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A Comparison on Solutions of Fifth-Order Boundary-Value Problems

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Abstract: A fast and accurate numerical scheme for the solution of fifth-order boundary-value problems has been investigated in this work. We apply the reproducing kernel method (RKM) for solving this problem. The analytical results of the equations have been acquired in terms of convergent series with easily computable components. We compare our results with the numerical methods: B-spline method, decomposition method, variational iteration method, Sinc-Galerkin method and homotopy perturbation method. The comparison of the results with exact ones is made to confirm the validity and efficiency.

Keywords: Reproducing kernel method, series solutions, fifth-order boundary-value problems, reproducing kernel space, numerical methods.

1 Introduction

Fifth-order boundary-value problems arise in the mathematical modelling of viscoelastic flows and other branches of mathematical, physical and engineering sciences [1,2]. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed in a book by Agarwal [3]. Khan [4] investigated the fifth-order boundary-value problems by using finite difference methods. Wazwaz [5] applied Adomian decomposition method for solution of such type of boundary-value problems. The use of spline function in the context of fifth-order boundary-value problems was studied by Fyfe [6], who used the quintic polynomial spline functions to develop consistency relations connecting the values of solution with fifth-order derivatives at the respective nodes. Polynomial sextic spline functions were used [7] to develop the smooth approximations to the solution of the fifth-order boundary-value problems. Caglar et al. [8] have used sixth-degree B-spline functions to develop first-order accurate method for the solution two-point special fifth-order boundary-value problems. Noor and Mohyud-Din [9,10] applied variational iteration and homotopy perturbation methods. Khan [11] has used the non-polynomial sextic spline functions and El-Gamel [12] employed the Sinc-Galerkin method for the solution of the fifth-order boundary-value problems. Lamnii et al. [13] developed and analyzed two sextic spline collocation methods for the problems. Siddiqi et al. [14,15] used the non-polynomial sextic spline method for special fifth-order problems. Wang et al. [16] attempted to obtain upper and lower approximate solutions of such problems by applying the sixth-degree B-spline residual correction method.

In this paper, RKM will be used to investigate the fifth-order boundary-value problems. In recent years, a lot of attention has been given to the study of RKM to investigate various scientific models. RKM which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The efficiency of the method was used by many authors to research several scientific applications. Cui et al. [17] investigated solutions to the definite solution problem of differential equations in space $W_2^l[0,1]$. The book [18] contains many useful reproducing kernel functions. Geng and Cui [19] and

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Zhou et al. [20] applied RKM to handle the second-order boundary value problems. Yao and Cui [21] and Wang et al. [22] investigated a class of singular boundary value problems by this method. In [23], the method was used to solve nonlinear infinite-delay-differential equations. Wang and Chao [24], Li and Cui [25], Zhou and Cui [26] independently employed RKM to variable-coefficient partial differential equations. Geng at al. [27] solved singularly perturbed multipantograph delay equations based on the reproducing kernel method. Du and Cui [28] investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and RKM. Lv and Cui [29] presented a new algorithm to solve linear fifth-order boundary value problems. In [30,31], authors developed a new existence proof of solutions for nonlinear boundary value problems. Li at al. [32] used reproducing kernel method for fractional Riccati differential equations. Wu and Li [33] applied iterative reproducing kernel method to obtain the analytical approximate solution of a nonlinear oscillator with discontinuities. Lan et al. [34] solved a class of singularly perturbed partial differential equation by using the perturbation method and RKM. Readers can check references [35,36,37,38,39,40] for more details of RKM.

In this work we consider the numerical approximation for the fifth-order boundary-value problems of the form

$$y^{(v)} = f(x)y + g(x), \quad x \in [a,b], \tag{1.1}$$

with boundary conditions

$$y(a) = A_0, \ y'(a) = A_1, \ y''(a) = A_2, \ y(b) = B_0, \ y'(b) = B_1, \tag{1.2}$$

where the functions f and g are continuous functions on $[a,b]$ and A_0, A_1, A_2, B_0, B_1 are finite real constants.

The paper is organized as follows. Section 2 introduces several reproducing kernel spaces. Linear operator and solution representation in ${}^oW_2^6[a,b]$ have been presented in Section 3. The main results, the exact and approximate solution of (1.1) are given in Section 4. We have proved that the approximate solution converges to the exact solution uniformly in this section. Some numerical experiments are illustrated in Section 5. There are some conclusions in the last section.

2 Preliminaries

In this section, we define some useful reproducing kernel spaces for succeeding sections.

Definition 2.1. *(Reproducing kernel function).* Let E be a nonempty abstract set. A function $K : E \times E \longrightarrow C$ is a reproducing kernel function of the Hilbert space H if and only if

$$\begin{cases} \forall t \in E, \ K(.,t) \in H, \\ \forall t \in E, \ \forall \varphi \in H, \ \langle \varphi(.), K(.,t) \rangle = \varphi(t). \end{cases} \tag{2.1}$$

The last condition is called "the reproducing property": the value of the function φ at the point t is reproduced by the inner product of φ with $K(.,t)$.

