing segment of ξ on ℓ if any one of the following holds:

- (i) z_i and z_{i+1} lie on (strictly) opposite sides of ℓ .
- (ii) $i = 0$ and $z_i \in \ell$.
- (iii) $i > 0$, $z_i \in \ell$ and $z_{i-1} \notin \ell$.
- (iv) $i = n - 1$, $z_i \notin \ell$ and $z_{i+1} \in \ell$.

Under these conditions we shall write $[z_i, z_{i+1}] \in X(\xi, \ell)$.

Definition 2(See [4]). Let $f : K \rightarrow \mathbb{C}$ and $\xi = [z_0, z_1, \dots, z_n]$ is an ordered finite list of element of K , the curve variation of f on the set ξ is defined to be

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$$\text{cvar}(f, \xi) = \sum_{i=1}^n |f(\mathbf{z}_i) - f(\mathbf{z}_{i-1})|.$$

Let $\text{vf}(\xi, \ell)$ denote the number of crossing segments of ξ on ℓ . The variation factor of ξ is defined to be

$$\text{vf}(\xi) = \max_{\ell} \text{vf}(\xi, \ell).$$

Note that $1 \leq \text{vf}(\xi) \leq n$. In [4], the authors also complete the definition for the case where $\xi = [\mathbf{z}_0]$, as well as set $\text{cvar}(f, [\mathbf{z}_0]) = 0$ and $\text{vf}([\mathbf{z}_0], \ell) = 1$ whenever $\mathbf{z}_0 \in \ell$.

Definition 3(See [4]). The two-dimensional variation of a functions $f : \mathbf{K} \rightarrow \mathbb{C}$ is defined to be

$$\text{var}(f, \mathbf{K}) = \sup_S \frac{\text{cvar}(f, \xi)}{\text{vf}(\xi)}, \quad (1)$$

where the supremum is taken over all finite ordered lists of elements of $\mathbf{K}$.

In [5] the authors proved that the space

$$BV(\mathbf{K}) = \{f : \mathbf{K} \rightarrow \mathbb{C} : \|f\|_{\infty} + \text{var}(f, \mathbf{K}) < \infty\} \quad (2)$$

is a Banach algebra with the pointwise operations and norm

$$\|f\|_{BV(\mathbf{K})} = \|f\|_{\infty} + \text{var}(f, \mathbf{K}). \quad (3)$$

It has always been of interest to study the generalization of the variation of a function in the plane, so the present paper aims to generalize the definition of bounded variation, basically combining the definition given in [4] and the definition given by Young in 1937.

3 Φ -variation of functions on compact subsets of $\mathbb{C}$

We shall denote by $\mathcal{N}$ the set of all functions $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ such that Φ is unbounded, continuous, convex and nondecreasing with $\Phi(0) = 0$. Such a function is said to satisfy condition Δ_2 if there exist a constant $1 \leq D < +\infty$ such that $\Phi(2t) \leq D\Phi(t)$ for $t \geq 0$.

The set $\mathcal{N}$ is closed under sums, products, multiplication by positive constants, compositions and taking inverse mappings (see [7, 8]).

Likewise, the notation $\mathcal{N}_{\infty}$ shall be used to denote the set of all functions $\Phi \in \mathcal{N}$, for which the Orlicz condition holds: $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = +\infty$. Following [9], functions in $\mathcal{N}$ shall be called Φ -functions.

Definition 4. Let $f : \mathbf{K} \rightarrow \mathbb{C}$ and $\xi = [\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_n]$ is an ordered finite list of element of $\mathbf{K}$, the curve variation of f on the set S is

$$\text{cvar}_{\Phi}(f, \xi) = \sum_{i=1}^n \Phi(|f(\mathbf{z}_i) - f(\mathbf{z}_{i-1})|),$$

and the variation of f is defined to be

$$\text{var}_{\Phi}(f, \mathbf{K}) = \sup_S \frac{\text{cvar}_{\Phi}(f, \xi)}{\text{vf}(\xi)}, \quad (4)$$

where the supremum is taken over all finite ordered lists of elements of $\mathbf{K}$, and $\text{vf}(S, \ell)$ denotes the number of crossing segments of S on ℓ . The variation factor of S is defined to be

$$\text{vf}(\xi) = \max_{\ell} \text{vf}(S, \ell). \quad (5)$$

Example 1. If f is constant on $\mathbf{K}$ then $\text{var}_{\Phi}(f, \mathbf{K}) = 0$.

Basic properties of the Φ -variation

Some basic properties of this new class of functions are presented below.

In all cases we refer to functions $f : \mathbf{K} \rightarrow \mathbb{C}$ and $\Phi \in \mathcal{N}$. Consider the following set:

$$\mathcal{V}_{\Phi}(\mathbf{K}) = \{f : \mathbf{K} \rightarrow \mathbb{C} : \text{var}_{\Phi}(f, \mathbf{K}) < \infty\}$$

1. If $z, z' \in \mathbf{K}$, then

$$\Phi(|f(z) - f(z')|) \leq \Phi(D(f, \mathbf{K}))$$

where $D(f, \mathbf{K}) := \text{diam} f(\mathbf{K}) = \sup\{|f(z) - f(z')| : z, z' \in \mathbf{K}\}$.

The result is obtained immediately by virtue of Φ being a monotone function.

2. If $\text{var}_{\Phi}(f, \mathbf{K}) < +\infty$ then $\text{var}_{\Phi}(\bar{f}, \mathbf{K}) < +\infty$, where $\bar{f}$ is the complex conjugate of f .

3. If Φ satisfies the Δ_2 condition then $\mathcal{V}_{\Phi}(\mathbf{K})$ is a real vector space, with the usual functions space operations.

Indeed, for any $f, g \in \mathcal{V}_{\Phi}(\mathbf{K})$, since Φ is increasing, we have

$$\begin{aligned} & \Phi(|(f+g)(\mathbf{z}) - (f+g)(\mathbf{z}')|) \\ &= \Phi(|(f(\mathbf{z}) - f(\mathbf{z}')) + (g(\mathbf{z}) - g(\mathbf{z}'))|) \\ &\leq \Phi(|f(\mathbf{z}) - f(\mathbf{z}')| + |g(\mathbf{z}) - g(\mathbf{z}')|) \\ &\leq \Phi(2 \max\{|f(\mathbf{z}) - f(\mathbf{z}')|, |g(\mathbf{z}) - g(\mathbf{z}')|\}) \\ &\leq D\Phi(\max\{|f(\mathbf{z}) - f(\mathbf{z}')|, |g(\mathbf{z}) - g(\mathbf{z}')|\}) \\ &\leq D[\Phi(|f(\mathbf{z}) - f(\mathbf{z}')|) + \Phi(|g(\mathbf{z}) - g(\mathbf{z}')|)]. \end{aligned}$$

As a result, from the latter inequality, it is easy to obtain

$$\text{cvar}_{\Phi}(f+g, \xi) \leq D[\text{cvar}_{\Phi}(f, \xi) + \text{cvar}_{\Phi}(g, \xi)], \quad (6)$$

for all $\xi = [z_0, z_1, \dots, z_n]$. So $f + g \in \mathcal{V}_\Phi(\mathbf{K})$.

Consequently, it is clear that

$$\frac{\text{cvar}_\Phi(f+g, \xi)}{\text{vf}(\xi)} \leq D \left(\frac{\text{cvar}_\Phi(f, \xi)}{\text{vf}(\xi)} + \frac{\text{cvar}_\Phi(g, \xi)}{\text{vf}(\xi)} \right),$$

so, taking the supreme over all finite ordered list of elements of $\mathbf{K}$ and using properties of the supreme, we get that

$$\text{var}_\Phi(f+g, \mathbf{K}) \leq D(\text{var}_\Phi(f, \mathbf{K}) + \text{var}_\Phi(g, \mathbf{K})).$$

This means that the sum is closed in $\mathcal{V}_\Phi(\mathbf{K})$.

Thus, for any $f \in \mathcal{V}_\Phi(\mathbf{K})$, $nf \in \mathcal{V}_\Phi(\mathbf{K})$ for all $n \in \mathbb{N}$. If $c \in \mathbb{R}$ then there exists $n \in \mathbb{N}$ such that $n > c$, Φ is increasing, it follows that $cf \in \mathcal{V}_\Phi(\mathbf{K})$ for any constant c .

4. Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be compact subsets in $\mathbb{C}$ such that $\mathbf{K}_1 \subseteq \mathbf{K}_2$. Every finite ordered list of elements of $\mathbf{K}_1$ is also an ordered finite list S of elements of $\mathbf{K}_2$, then

$$\begin{aligned} \text{var}_\Phi(f, \mathbf{K}_1) &= \sup_{S \subseteq \mathbf{K}_1} \frac{\text{cvar}(f, \xi)}{\text{vf}(\xi)} \\ &\leq \sup_{S \subseteq \mathbf{K}_2} \frac{\text{cvar}(f, \xi)}{\text{vf}(\xi)} \\ &= \text{var}_\Phi(f, \mathbf{K}_2). \end{aligned}$$

5. The functional $\text{var}_\Phi(f, \mathbf{K})$ is convex, so $\mathcal{V}_\Phi(\mathbf{K})$ is a convex set.

Indeed, letting $\alpha \in [0, 1]$ and $f, g \in \mathcal{V}_\Phi(\mathbf{K})$, then for each $\xi = [z_0, z_1, \dots, z_n]$ finite ordered list of elements of $\mathbf{K}$, we have

$$\begin{aligned} &\text{cvar}_\Phi(\alpha f + (1-\alpha)g, \xi) \\ &= \sum_{i=1}^n \Phi(|\alpha(f(z_i) - f(z_{i-1})) + (1-\alpha)(g(z_i) - g(z_{i-1})))|) \\ &\leq \sum_{i=1}^n \alpha \Phi(|f(z_i) - f(z_{i-1})|) + (1-\alpha) \sum_{i=1}^n \Phi(|g(z_i) - g(z_{i-1})|) \\ &= \alpha \text{cvar}_\Phi(f, \xi) + (1-\alpha) \text{cvar}_\Phi(g, \xi). \end{aligned}$$

Thus, we have

$$\begin{aligned} \text{var}_\Phi(\alpha f + (1-\alpha)g, \mathbf{K}) &\leq \alpha \text{var}_\Phi(f, \mathbf{K}) \\ &\quad + (1-\alpha) \text{var}_\Phi(g, \mathbf{K}). \end{aligned}$$

From the last inequality we get that $\text{var}_\Phi(\cdot)$ is a convex functional and that $\mathcal{V}_\Phi(\mathbf{K})$ is a convex set.