Definition 2.2. We define the space ${}^oW_2^6[a,b]$ by

$${}^oW_2^6[a,b] = \left\{ u \middle| \begin{array}{l} u, \ u', \ u'', \ u''', \ u^{(4)}, \ u^{(5)} \\ \quad \text{are absolutely continuous in } [a,b], \\ \\ u^{(6)} \in L^2[a,b], \ x \in [a,b], \\ u(a) = u(b) = u'(a) = u'(b) = u''(a) = 0. \end{array} \right\}$$

The sixth derivative of u exists almost everywhere since $u^{(5)}$ is absolutely continuous. The inner product and the norm in ${}^oW_2^6[a,b]$ are defined by

$$\langle u,v \rangle_{{}^oW_2^6} = \sum_{i=0}^{5} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(6)}(x)v^{(6)}(x)dx,$$

and

$$\|u\|_{W_2^6} = \sqrt{\langle u,u \rangle_{{}^oW_2^6}}, \quad u,v \in {}^o W_2^6[a,b].$$

The space ${}^oW_2^6[a,b]$ is a reproducing kernel space, i.e., for each fixed $y \in [a,b]$ and any $u \in {}^o W_2^6[a,b]$, there exists a function R_y such that

$$u = \langle u, R_y \rangle_{{}^oW_2^6}.$$

Definition 2.3. We define the space $H_2^1[a,b]$ by

$$H_2^1[a,b] = \left\{ u \middle| u \text{ is absolutely continuous in } [a,b] \right\}$$

where $u'(x) \in L^2[a,b]$, $x \in [a,b]$ and the inner product and the norm in $H_2^1[a,b]$ are defined by

$$\langle u,v \rangle_{H_2^1} = u(a)v(a) + \int_a^b u'(x)v'(x)dx, \ \ u,v \in H_2^1[a,b]$$

and

$$\|u\|_{H_2^1} = \sqrt{\langle u,u \rangle_{H_2^1}}, \ \ u \in H_2^1[a,b].$$

The space $H_2^1[a,b]$ is a reproducing kernel space and its reproducing kernel function T_x is given by

$$T_x(y) = \begin{cases} 1-a+x, & x \le y, \\ \\ 1-a+y, & x > y. \end{cases} \tag{2.2}$$

Theorem 2.1. The space ${}^oW_2^6[a,b]$ is a reproducing kernel Hilbert space whose reproducing kernel function is given by,

$$R_y(x) = \begin{cases} \sum_{i=1}^{12} c_i(y)x^{i-1}, & x \le y, \\ \\ \sum_{i=1}^{12} d_i(y)x^{i-1}, & x > y. \end{cases}$$

Proof: Let $u \in {}^{o}W_2^6[a,b]$. By Definition 2.2 we have

$$\langle u, R_y \rangle_{{}_{o}W_2^6} = \sum_{i=0}^{5} u^{(i)}(a) R_y^{(i)}(a) + \int_a^b u^{(6)}(x) R_y^{(6)}(x) dx. \tag{2.3}$$

Through several integrations by parts for (2.3) we have

$$\begin{aligned} \langle u, R_y \rangle_{{}_{o}W_2^6} &= \textstyle\sum_{i=0}^{5} u^{(i)}(a) \left[R_y^{(i)}(a) - (-1)^{(5-i)} R_y^{(11-i)}(a) \right] \\ &+ \textstyle\sum_{i=0}^{5} (-1)^{(5-i)} u^{(i)}(b) R_y^{(11-i)}(b) + \int_a^b u(x) R_y^{(12)}(x) dx. \end{aligned} \tag{2.4}$$

Note that property of the reproducing kernel

$$\langle u, R_y \rangle_{{}_{o}W_2^6} = u(y).$$

Now, if

$$\begin{cases} R_y^{(5)}(a) - R_y^{(6)}(a) = 0, \\ R_y^{(4)}(a) + R_y^{(7)}(a) = 0, \\ R_y'''(a) - R_y^{(8)}(a) = 0, \\ R_y^{(6)}(b) = 0, \\ R_y^{(7)}(b) = 0, \\ R_y^{(8)}(b) = 0, \\ R_y^{(9)}(b) = 0, \end{cases} \tag{2.5}$$

then (2.4) implies that,

$$R_y^{(12)}(x) = \delta(x-y),$$

when $x \neq y$, then

$$R_y^{(12)}(x) = 0,$$

and therefore

$$R_y(x) = \begin{cases} \sum_{i=1}^{12} c_i(y) x^{i-1}, & x \leq y, \\ \sum_{i=1}^{12} d_i(y) x^{i-1}, & x > y. \end{cases}$$

Since

$$R_y^{(12)}(x) = \delta(x-y),$$

we have

$$R_{y^+}^{(k)}(y) = R_{y^-}^{(k)}(y), \quad k = 0,1,2,3,4,5,6,7,8,9,10, \tag{2.6}$$

and

$$R_{y^+}^{(11)}(y) - R_{y^-}^{(11)}(y) = 1. \tag{2.7}$$

Since $R_y \in W_2^6[0,1]$, it follows that

$$R_y(a) = 0, \ R_y(b) = 0, \ R_y'(a) = 0, \ R_y'(b) = 0, R_y''(a) = 0. \tag{2.8}$$

From (2.5)-(2.8), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1,2,...,12)$ can be obtained. This completes the proof. □

3 Solution Representation in ${}^{o}W_2^6[a,b]$

In this section, the solution of equation (1.1) is given in the reproducing kernel space ${}^{o}W_2^6[a,b]$.