6. If there exists $\lambda > 0$ such that $\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) < +\infty$ the

f is a bounded function.

If z_0 is a fixed arbitrary point in $\mathbf{K}$, then

$$\begin{aligned} \Phi\left(\left|\frac{f}{2\lambda}(z)\right|\right) &\leq \frac{1}{2}\Phi\left(\left|\frac{f}{\lambda}(z) - \frac{f}{\lambda}(z_0)\right|\right) + \frac{1}{2}\Phi\left(\left|\frac{f}{\lambda}(z_0)\right|\right) \\ &\leq \frac{1}{2}\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) + \frac{1}{2}\Phi\left(\left|\frac{f}{\lambda}(z_0)\right|\right). \end{aligned}$$

Therefore,

$$|f(z)| \leq \Phi^{-1}\left(\frac{1}{2}\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) + \frac{1}{2}\Phi\left(\left|\frac{f}{\lambda}(z_0)\right|\right)\right) 2\lambda.$$

7. Since $f(z) = f_1(z) + if_2(z)$ where $f_1 : \mathbf{K} \rightarrow \mathbb{R}$ and $f_2 : \mathbf{K} \rightarrow \mathbb{R}$, then by virtue of the module properties of a complex number, we have that

$$\text{var}_\Phi(f_1, \mathbf{K}) \leq \text{var}_\Phi(f, \mathbf{K})$$

and

$$\text{var}_\Phi(f_2, \mathbf{K}) \leq \text{var}_\Phi(f, \mathbf{K}).$$

8. If $\Phi, \Psi \in \mathcal{N}$, $f : \mathbf{K} \rightarrow \mathbb{C}$ and $\xi = [z_0, z_1, \dots, z_n]$ as before, then

$$\text{var}_{\Phi+\Psi}(f, \mathbf{K}) \leq \text{var}_\Phi(f, \mathbf{K}) + \text{var}_\Psi(f, \mathbf{K}), \quad (7)$$

$$\text{var}_{\Phi\Psi}(f, \mathbf{K}) \leq \text{var}_\Phi(f, \mathbf{K}) \text{var}_\Psi(f, \mathbf{K}), \quad (8)$$

and

$$\text{var}_{c\Phi}(f, \mathbf{K}) = c \text{var}_\Phi(f, \mathbf{K}). \quad (9)$$

Indeed,

$$\begin{aligned} \text{cvar}_{\Phi+\Psi}(f, \xi) &= \sum_{i=1}^n (\Phi + \Psi)(|f(z_i) - f(z_{i-1})|) \\ &= \sum_{i=1}^n \Phi(|f(z_i) - f(z_{i-1})|) \\ &\quad + \sum_{i=1}^n \Psi(|f(z_i) - f(z_{i-1})|) \\ &= \text{cvar}_\Phi(f, \xi) + \text{cvar}_\Psi(f, \xi). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\text{cvar}_{\Phi+\Psi}(f, \xi)}{\text{vf}(\xi)} &= \frac{\text{cvar}_\Phi(f, \xi)}{\text{vf}(\xi)} + \frac{\text{cvar}_\Psi(f, \xi)}{\text{vf}(\xi)} \\ &\leq \text{var}_\Phi(f, \mathbf{K}) + \text{var}_\Psi(f, \mathbf{K}). \end{aligned}$$

As a result, from the last inequality, we get (7), (8) and (9) are obtained in a way similar to (7).

4 A Banach algebra

It is natural to wonder if this new class of functions is a vector space when Φ does not satisfy the condition Δ_2 . To address this concern and considering an arbitrary $\Phi \in \mathcal{N}$, is easy to verify that

$$\mathbb{A}_\Phi(\mathbf{K}) = \left\{ f : \mathbf{K} \rightarrow \mathbb{C} : \frac{f}{\lambda} \in \mathcal{V}_\Phi(\mathbf{K}) \text{ for some } \lambda > 0 \right\}$$

is a vector space and we now proceed to verify that $\mathbb{A}_\Phi(\mathbf{K})$ is a Banach algebra under pointwise operations.

Theorem 1. The subset

$$\Omega = \{f \in \mathcal{V}_\Phi(\mathbf{K}) : \text{var}_\Phi(f, \mathbf{K}) \leq 1\}$$

is a non-empty, convex, symmetrical and absorbent set of $\mathbb{A}_\Phi(\mathbf{K})$.

Proof. Example 1 ensures that Ω is a non-empty set.

The convexity and symmetry is an immediate consequence of the convexity of the functional $\text{var}_\Phi(f, \mathbf{K})$.

On the other hand, if $f \in \mathbb{A}_\Phi(\mathbf{K})$, then there exists $\lambda > 0$ such that $\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) < +\infty$.

If $\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) \leq 1$, there is nothing to prove.

In the other case, where $f \in \mathbb{A}_\Phi(\mathbf{K})$ and $\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) > 1$, then

$$\text{var}_\Phi\left(\frac{f}{\lambda \text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right)}, \mathbf{K}\right) \leq \frac{1}{\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right)} \text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) = 1,$$

which verifies that Ω is an absorbent set.

Given that Ω is a convex, symmetrical and absorbent set of $\mathbb{A}_\Phi(\mathbf{K})$, the Minkowski functional

$$\rho_\Phi(f) := \inf \left\{ \lambda > 0 : \frac{f}{\lambda} \in \Omega \right\}$$

is a semi-norm over $\mathbb{A}_\Phi(\mathbf{K})$, see [10].

Theorem 2. Let $f \in \mathbb{A}_\Phi(\mathbf{K})$, then

1. $\rho_\Phi(f) = 0$ if and only if f is a constant function in $\mathbf{K}$.
 2. If f is not a constant function, then
- $$\text{var}_\Phi\left(\frac{f}{\rho_\Phi(f)}, \mathbf{K}\right) \leq 1.$$

Proof. If f is a constant function, is clear that $\rho_\Phi(f) = 0$.

Reciprocally, if $\rho_\Phi(f) = 0$ and f is not a constant function in $\mathbf{K}$, there are $\mathbf{z}_0, \mathbf{z}_1 \in \mathbf{K}$ such that $f(\mathbf{z}_0) \neq f(\mathbf{z}_1)$.

In addition, for each $n \in \mathbb{N}$ there exists $\lambda_n > 0$ such that

$$\text{var}_\Phi\left(\frac{f}{\lambda_n}, \mathbf{K}\right) \leq 1 \text{ and } 0 < \lambda_n < \frac{1}{n}.$$

This implies that for every $n \in \mathbb{N}$ we have

$$\frac{\Phi\left(\left|\frac{f}{\lambda_n}(\mathbf{z}_0) - \frac{f}{\lambda_n}(\mathbf{z}_1)\right|\right)}{\text{vf}(\xi)} \leq \text{var}_\Phi\left(\frac{f}{\lambda_n}, \mathbf{K}\right) \leq 1,$$

for $S = [\mathbf{z}_0, \mathbf{z}_1]$.

If in the previous inequality, as $n \rightarrow +\infty$, the numerator of its left side tends to $+\infty$ while its denominator remains constant, which is a contradiction since the expression is bounded by 1.

From this contradiction, we get that f is a constant function in $\mathbf{K}$.

To demonstrate (2), for every $n \in \mathbb{N}$ there exists $\lambda_n \in \left\{ \lambda > 0 : \text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) \leq 1 \right\}$ such that

$$\rho_\Phi(f) \leq \lambda_n \leq \rho_\Phi(f) + \frac{1}{n}.$$

So, for each $\xi = [\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_n]$,

$$\frac{\text{cvar}_\Phi\left(\frac{f}{\lambda_n}, S\right)}{\text{vf}(\xi)} \leq 1,$$

and by virtue of the continuity of Φ , we get that

$$\frac{\text{cvar}_\Phi\left(\frac{f}{\rho_\Phi(f)}, S\right)}{\text{vf}(\xi)} = \lim_{n \rightarrow \infty} \frac{\text{cvar}_\Phi\left(\frac{f}{\lambda_n}, S\right)}{\text{vf}(\xi)} \leq 1.$$

As this inequality is true for each finite ordered list of elements in $\mathbb{C}$, then $\text{var}_\Phi\left(\frac{f}{\rho_\Phi(f)}, S\right) \leq 1$.

Theorem 3. Let Φ be a ϕ -function. Let $f : \mathbf{K} \rightarrow \mathbb{C}$ and $g : \mathbb{C} \rightarrow \mathbf{K}$ be functions with the properties that $f \in \mathbb{A}_\Phi(\mathbf{K})$ and g satisfies a Lipschitz condition. Then the composition $g \circ f \in \mathbb{A}_\Phi(\mathbf{K})$.

Proof. Since g satisfies a Lipschitz condition, there is a constant $M > 0$ such that

$$|g(z) - g(w)| \leq M|z - w| \text{ for all } z, w \in \mathbb{C}.$$

Also, if $f \in \mathbb{A}_\Phi(\mathbf{K})$, then there exist a constant $\lambda > 0$ such that $\text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right) < +\infty$.

If $\xi = [\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_n]$, an ordered finite list of elements of $\mathbb{C}$, then

$$\begin{aligned} \text{cvar}_\Phi\left(\frac{g \circ f}{M\lambda}, S\right) &= \sum_{i=1}^n \Phi\left(\left|\frac{(g \circ f)(\mathbf{z}_i)}{M\lambda} - \frac{(g \circ f)(\mathbf{z}_{i-1})}{M\lambda}\right|\right) \\ &= \sum_{i=1}^n \Phi\left(\frac{1}{M\lambda} |g(f(\mathbf{z}_i)) - g(f(\mathbf{z}_{i-1}))|\right) \\ &\leq \sum_{i=1}^n \Phi\left(\frac{1}{M\lambda} M |f(\mathbf{z}_i) - f(\mathbf{z}_{i-1})|\right) \\ &= \sum_{i=1}^n \Phi\left(\left|\frac{f}{\lambda}(\mathbf{z}_i) - \frac{f}{\lambda}(\mathbf{z}_{i-1})\right|\right). \end{aligned}$$

From here, we get

$$\text{cvar}_\Phi\left(\frac{g \circ f}{M\lambda}, S\right) \leq \text{cvar}_\Phi\left(\frac{f}{\lambda}, S\right),$$

so

$$\text{var}_\Phi\left(\frac{g \circ f}{M\lambda}, \mathbf{K}\right) \leq \text{var}_\Phi\left(\frac{f}{\lambda}, \mathbf{K}\right).$$

Thus, $g \circ f \in \mathbb{A}_\Phi$.