On defining the linear operator $L : {}^{o}W_2^6[a,b] \to H_2^1[a,b]$ as

$$Lu = u^{(5)}(x) - f(x)u(x).$$

Model problem (1.1) changes the following problem:

$$\begin{cases} Lu = K(x), \ x \in [0,1] \\ u(a) = 0, \ u'(a) = 0, \ u''(a) = 0, \ u(b) = 0, \ u'(b) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1. If $u \in {}^{o}W_2^6[a,b]$, then $\left\| u^{(k)} \right\|_{L^\infty} \leq M_k \|u\|_{{}_{o}W_2^6}$, where M_k $(k = 0,1,...,5)$ are positive constants.

Proof: For any $x \in [a,b]$ it holds that

$$\|R_x(y)\|_{{}_{o}W_2^6} = \sqrt{\langle R_x(y), R_x(y) \rangle_{{}_{o}W_2^6}} = \sqrt{R_x(x)},$$

from the continuity of R_x, there exists a constant $M_0 > 0$, such that $\|R_x(y)\|_{{}_{o}W_2^6} \leq M_0$. By (2.1) one gets

$$\begin{aligned} |u(x)| &= \left| \langle u(y), R_x(y) \rangle_{{}_{o}W_2^6} \right| \\ &\leq \|R_x(y)\|_{{}_{o}W_2^6} \|u(y)\|_{{}_{o}W_2^6} = M_0 \|u(y)\|_{{}_{o}W_2^6}. \end{aligned} \tag{3.2}$$

For any $x, y \in [a,b]$, there exists M_k $(k = 1,2,...,5)$, such that

$$\left\| R_x^{(k)}(y) \right\|_{{}_{o}W_2^6} \leq M_k, \quad (k = 1,2,...,5).$$

By reproducing property and Cauchy-Schwarz inequality we have

$$\begin{aligned} \left| u^{(k)}(x) \right| = \left| \left\langle u(y), R_x^{(k)}(y) \right\rangle_{{}_{o}W_2^6} \right| &\leq \left\| R_x^{(k)}(y) \right\|_{{}_{o}W_2^6} \|u(y)\|_{{}_{o}W_2^6} \\ &= M_k \|u(y)\|_{{}_{o}W_2^6} \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), it follows that

$$\left\| u^{(k)}(x) \right\|_{L^\infty} \leq M_k \|u(y)\|_{{}_{o}W_2^6} \ (k = 1,2,...,5).$$

This completes the proof. □

Theorem 3.1. Suppose $f_i' \in L^2[a,b]$ $(i = 0,1,2,3,4)$. Then $L : {}^{o}W_2^6[a,b] \to H_2^1[a,b]$ is a bounded linear operator.

Proof: (i) By the definition of the operator it is clear that L is a linear operator.

(ii) Due to the definition of $H_2^1[a,b]$, we have

$$\|(Lu)(x)\|^2_{H_2^1} = \langle (Lu)(x),(Lu)(x)\rangle_{H_2^1}$$
$$= [(Lu)(a)]^2 + \int_a^b [(Lu)'(x)]^2 dx$$
$$= \left[\sum_{i=0}^{5} f_i(a)u^{(i)}(a)\right]^2 + \int_a^b \left[\left(\sum_{i=0}^{5} f_i(x)u^{(i)}(x)\right)'\right]^2 dx.$$

$$\int_a^b [(Lu)'(x)]^2 dx = \int_a^b \left[u^{(6)}(x) + \sum_{i=0}^{4}(f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x))\right]^2 dx$$
$$= \int_a^b \left[u^{(6)}(x)\right]^2 dx + 2\int_a^b \left[u^{(6)}(x)\sum_{i=0}^{4}(f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x))\right] dx$$
$$+ \int_a^b \left[\sum_{i=0}^{4}(f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x))\right]^2 dx,$$

where

$$\int_a^b \left[u^{(6)}(x)\right]^2 dx \le \|u(x)\|^2_{{}^oW_2^6},$$

and

$$\int_a^b \left[u^{(6)}(x)\sum_{i=0}^{4}(f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x))\right] dx$$
$$\le \left\{\int_a^b \left[u^{(6)}(x)\right]^2 dx\right\}^2 \left\{\int_a^b \left[\sum_{i=0}^{4}(f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x))\right]^2 dx\right\}^2.$$

By Lemma 3.1 and $f_i'(x) \in L^2[a,b]$, we can obtain a constant $N > 0$, satisfying

$$\int_a^b \left[\sum_{i=0}^{4}(f_i'(x)u^{(i)}(x) + f_i(x)u^{(i+1)}(x))\right]^2 dx \le N(b-a)\|u(x)\|^2_{{}^oW_2^6}.$$

Furthermore one gets

$$\int_a^b [(Lu)'(x)]^2 dx \le \|u(x)\|^2_{{}^oW_2^6} + 2\sqrt{N(b-a)}\|u(x)\|^2_{{}^oW_2^6} + N(b-a)\|u(x)\|^2_{{}^oW_2^6},$$

let $G = \left(1 + 2\sqrt{N(b-a)} + N(b-a)\right) > 0$, then

$$\int_a^b [(Lu)'(x)]^2 dx \le G\|u(x)\|^2_{{}^oW_2^6}.$$

Therefore L is a bounded operator. This completes the proof. □

4 The Normal Orthogonal Function System of ${}^oW_2^6[a,b]$

Let $\{x_i\}_{i=1}^{\infty}$ as any dense set in $[a,b]$ and $\Psi_x(y) = L^*T_x(y)$, where L^* is conjugate operator of L and T_x is given by (2.2). Furthermore, for simplicity let $\Psi_i(x) = \Psi_{x_i}(x)$, namely,

$$\Psi_i(x) \overset{def}{=} \Psi_{x_i}(x) = L^*T_{x_i}(x).$$

Lemma 4.1. $\{\Psi_i(x)\}_{i=1}^{\infty}$ is complete system of ${}^oW_2^6[a,b]$.