Theorem 4. $\mathbb{A}_\Phi(\mathbf{K})$ is a Banach space with norm $\|f\|_{\mathbb{A}_\Phi} := \|f\|_\infty + \rho_\Phi(f)$.

Proof. Since $\|\cdot\|_\infty$ defines a norm on the functions space and $\rho_\Phi(\cdot)$ is a seminorm in $\mathbb{A}_\Phi(\mathbf{K})$, is easy to verify that $\|\cdot\|_{\mathbb{A}_\Phi}$ is a norm on $\mathbb{A}_\Phi(\mathbf{K})$.

Let $\{f_n\}_{n=1}^\infty$ be a Cauchy succession in $\mathbb{A}_\Phi(\mathbf{K})$ and $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{\mathbb{A}_\Phi} < \varepsilon \text{ for all } m > n \geq N.$$

However, this implies that

$$\|f_n - f_m\|_\infty < \varepsilon \text{ for all } m > n \geq N.$$

Therefore, there exists a function $f : \mathbf{K} \rightarrow \mathbb{C}$ such that f_n converges uniformly to f .

Now, we verify that $\|f_n - f\|_{\mathbb{A}_\Phi} \rightarrow 0$ as $n \rightarrow \infty$.

Indeed, for $\varepsilon > 0$, as above there exists $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} \rho_\Phi(f_n - f_m) &< \|f_n - f_m\|_{\mathbb{A}_\Phi} < \varepsilon \text{ for all } m > n \geq N_1 \\ \|f_n - f\|_\infty &< \varepsilon \text{ for all } n \geq N_2 \end{aligned}$$

This implies that there exists a λ_0 such that for all $m > n \geq N_1$

$$\text{var}_\Phi\left(\frac{f_n - f_m}{\lambda_0}, \mathbf{K}\right) \leq 1 \text{ and } \lambda_0 < \varepsilon.$$

So, for all $m > n \geq N_1$

$$\begin{aligned} \text{var}_\Phi\left(\frac{f_n - f_m}{\varepsilon}, \mathbf{K}\right) &= \text{var}_\Phi\left(\frac{f_n - f_m}{\lambda_0} \frac{\lambda_0}{\varepsilon}, \mathbf{K}\right) \\ &\leq \frac{\lambda_0}{\varepsilon} \text{var}_\Phi\left(\frac{f_n - f_m}{\lambda_0}, \mathbf{K}\right) \leq \frac{\lambda_0}{\varepsilon}. \end{aligned}$$

However, if $\xi = [z_0, z_1, \dots, z_n]$ is an ordered finite list of elements of $\mathbf{K}$, then

$$\frac{\sum_{i=1}^n \Phi\left(\left|\frac{f_n - f_m}{\varepsilon}(z_{i+1}) - \frac{f_n - f_m}{\varepsilon}(z_i)\right|\right)}{\text{vf}(\xi)} \leq 1 \text{ for } m > n \geq N_1.$$

letting $m \rightarrow \infty$ in the last inequality, we obtain

$$\frac{\sum_{i=1}^n \Phi\left(\left|\frac{f_n - f}{\varepsilon}(z_{i+1}) - \frac{f_n - f}{\varepsilon}(z_i)\right|\right)}{\text{vf}(\xi)} \leq 1.$$

for $n \geq \max\{N_1, N_2\}$.

Since, the latter inequality is valid for any finite ordered list of elements of $\mathbb{C}$, we obtain

$$\text{var}_\Phi\left(\frac{f_n - f}{\varepsilon}, \mathbf{K}\right) \leq 1 \text{ for } n \geq \max\{N_1, N_2\},$$

so

$$\rho_\Phi(f_n - f) < \varepsilon \text{ for } n \geq \max\{N_1, N_2\}.$$

Therefore, if $n \geq \max\{N_1, N_2\}$, then

$$\|f_n - f\|_{\mathbb{A}_\Phi} = \|f_n - f\|_\infty + \rho_\Phi(f_n - f) < 2\varepsilon,$$

and the proof is completed.

Theorem 5. The space $\mathbb{A}_\Phi$ is a commutative Banach algebra under pointwise operations.

Proof. Let $f, g \in \mathbb{A}_\Phi$. Note that

$$\begin{aligned} &|fg(z) - fg(z')| \\ &= |f(z)g(z) - f(z)g(z') + f(z)g(z') - f(z')g(z')| \\ &= |f(z)(g(z) - g(z')) + g(z')(f(z) - f(z'))| \\ &\leq \|f\|_\infty |g(z) - g(z')| + \|g\|_\infty |f(z) - f(z')|. \end{aligned}$$

If $\xi = [z_0, z_1, \dots, z_n]$ is an ordered finite list of elements of $\mathbf{K}$, then

$$\begin{aligned} &\Phi\left(\left|\frac{fg}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty}(z_{i+1}) - \frac{fg}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty}(z_i)\right|\right) \\ &\leq \Phi\left(\left|\frac{f}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty}\right|_\infty |g(z_{i+1}) - g(z_i)| + \left|\frac{g}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty}\right|_\infty |f(z_{i+1}) - f(z_i)|\right) \\ &= \Phi\left(\frac{\rho_\Phi(g)\|f\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \left|\frac{g}{\rho_\Phi(g)}(z_{i+1}) - \frac{g}{\rho_\Phi(g)}(z_i)\right| + \frac{\rho_\Phi(f)\|g\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \left|\frac{f}{\rho_\Phi(f)}(z_{i+1}) - \frac{f}{\rho_\Phi(f)}(z_i)\right|\right) \\ &\leq \frac{\rho_\Phi(g)\|f\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \Phi\left(\left|\frac{g}{\rho_\Phi(g)}(z_{i+1}) - \frac{g}{\rho_\Phi(g)}(z_i)\right|\right) + \frac{\rho_\Phi(f)\|g\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \Phi\left(\left|\frac{f}{\rho_\Phi(f)}(z_{i+1}) - \frac{f}{\rho_\Phi(f)}(z_i)\right|\right) \\ &\leq \frac{\rho_\Phi(g)\|f\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \text{cvar}\left(\frac{g}{\rho_\Phi(g)}, S\right) + \frac{\rho_\Phi(f)\|g\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \text{cvar}\left(\frac{f}{\rho_\Phi(f)}, S\right), \end{aligned}$$

where we get the following

$$\begin{aligned} &\frac{\text{cvar}\left(\frac{fg}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty}\right)}{\text{vf}(\xi)} \\ &\leq \frac{\rho_\Phi(g)\|f\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \text{cvar}\left(\frac{g}{\rho_\Phi(g)}, S\right) + \frac{\rho_\Phi(f)\|g\|_\infty}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty} \text{cvar}\left(\frac{f}{\rho_\Phi(f)}, S\right) \leq 1. \end{aligned}$$

and

$$\text{var}_\Phi\left(\frac{fg}{\rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty}, \mathbf{K}\right) \leq 1.$$

This means that

$$\begin{aligned} \rho_\Phi(fg) &\leq \rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty \\ \|fg\|_\infty + \rho_\Phi(fg) &\leq \|fg\|_\infty + \rho_\Phi(f)\|g\|_\infty + \rho_\Phi(g)\|f\|_\infty \\ \|fg\|_{\mathbb{A}_\Phi} &\leq \|f\|_{\mathbb{A}_\Phi} \|g\|_{\mathbb{A}_\Phi}. \end{aligned}$$

5 Some examples

Let $\mathbf{K} \subseteq \mathbb{R} \subseteq \mathbb{C}$ be a compact set, $\Phi \in \mathcal{N}$, let I be the smallest closed interval containing $\mathbf{K}$ and λ the length of I . Then, for $f : \mathbf{K} \rightarrow \mathbb{C}$ defined by $f(z) = z$ and all ordered finite list of elements $\xi = [z_0, z_1, \dots, z_n]$ of $\mathbf{K}$, we have

$$\begin{aligned}
 \frac{\text{cvar}_{\Phi}\left(\frac{f}{\lambda}, \xi\right)}{\text{vf}(\xi)} &= \frac{1}{\text{vf}(\xi)} \sum_{i=1}^n \Phi\left(\left|\frac{f}{\lambda}(\mathbf{z}_i) - \frac{f}{\lambda}(\mathbf{z}_{i-1})\right|\right) \\
 &= \frac{1}{\text{vf}(\xi)} \sum_{i=1}^n \Phi\left(\left|\frac{\mathbf{z}_i - \mathbf{z}_{i-1}}{\lambda}\right|\right) \\
 &\leq \frac{1}{\text{vf}(\xi)} \sum_{i=1}^n \left|\frac{\mathbf{z}_i - \mathbf{z}_{i-1}}{\lambda}\right| \Phi(1) \\
 &= \frac{1}{\text{vf}(\xi)} \frac{\Phi(1)}{\lambda} \sum_{i=1}^n |\mathbf{z}_i - \mathbf{z}_{i-1}| \\
 &\leq \frac{1}{\text{vf}(\xi)} \frac{\Phi(1)}{\lambda} \sum_{j=1}^k \text{vf}(\xi) |\mathbf{z}_{i_j} - \mathbf{z}_{i_{j-1}}| \\
 &\leq \Phi(1).
 \end{aligned}$$

where $[\mathbf{z}_0 = \mathbf{z}_{i_0}, \mathbf{z}_{i_1}, \dots, \mathbf{z}_{i_k}]$ corresponds to the polygonal with different points in ξ .

This means that the function $f(\mathbf{z}) = \mathbf{z}$ is in $\mathbb{A}_{\Phi}(\mathbf{K})$ when defined in a compact set of $\mathbb{R}$.