Proof: For $u \in {}^o W_2^6[a,b]$, let $\langle u, \Psi_i\rangle = 0$ $(i = 1,2,...)$, that is

$$\langle u, L^*T_{x_i}\rangle = (Lu)(x_i) = 0.$$

Note that $\{x_i\}_{i=1}^{\infty}$ is the dense set in $[a,b]$, therefore $(Lu)(x) = 0$. It follows that $u(x) = 0$ from the existence of L^{-1}. □

Lemma 4.2. Subscript η of operator $L\eta$ indicates that the operator L applies to function of η such that we have

$$\Psi_i(x) = (L\eta R_x(\eta))(x_i).$$

Proof: By reproducing property and property of conjugate operator we get

$$\Psi_i(x) = \langle \Psi_i(\xi), R_x(\xi)\rangle_{{}^oW_2^6[a,b]}$$
$$= \langle L^* T_{x_i}(\xi), R_x(\xi)\rangle_{{}^oW_2^6[a,b]}$$
$$= \langle (T_{x_i})(\xi), (L_\eta R_x(\eta))(\xi)\rangle_{W_2^1[a,b]}$$
$$= (L_\eta R_x(\eta))(x_i).$$

This completes the proof. □

Remark 4.1. The orthonormal system $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ of ${}^oW_2^6[a,b]$ can be derived from Gram-Schmidt orthogonalization process of $\{\Psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\Psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\Psi_k(x), \quad (\beta_{ii} > 0, \quad i = 1,2,...), \tag{4.1}$$

where β_{ik} are orthogonal coefficients.

Theorem 4.1. If u is the exact solution of (3.1) then

$$u = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}K(x_k)\overline{\Psi}_i(x), \tag{4.2}$$

where $\{x_i\}_{i=1}^{\infty}$ is a dense set in $[a,b]$.

Proof: From the (3.7) and uniqueness of solution of (3.1), we have

$$u = \sum_{i=1}^{\infty}\langle u, \overline{\Psi}_i\rangle_{{}^oW_2^6}\overline{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u, L^*T_{x_k}\rangle_{{}^oW_2^6}\overline{\Psi}_i$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu, T_{x_k}\rangle_{W_2^1}\overline{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle K, T_{x_k}\rangle_{W_2^1}\overline{\Psi}_i$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}K(x_k)\overline{\Psi}_i(x).$$

This completes the proof. □

Now the approximate solution u_n can be obtained by truncating the $n-$ term of the exact solution u as

$$u_n = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} K(x_k)\overline{\Psi}_i(x). \tag{4.3}$$

Lemma 4.3. Assume u is the solution of (3.1) and r_n is the error between the approximate solution u_n and the exact solution u. Then the error sequence r_n is monotone decreasing in the sense of $\|.\|_{{}_oW_2^6}$ and $\|r_n(x)\|_{{}_oW_2^6} \to 0$.

Proof: From (4.2) and (4.3), we obtain

$$\|u-u_n\|_{{}_oW_2^6} = \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i} \beta_{ik} K(x_k)\overline{\Psi}_i(x) \right\|_{{}_oW_2^6}.$$

Thus

$$\|u-u_n\|_{{}_oW_2^6} \to 0, \quad n \to \infty.$$

In addition

$$\begin{aligned}\|u-u_n\|^2_{{}_oW_2^6} &= \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i} \beta_{ik} K(x_k)\overline{\Psi}_i(x) \right\|^2_{{}_oW_2^6} \\ &= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} K(x_k)\overline{\Psi}_i(x) \right)^2.\end{aligned}$$

Then, $\|u-u_n\|_{{}_oW_2^6}$ is monotonically decreasing in n.

5 Numerical Results

In this section, four numerical examples are provided to show the accuracy of the present method. All computations are performed by Maple. The RKM does not require discretization of the variables, i.e., time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of RKM for the fifth-order boundary value problems is controllable and absolute errors are small with present choice of x (see Tables 1-4). The numerical results we obtained justify the advantage of this methodology.

Example 5.1. We first consider the linear boundary value problem

$$\begin{cases} y^{(5)}(x) = y - 15e^x - 10xe^x, \quad 0 < x < 1, \\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y(1) = 0,\ y'(1) = -e. \end{cases} \tag{5.1}$$

The exact solution of (5.1) is given as [5,8,9]

$$y(x) = x(1-x)e^x.$$

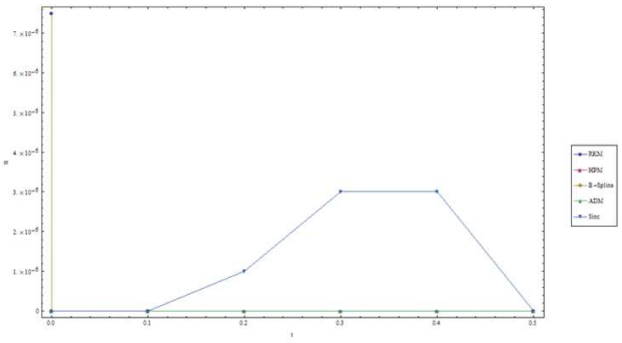

Fig. 1: Absolute Error for Example 5.1.

After homogenizing the boundary conditions of (5.1), we acquire

$$\begin{cases} u^{(5)}(x) - u(x) = 1 - 5e^x\left[2 - 3x + 3x^2\left(2 - \frac{5}{e}\right) + 4x^3\left(-\frac{3}{2} + \frac{4}{e}\right)\right] \\ \qquad -10e^x\left[-3 + 6x\left(2 - \frac{5}{e}\right) + 12x^2\left(-\frac{3}{2} + \frac{4}{e}\right)\right] \\ -10e^x\left[6\left(2 - \frac{5}{e}\right) + 24x\left(-\frac{3}{2} + \frac{4}{e}\right)\right] - 5e^x\left[24\left(-\frac{3}{2} + \frac{4}{e}\right)\right] \\ \qquad -15e^x - 10xe^x, \qquad 0 < x < 1 \\ \\ u(0) = 0,\ u'(0) = 0,\ u''(0) = 0,\ u(1) = 0,\ u'(1) = 0. \end{cases} \tag{5.2}$$

Using RKM for this example we obtain Table 5.1 and Figure 5.1.

Example 5.2. We now consider the nonlinear boundary value problem

$$\begin{cases} y^{(5)}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \\ y(0) = 1,\ y'(0) = 1,\ y''(0) = 1,\ y(1) = e,\ y'(1) = e. \end{cases} \tag{5.3}$$

The exact solution of (5.3) is given as [5,9]

$$y(x) = e^x.$$

Table 5.1. The absolute error of Example 5.1 for boundary conditions at $0.0 \leq x \leq 1.0$.

x	Exact Solution	RKM	HPM [10]	B-Spline [8]	ADM [5]	Sinc [12]
0.0	0.0000	0.000	0.0000	0.0000	0.0000	0.0000
0.1	0.099465382	5.89×10^{-11}	3×10^{-11}	8.0×10^{-3}	3×10^{-11}	0.0000
0.2	0.195424441	1.73×10^{-11}	2×10^{-10}	1.2×10^{-3}	2×10^{-10}	0.1×10^{-5}
0.3	0.283470349	6.02×10^{-10}	4×10^{-10}	5.0×10^{-3}	4×10^{-10}	0.3×10^{-5}
0.4	0.358037927	7.42×10^{-10}	8×10^{-10}	3.0×10^{-3}	8×10^{-10}	0.3×10^{-5}
0.5	0.412180317	3.32×10^{-11}	1.2×10^{-9}	8.0×10^{-3}	1.2×10^{-9}	0.0000
0.6	0.437308512	3.10×10^{-10}	2×10^{-9}	6.0×10^{-3}	2×10^{-9}	0.5×10^{-5}
0.7	0.422888068	3.08×10^{-10}	2.2×10^{-9}	0.000	2.2×10^{-9}	0.9×10^{-5}
0.8	0.356086548	4.58×10^{-9}	1.9×10^{-9}	9.0×10^{-3}	1.9×10^{-9}	0.2×10^{-5}
0.9	0.221364280	4.30×10^{-9}	1.4×10^{-9}	9.0×10^{-3}	1.4×10^{-9}	0.1×10^{-5}
1.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 5.2. The absolute error of Example 5.2 for boundary conditions at $0.0 \leq x \leq 1.0$.

x	Exact Solution	RKM	HPM [10]	B-Spline[8]	ADM [5]	VIM [9]
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	1.105170918	5.19×10^{-9}	1×10^{-9}	7.0×10^{-4}	1×10^{-9}	1×10^{-9}
0.2	1.221402758	0.60×10^{-9}	2×10^{-9}	7.2×10^{-4}	2×10^{-9}	2×10^{-9}
0.3	1.349858808	3.19×10^{-9}	1×10^{-9}	4.1×10^{-4}	1×10^{-9}	1×10^{-9}
0.4	1.491824698	2.50×10^{-9}	2×10^{-8}	4.6×10^{-4}	2×10^{-8}	2×10^{-8}
0.5	1.648721271	3.03×10^{-9}	3.1×10^{-8}	4.7×10^{-4}	3.1×10^{-8}	3.1×10^{-8}
0.6	1.822118800	9.60×10^{-9}	3.7×10^{-8}	4.8×10^{-4}	3.7×10^{-8}	3.7×10^{-8}
0.7	2.013752707	4.20×10^{-8}	4.1×10^{-8}	3.9×10^{-4}	4.1×10^{-8}	4.1×10^{-8}
0.8	2.225540928	4.09×10^{-9}	3.1×10^{-8}	3.1×10^{-4}	3.1×10^{-8}	3.1×10^{-8}
0.9	2.459603111	5.46×10^{-8}	1.4×10^{-8}	1.6×10^{-4}	1.4×10^{-8}	1.4×10^{-8}
1.0	2.718281828	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5.3. The absolute error (AE) of Example 5.3 for boundary conditions at $0.0 \leq x \leq 1.0$.

x	Exact Solution	Approximate Solution	AE, $1.0E-8$ RKM	AE,$1.0E-4$ [12] Sinc-Galerkin
0.0	0.0	0.0	0.0	0.0
0.0806	0.07751644243	0.07751634304	0.003	0.0
0.1648	0.1525493985	0.1525493860	1.25	0.2
0.2285	0.2057939130	0.2057939053	0.77	0.2
0.3999	0.3364008055	0.3364007134	9.21	0.4
0.5	0.4054651081	0.4054650667	4.14	0.1
0.6923	0.5260885504	0.5260885142	3.62	0.2
0.7714	0.5717701944	0.5717701752	1.92	0.3
0.8836	0.6331848394	0.6331848843	4.49	0.2
0.9447	0.6651077235	0.6651077287	0.52	0.5
1.0	0.6931471806	0.6931471783	0.23	0.0