As a second example, let $\mathbf{K} \subseteq \mathbb{C}$ be a compact set, $\Phi \in \mathcal{N}$ and $\chi_{\mathbf{w}} : \mathbf{K} \rightarrow \mathbb{C}$ the characteristic function defined by $\chi_{\mathbf{w}}(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathbf{z} = \mathbf{w} \\ 0 & \text{other case.} \end{cases}$. Then, for all ordered finite list of elements $\xi = [\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_n]$ of $\mathbf{K}$, if we let m be the number of segments $[\mathbf{z}_{i-1}, \mathbf{z}_i]$ whose extreme points are unequal and one of them is $\mathbf{w}$, then

$$\begin{aligned}
 \frac{\text{cvar}_{\Phi}(\chi_{\mathbf{w}}, \xi)}{\text{vf}(\xi)} &= \frac{1}{\text{vf}(\xi)} \sum_{i=1}^n \Phi(|\chi_{\mathbf{w}}(\mathbf{z}_i) - \chi_{\mathbf{w}}(\mathbf{z}_{i-1})|) \\
 &= \frac{1}{\text{vf}(\xi)} \sum_{i=1}^m \Phi(1) \\
 &= \frac{m\Phi(1)}{\text{vf}(\xi)}. \tag{10}
 \end{aligned}$$

However, for any straight line ℓ containing $\mathbf{w}$ we have that the extreme points of these m segments will have extreme points, different to $\mathbf{w}$, are either on one of the half-planes determined by ℓ or on it.

Suppose that, the largest number of these extreme points (different to $\mathbf{w}$), say r , are on the half-planes which we will denote by Π . Under this condition, there is a straight line parallel to ℓ at a distance

$$d = \min\{d(\ell, \mathbf{z}) : \mathbf{z} \in \xi \text{ and } \mathbf{z} \in \Pi\}.$$

Therefore, we have that $r \leq m \leq \text{vf}(\xi)$.

With a similar reasoning, it can be obtained that if s is the largest number of these extreme points that are in a same semi-line, that start at $\mathbf{w}$, we have that $s \leq \text{vf}(\xi)$. Therefore, we get that $m \leq 2r + 2s \leq 4\text{vf}(\xi)$. Consequently, from (10), we get

$$\begin{aligned}
 \frac{\text{cvar}_{\Phi}(\chi_{\mathbf{w}}, \xi)}{\text{vf}(\xi)} &= \frac{1}{\text{vf}(\xi)} \sum_{i=1}^n \Phi(|\chi_{\mathbf{w}}(\mathbf{z}_i) - \chi_{\mathbf{w}}(\mathbf{z}_{i-1})|) \\
 &\leq 4\Phi(1).
 \end{aligned}$$

Hence $\chi_{\mathbf{w}} \in \mathbb{A}_{\Phi}(\mathbf{K})$.

6 Important consequences

Recall that a subset I of a commutative Banach algebra A is said to be an ideal of A if I is a vector subspace of A and $xa \in I$ for all $x \in I$ and $a \in A$. An ideal $I \neq A$ is said to be a proper ideal. A proper ideal which is not contained in any larger proper ideal is said to be a maximal ideal.

In [11], the maximal ideal space BV^{Φ} is determined. In the following lines, we briefly present the properties that continue to be fulfilled in our new class of functions.

Theorem 6. Let $f \in \mathbb{A}_{\Phi}(\mathbf{K})$. Then,

1. If $\inf_{\mathbf{z} \in \mathbf{K}} |f(\mathbf{z})| > 0$, then $\frac{1}{f} \in \mathbb{A}_{\Phi}(\mathbf{K})$.
2. If $\inf_{\mathbf{z} \in \mathbf{K}} |f(\mathbf{z})| = 0$, then $\frac{1}{f} \notin \mathbb{A}_{\Phi}(\mathbf{K})$.

Proof. To demonstrate the first part, suppose that $f \in \mathbb{A}_{\Phi}(\mathbf{K})$, then there exists $\lambda > 0$ such that

$$\text{var}_{\Phi}\left(\frac{f}{\lambda}, \mathbf{K}\right) < +\infty.$$

Since $\inf_{\mathbf{z} \in \mathbf{K}} |f(\mathbf{z})| > 0$, let $m \in \mathbb{N}$ such that $m(\inf_{\mathbf{z} \in \mathbf{K}} |f(\mathbf{z})|)^2 \geq 1$, then if $\xi = [\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_n]$ is an ordered finite list of element of $\mathbf{K}$

$$\begin{aligned}
 &\text{cvar}_{\Phi}\left(\frac{1}{m\lambda f}, S\right) \\
 &= \sum_{i=1}^n \Phi\left(\left|\frac{1}{m\lambda f}(\mathbf{z}_i) - \frac{1}{m\lambda f}(\mathbf{z}_{i-1})\right|\right) \\
 &= \sum_{i=1}^n \Phi\left(\left|\frac{f(\mathbf{z}_{i-1}) - f(\mathbf{z}_i)}{m\lambda f(\mathbf{z}_i)f(\mathbf{z}_{i-1})}\right|\right) \\
 &\leq \sum_{i=1}^n \Phi\left(\frac{1}{m(\inf_{\mathbf{z} \in \mathbf{K}} |f(\mathbf{z})|)^2} \left|\frac{f(\mathbf{z}_{i-1}) - f(\mathbf{z}_i)}{\lambda}\right|\right) \\
 &\leq \sum_{i=1}^n \Phi\left(\left|\frac{f}{\lambda}(\mathbf{z}_{i-1}) - \frac{f}{\lambda}(\mathbf{z}_i)\right|\right) = \text{cvar}_{\Phi}\left(\frac{f}{\lambda}, S\right).
 \end{aligned}$$

So, we get that

$$\text{var}_{\Phi}\left(\frac{1}{m\lambda f}, \mathbf{K}\right) \leq \text{var}_{\Phi}\left(\frac{f}{\lambda}, \mathbf{K}\right) < +\infty,$$

so $\frac{1}{f} \in \mathbb{A}_{\Phi}(\mathbf{K})$.