After homogenizing the boundary conditions of (5.3), we get

Table 5.4. The absolute error of Example 5.4 for boundary conditions at $0.0 \leq x \leq 1.0$.

x	Exact Solution	Approximate Solution	AE RKM	AE, $1.0E-3$ [12] Sinc-Gal.	AE [43]
0.0	1.0	1.0	0.0	0.0	0.0
0.0100	1.105170918	1.105170918	0.0	0.0	7.79×10^{-10}
0.1184	1.125694299	1.125694299	0.0	0.0	7.83×10^{-7}
0.1517	1.163811041	1.163811041	0.0	0.1	1.36×10^{-6}
0.2410	1.272521035	1.272521035	0.0	0.0	3.005×10^{-6}
0.3604	1.433902861	1.433902861	0.0	0.1	2.52×10^{-6}
0.4287	1.535260387	1.535260387	0.0	0.0	2.57×10^{-7}
0.5000	1.648721271	1.648721271	0.0	0.2	5.04×10^{-6}
0.6395	1.895532876	1.895532876	0.0	0.1	1.58×10^{-5}
0.8482	2.335439276	2.335439276	0.0	0.2	1.33×10^{-5}
0.9996	2.717194733	2.7171947329	1×10^{-10}	0.2	2.10×10^{-10}
1.0	2.718281828	2.718281828	0.0	0.0	0.0

$$\begin{cases} u^{(5)}(x)-2e^{-x}\left[1+e^{x}\left(x-\frac{x^2}{2}+x^3\left(2-\frac{5}{e}\right)+x^4\left(-\frac{3}{2}+\frac{4}{e}\right)\right)\right]u(x) \\ =e^{-x}u^2(x)+e^{-x}\left[1+e^{x}\left(x-\frac{x^2}{2}+x^3\left(2-\frac{5}{e}\right)+x^4\left(-\frac{3}{2}+\frac{4}{e}\right)\right)\right]^2 \\ \quad -e^{x}\left[x-\frac{x^2}{2}+x^3\left(2-\frac{5}{e}\right)+x^4\left(-\frac{3}{2}+\frac{4}{e}\right)\right] \\ \quad -5e^{x}\left[1-x+3x^2\left(2-\frac{5}{e}\right)+4x^3\left(-\frac{3}{2}+\frac{4}{e}\right)\right] \\ \quad -10e^{x}\left[-1+6x\left(2-\frac{5}{e}\right)+12x^2\left(-\frac{3}{2}+\frac{4}{e}\right)\right] \\ \quad -10e^{x}\left[6\left(2-\frac{5}{e}\right)+24x\left(-\frac{3}{2}+\frac{4}{e}\right)\right] \\ \quad -5e^{x}\left[24\left(-\frac{3}{2}+\frac{4}{e}\right)\right], \quad 0<x<1, \\ u(0)=0,\ u'(0)=0,\ u''(0)=0,\ u(1)=0,\ u'(1)=0. \end{cases} \tag{5.4}$$

Using RKM for this example we obtain Table 5.2.

Example 5.3. Consider the nonlinear boundary value problem

$$\begin{cases} y^{(5)}(x)=-24e^{-y(x)}+\frac{48}{(1+x)^5}, \quad 0<x<1, \\ y(0)=0,\ y'(0)=1,\ y''(0)=-1,\ y(1)=\ln 2,\ y'(1)=0.5. \end{cases} \tag{5.5}$$

The exact solution of (5.5) is given as [12]

$$y(x)=\ln(x+1).$$

We use the following transformation to homogenize the boundary conditions:

$$y(x)=u(x)-x+\frac{x^2}{2}-x^3\left(4\ln 2-\frac{5}{2}\right)-x^4(2-3\ln 2).$$

After homogenizing the boundary conditions of (5.5), we have

$$\begin{cases} u^{(5)}(x)=-24e^{-\left(u(x)+x-\frac{x^2}{2}+x^3\left(4\ln 2-\frac{5}{2}\right)+x^4(2-3\ln 2)\right)}+\frac{48}{(1+x)^5}, \\ u(0)=0,\ u'(0)=0,\ u''(0)=0,\ u(1)=0,\ u'(1)=0. \end{cases} \tag{5.6}$$

Using RKM for this example we obtain Table 5.3 and Figure 5.2.

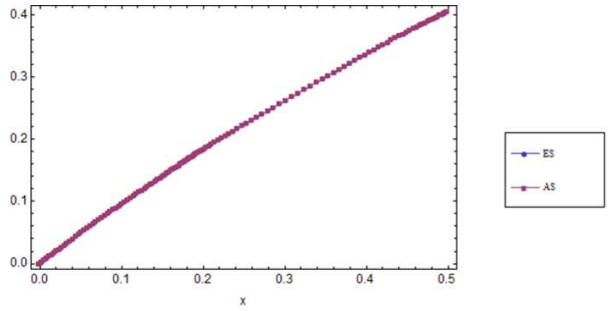

Fig. 2: Exact solution (AE) and Approximate Solution (AS) for Example 5.3.

Example 5.4. This is the nonlinear boundary value problem

$$\begin{cases} y^{(5)}(x)+y^{(4)}(x)+e^{-2x}y^2(x)=2e^{x}+1 \quad 0<x<1, \\ y(0)=0,\ y'(0)=1,\ y''(0)=1,\ y(1)=e,\ y'(1)=e. \end{cases} \tag{5.7}$$

The exact solution of (5.7) is given as [12]

$$y(x)=e^{x}.$$

After homogenizing the boundary conditions of (5.7), we obtain

$$\begin{cases} u^{(5)}(x)+u^{(4)}(x)=-e^{-2x}\left(u(x)+1+x+\frac{x^2}{2}+x^3(3e-8)+x^4\left(\frac{11}{2}-2e\right)\right)^2 \\ \qquad +2e^x+48e-131, \\ u(0)=0,\ u'(0)=0,\ u''(0)=0,\ u(1)=0,\ u'(1)=0. \end{cases} \tag{5.8}$$

Using RKM for this example, we obtain Table 5.4.

Remark 5.1. Lamnii et al. [13] solved the problem (3.1) by using sextic spline collocation method. He obtained the accurate approximate solutions of this problem for the small h values. Zhang [41] investigated approximate solution of the problem (3.1) by using variational iteration method. In addition, the same problem is solved by Noor and Mohyud-Din [9] previously and they got better results by using the variational iteration method. Lv and Cui [29] and Wang at al. [42] studied only the linear fifth-order two-point boundary value problems and Akram and Rehman [43] solved two examples by RKM. We chose different reproducing kernel functions and we obtained better results. We also showed how to homogenize the boundary conditions for Example 5.3. Homogenizing the boundary conditions is necessary for this method. Therefore, this work will be a new contribution for solving boundary value problems by RKM.

Using our method we chose 36 points on $[0,1]$. In Tables 1-4, we computed the absolute errors $|u(x,t)-u_n(x,t)|$ at the points $\{(x_i):\ x_i=i,\ \ i=0.0,0.1,...,1.0\}$.

Remark 4.2. RKM tested on four problems, one linear and three nonlinear. A comparison with decomposition method by Wazwaz [5], sixth B-spline method by Caglar et al. [8], variational iteration and homotopy perturbation methods by Noor and Mohyid-Din [9,10] and Sinc-Galerkin method by Gamel [12] were made and it was seen that the present method yields good results (see Tables 1-4).

6 Conclusion

In this paper, we introduced an algorithm for solving the fifth-order problem with boundary conditions. For illustration purposes, we chose four examples which were selected to show the computational accuracy. It may be concluded that, the RKM is very powerful and efficient in finding exact solution for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. The approximate solution obtained by the present method is uniformly convergent.

Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the RKM does not require discretization or perturbation and it does not make closure approximation. Results of numerical examples show that the present method is an accurate and reliable analytical method for the fifth order problem with boundary conditions.

References

[1] A. R. Davies, A. Karageorghis, T.N. Phillips, Spectral Galerkin method for primary two-point boundary-value problems in modelling viscoelastic flows, Int. J. Num. Methods Eng. 26 (1988) 647–662.
[2] A. Karageorghis, T. N. Phillips, A.R. Davies, Spectral collocation methods for the primary two-point boundary-value problems in modelling viscoelastic flows, Int. J. Num. Methods Eng. 26 (1988) 805–813.
[3] R. P. Agarwal, Boundary Value Problems for High Order Differential Equations, World Scientific, Singapore, 1986.
[4] M. S. Khan, Finite-difference solutions of fifth-order boundary-value problems, Ph.D. Thesis, Brunel University, England, 1994.
[5] A. M. Wazwaz, The numerical solution of fifth-order boundary-value problems by the decomposition method, J. Comp. Appl. Math. 136 (2001) 259–270.
[6] D. J. Fyfe, Linear dependence relations connecting equal interval Nth degree splines and their derivatives, J. Inst. Math. Appl. 7 (1971) 398–406.
[7] Siraj-ul-Islam, M.A. Khan, A numerical method based on non-polynomial sextic spline functions for the solution of special fifth-order boundary-value problems, Appl. Math. Comput. 181 (2006) 356–361.
[8] H. N. Caglar, S.H. Caglar, E.E. Twizell, The numerical solution of fifth-order boundary-value problems with sixth degree B-spline functions, Appl. Math. Lett. 12 (1999) 25–30.
[9] M. A. Noor, S.T. Mohyud-Din, Variational iteration technique for solving fifth-order boundary-value problems, Appl. Math. Comput. 189 (2007) 1929–1942.
[10] M. A. Noor, S.T. Mohyud-Din, An efficient algorithm for solving fifth-order boundary-value problems, Math. Comput. Modelling 45 (2007) 954–964.
[11] M. A. Khan, Siraj-ul-Islam, I.A. Tirmizi, E.H. Twizell, S. Ashraf, A class of methods based on non-polynomial sextic spline functions for the solution of a special fifth-order boundary-value problems, J. Math. Anal. Appl.321 (2006) 651–660.
[12] M. El-Gamel, Sinc and numerical solution of fifth-order boundary-value problems, Appl. Math. Comput. 187 (2007) 1417–1433.
[13] A. Lamnii, H. Mraoui, D. Sbibih, A. Tijini, Sextic spline solution of fifth-order boundary-value problems, Math. Comput. Simulation 77 (2008) 237–246.
[14] S. Siddiqi, G. Akram, Solution of fifth-order boundary-value problems using nonpolynomial spline technique, Appl. Math. Comput. 175 (2006) 1574–1581.
[15] S. Siddiqi, G. Akram, Sextic spline solutions of fifth-order boundary-value problems, Appl. Math. Lett. 20 (2007) 591–597.