Now, to prove the second part, if $f(\mathbf{z}) = 0$ for some $\mathbf{z} \in \mathbf{K}$, then $\frac{1}{f}$ is not defined in $\mathbf{K}$, so $\frac{1}{f} \notin \mathbb{A}_{\Phi}(\mathbf{K})$.

If $f(\mathbf{z}) \neq 0$ for each $\mathbf{z} \in \mathbf{K}$ and $\inf_{\mathbf{z} \in \mathbf{K}} |f(\mathbf{z})| = 0$, then for each $n \in \mathbb{N}$, there exist $\mathbf{z}_n \in \mathbf{K}$ such that

$$0 < |f(\mathbf{z}_n)| \leq \frac{1}{n}.$$

Therefore, $\frac{1}{f}$ is not a bounded function, since by virtue of

Basic properties (6), $\frac{1}{f} \notin \mathbb{A}_{\Phi}(\mathbf{K})$.

This means that each ideal of $\mathbb{A}_\Phi(\mathbf{K})$ must be a subset of

$$\mathbb{I} := \{f \in \mathbb{A}_\Phi(\mathbf{K}) : \inf_{z \in \mathbf{K}} |f(z)| = 0\}.$$

Note also that for each $z_0 \in \mathbf{K}$, it is easy to verify that $h_{z_0} : \mathbb{A}_\Phi(\mathbf{K}) \rightarrow \mathbb{C}$ is defined by $h_{z_0}(f) = f(z_0)$ is a complex homomorphism whose kernel is I_{z_0} .

Thus, for each $z_0 \in \mathbf{K}$, the set

$$I_{z_0} = \{f \in \mathbb{A}_\Phi(\mathbf{K}) : f(z_0) = 0\}$$

is a maximal ideal of $\mathbb{A}_\Phi(\mathbf{K})$ (see [12,6]), i.e. every element f in a nontrivial ideal I must have 0 in the closure of its range; just as in the case of actual real variable functions ([11])

Theorem 7. Let I be a nontrivial ideal of $\mathbb{A}_\Phi(\mathbf{K})$, then there exists a $z_0 \in \mathbf{K}$ such that for all $f \in I$, $\inf_{w \in U_{z_0}} |f(w)| = 0$ for all neighbourhoods U_{z_0} of z_0 .

Proof. Assume that for each $z \in \mathbf{K}$, there exists $f_z \in I$ with $|f_z(w)| = \delta_z > 0$

for all w in some neighbourhood U_z of z .

Being $\{U_z\}_{z \in \mathbf{K}}$ an open cover of the compact set $\mathbf{K}$, by the compactness of $\mathbf{K}$, it can be covered by finitely many such neighborhoods. Hence, we can find $\{z_0, z_1, \dots, z_n\} \subseteq$

$\mathbf{K}$ such that $\mathbf{K} \subseteq \bigcup_{i=0}^n U(z_i)$.

Note that if $f \in \mathbb{A}_\Phi(\mathbf{K})$, also the complex conjugate $\overline{f} \in \mathbb{A}_\Phi(\mathbf{K})$. Being I an ideal, this means that if $f \in I$, also the function $f\overline{f} = |f|^2 \in I$. Hence, the function $f : \mathbf{K} \rightarrow \mathbb{C}$ defined by

$$f(z) := \sum_{i=0}^n f_{z_i}(z) \overline{f_{z_i}(z)} = \sum_{i=0}^n |f_{z_i}(z)|^2 \geq \min_{0 \leq i \leq n} \delta_{z_i}^2 > 0.$$

This is a contradiction because, $f \in I$ is invertible. Hence, for any nontrivial ideal, there exists a point $z_0 \in \mathbf{K}$ such that for all f in to ideal, we'll have to $\inf_{z \in U(z_0)} |f(z)| = 0$ for all neighbourhoods $U(z_0)$ of z_0 .

7 Applications

The spaces of real or complex functions of limited variation defined on a compact space have a prominent role in Functional Analysis. An example of this is that many mathematicians have dedicated a big deal of time to the study of these function spaces (For example, [5,8]). And, specifically, within Banach's study of spaces, the notion of linear operators stands out, and in particular multiplication and superposition operators that allow solving some differential equations having solutions in these spaces. This makes us sure about the potential of the new theory to face the problems of determining the solution of differential equations in this new space of limited variation, applicable in many branches of science, applied mathematics and engineering.

8 Conclusion

A new class of bounded variation functions have been defined (from $\mathbf{K}$ to $\mathbb{C}$, where $\mathbf{K}$ is a compact subset of $\mathbb{C}$), generalizing the concept of variation presented in [5,4]. We have shown that this class of bounded variation functions is a Banach algebra. Previously, its properties were shown and the Minkowski functional was used to provide this class of functions with a norm that makes it a Banach space. Finally, some maximal ideals were described.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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