[16] C. Wang, Z.X. Lee, Y. Kuo, Application of residual correction method in calculating upper and lower approximate solutions fifth-order boundary-value problems, Appl. Math. Comput. 199 (2008) 677–690.
[17] M. Cui, Z. Deng, Solutions to the definite solution problem of differential equations in space $W_2^l[0,1]$, Adv. Math. 17 (1986) 327–328.
[18] M. Cui, Y. Lin, Nonlinear Numerical Analysis in the Reproducing Kernel Spaces, Nova Science Publishers, New York, 2009.
[19] F. Geng, M. Cui, Solving a nonlinear system of second order boundary value problems, J. Math. Anal. Appl. 327 (2007) 1167–1181.
[20] Y. Zhou, Y. Lin, M. Cui, An efficient computational method for second order boundary value problems of nonlinear differential equations, Appl. Math. Comput. 194 (2007) 357–365.
[21] H. Yao, M. Cui, a new algorithm for a class of singular boundary value problems, Appl. Math. Comput. 186 (2007) 1183–1191.
[22] W. Wang, M. Cui, B. Han, A new method for solving a class of singular two-point boundary value problems, Appl. Math. Comput. 206 (2008) 721–727.
[23] X. Lü, M. Cui, Analytic solutions to a class of nonlinear infinite-delay-differential equations, J. Math. Anal. Appl. 343 (2008) 724–732.
[24] Y. L. Wang, L. Chao, Using reproducing kernel for solving a class of partial differential equation with variable-coefficients, Appl. Math. Mech. 29 (2008) 129–137.
[25] F. Li, M. Cui, A best approximation for the solution of one-dimensional variable-coefficient Burgers' equation, Num. Meth. Partial Dif. Eq. 25 (2009) 1353–1365.
[26] S. Zhou, M. Cui, Approximate solution for a variable-coefficient semilinear heat equation with nonlocal boundary conditions, Int. J. Comp. Math. 86 (2009) 2248–2258.
[27] F. Z. Geng and S. P. Qian, Solving singularly perturbed multipantograph delay equations based on the reproducing kernel method, Abst Appl.Anal ,Volume 2014, Article ID 794716, 6 pages.
[28] J. Du, M. Cui, Solving the forced Duffing equations with integral boundary conditions in the reproducing kernel space, Int. J. Comp. Math. 87 (2010) 2088–2100.
[29] X. Lv, M. Cui, An efficient computational method for linear fifth-order two-point boundary value problems, J. Comp. Appl. Math. 234 (2010) 1551–1558.
[30] W. Jiang, M. Cui, Constructive proof for existence of nonlinear two-point boundary value problems, Appl. Math. Comput. 215 (2009) 1937–1948.
[31] J. Du, M. Cui, Constructive proof of existence for a class of fourth-order nonlinear BVPs, Comp. Math. Appl. 59 (2010) 903–911.
[32] X. Y. Li, B. Y. Wu, and R. T.Wang, Reproducing kernel method for fractional Riccati differential equations, Abst Appl.Anal., Volume 2014, Article ID 970967, 6 pages.
[33] B.Y. Wu, X.Y. Li, Iterative reproducing kernel method for nonlinear oscillator with discontinuity, Appl. Math. Lett. 23 (2010) 1301–1304.
[34] Y. Lan.,H. Yu, F. Gui Tan, and S. Qu, Solving a class of singularly perturbed partial differential equation by using the perturbation method and reproducing kernel method, Abst Appl.Anal., Volume 2014, Article ID 615840, 5 pages.
[35] M. Inc, A. Akgül, and A. Kilicman. Numerical solutions of the second-order one-dimensional telegraph equation based on reproducing kernel Hilbert space method. Abstr. Appl. Anal., Volume 2013, Article ID 768963, 13 pages.
[36] M. Inc and A. Akgül. The reproducing kernel Hilbert space method for solving Troesch's problem. Journal of the Association of Arab Universities for Basic and Applied Sciences, (2013) 14, 19–27.
[37] M. Inc, A. Akgül, and F. Geng. Reproducing kernel Hilbert space method for solving Bratu's problem. Bul. Malays. Math. Sci. Soc., (2015) 38(1):271–287.
[38] M. Inc, A. Akgül, and A. Kilicman. Explicit solution of telegraph equation based on reproducing kernel method. J. Funct. Space Appl., 2012:23, 2012.
[39] M. Inc, A. Akgül, and A. Kilicman. A new application of the reproducing kernel Hilbert space method to solve MHD Jeffery-Hamel flows problem in non-parallel walls. Abstr. Appl. Anal., Volume 2013, Article ID 239454, 12 pages.
[40] M. Inc, A. Akgül, and A. Kilicman. On solving KdV equation using reproducing kernel Hilbert space method. Abstr. Appl. Anal., Volume 2013, Article ID 578942, 11 pages.
[41] J. Zhang, The numerical solution of fifth-order boundary value problems by the variational iteration method, Comp. Math. Appl. 58 (2009) 2347–2350.
[42] Wang Y., Lu, S., Tan, F., Du, M., Yu., H., Solving a class of singular fifth-order boundary value problems using reproducing kernel Hilbert space method, Abst Appl.Anal., Volume 2013, Article ID 925192, 6 pages.
[43] G. Akram and H. Rehman, Solution of fifth order boundary value problems in reproducing kernel space, Middle- East Journal of Scientific Research 10 (2): (2011) 191-195.

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