

Bernstein Type Inequalities between Composite Polynomials

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Abstract: In this paper we consider a more general class of polynomials $P(R(z))$ introduced by Shah and Liman [10] of degree mr , where $R(z)$ is a polynomial of degree at most r and prove number of inequalities concerning polynomials in the complex domain.

Keywords: Composite Polynomials, Complex Domain, Zeros, Inequalities.

1 Introduction

In a prize-winning essay on a problem of best approximation, Bernstein [2],[3] proved and made considerable use of an inequality concerning derivative of polynomial. However the first result in this area was connected with investigations of the well-known chemist Mendeleev, who asked "If the bound of a rational polynomial over a given interval is known, how large may its derivative in this interval?" He then conveyed this result to A. A. Markov, who generalized the result to polynomials of degree n and the analogue to Markov's result for the unit disk in the complex plane, instead of interval $[-1,1]$ was formulated by Bernstein, who proved that if $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree at most n and $P'(z)$ is its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \quad (1.1)$$

Since the answer is provided in the form of inequality, therefore there is always desire to refine this inequality, which has motivated mathematicians to work in this field. With this inspiration here we prove some Bernstein type inequalities concerning the class of composite polynomials which in turn yields the results for algebraic polynomials as special cases.

The next inequality is a simple deduction from the maximum modulus principle [9, p.346](see also [4] and

[6]), which states that if $P(z)$ is a polynomial of degree at most n , then

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \quad (1.2)$$

In both inequalities (1.1) and (1.2) equality holds only when $P(z)$ is a constant multiple of z^n .

If we restrict ourselves to a class of polynomials having no zero in $|z| < 1$, then the above inequality can be sharpened. In fact, Erdős conjectured and latter Lax [8] proved that if $P(z) \neq 0$ in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \quad (1.3)$$

and

$$\max_{|z|=R>1} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|. \quad (1.4)$$

Turán [11] proved that, if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \quad (1.5)$$

Concerning the minimum modulus of a polynomial $P(z)$ and its derivative $P'(z)$, Aziz and Dawood [1] proved that, if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)|, \quad (1.6)$$

and

$$\min_{|z|=R>1} |P(z)| \geq R^n \min_{|z|=1} |P(z)|. \quad (1.7)$$

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As an improvement of Inequality (1.4), Aziz and Dawood [1], proved if $P(z)$ does not vanish in the disk $|z| < 1$, then

$$\max_{|z|=R>1} |P(z)| \leq \left(\frac{R^n + 1}{2} \right) \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{2} \right) \min_{|z|=1} |P(z)|. \quad (1.8)$$

In this paper we consider the more generalized class of polynomials $P(Q(z))$, introduced by Shah and Liman [10], where $Q(z)$ is a polynomial of degree at most m defined by $(PoQ)(z) = P(Q(z))$, so that $PoQ \in P_{nm}$ and prove the following interesting results:

First we prove the following inequality which not only includes inequality (1.1), as a special case, but also gives the compact generalization of inequality (1.2).

Theorem 1: If $P(Q(z)) \in P_{nm}$, then for any real or complex number β , with $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$|P(Q(Rz)) - \beta P(Q(rz))| \leq |R^{nm} - r^{nm} \beta| |z|^{nm} \max_{|z|=1} |P(Q(z))|, \quad \text{for } |z| \geq 1. \quad (1.9)$$

The result is best possible and equality in inequality (1.9), holds for $P(Q(z)) = \alpha z^{nm}$, where $Q(z) = z^m$.

Dividing both side inequality (1.9) by $R - 1$ and take the limit as $R \rightarrow 1$ along with $\beta = r = 1$, we get the following result:

Corollary 1: If $P(Q(z)) \in P_{nm}$, then for $|z| \geq 1$, we have

$$|P'(Q(z))| \leq \frac{nm}{|Q'(z)|} |z|^{nm} \max_{|z|=1} |P(Q(z))| \quad (1.10).$$

Substituting $Q(z) = z$ in inequality (1.10), we get Bernstein's inequality.

Taking $\beta = 0$ in the inequality (1.9), we get the following result:

Corollary 2: If $P(Q(z)) \in P_{nm}$, then for $R \geq 1$,

$$|P(Q(Rz))| \leq |R|^{nm} |z|^{nm} \max_{|z|=1} |P(Q(z))|, \quad \text{for } |z| \geq 1. \quad (1.11)$$

For $Q(z) = z$, inequality (1.11) reduce to inequality (1.2).

Next, we prove the following which includes the result of Govil and Rahman [7] as a special case.

Theorem 2: If $P(S(z))$ is a polynomial of degree nm such that $P(S(0)) \neq 0$ and $Q(S(z)) = z^{nm} \overline{P(S(\frac{1}{z}))}$, then for any complex number β with $|\beta| \leq 1$ and $R > r \geq 1$

$$|P(S(Rz)) - \beta P(S(rz))| + |Q(S(Rz)) - \beta Q(S(rz))| \leq \{|R^{nm} - r^{nm} \beta| |z|^{nm} + |1 - \beta|\} \max_{|z|=1} |P(S(z))|, \quad \text{for } |z| \geq 1. \quad (1.12)$$

The result is best possible and equality in inequality (1.12), holds for $P(Q(z)) = z^{nm} + 1$, where $Q(z) = z^m$.

Dividing both side inequality (1.12) by $R - 1$ and take the limit as $R \rightarrow 1$ along with $\beta = r = 1$, we get the following result:

Corollary 3: If $P(S(z))$ is a polynomial of degree nm such that $P(S(0)) \neq 0$ and $Q(S(z)) = z^{nm} \overline{P(S(\frac{1}{z}))}$, then for $|z| \geq 1$, we have

$$|P'(S(z))| + |Q'(S(z))| \leq \left\{ \frac{nm|z|^{nm} + 1}{|S'(z)|} \right\} \max_{|z|=1} |P(S(z))|.$$

For $S(z) = z$, we have

$$|P'(z)| + |Q'(z)| \leq \{n|z|^n + 1\} \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1. \quad (1.13)$$

Inequality (1.13), is a special case of result proved by Govil and Rahman [7, Inequality (2.3)].

Substituting $\beta = 0$ in inequality (1.12), we get the following result:

Corollary 4: If $P(S(z))$ is a polynomial of degree nm such that $P(S(0)) \neq 0$ and $Q(S(z)) = z^{nm} \overline{P(S(\frac{1}{z}))}$, then for $R > 1$, we have

$$|P(S(Rz))| + |Q(S(Rz))| \leq \{|R^{nm}|z|^{nm} + 1\} \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1. \quad (1.14)$$

For $S(z) = z$, inequality (1.14) gives

$$|P(Rz)| + |Q(Rz)| \leq \{|R^n||z|^n + 1\} \max_{|z|=1} |P(S(z))|, \quad \text{for } |z| \geq 1. \quad (1.15)$$

Letting $R \rightarrow 1$ in inequality (1.15), we have

$$|P(z)| + |Q(z)| \leq \{|z|^n + 1\} \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1. \quad (1.16)$$

Next, we prove a result for the class of polynomials, which does not vanish outside the unit disc and obtain compact generalization of inequalities (1.6) and (1.7).

Theorem 3: If $P(Q(z))$ is a polynomial of degree nm having all its zeros in $|z| \leq 1$, then for any complex number β with $|\beta| \leq 1$ and $R > r \geq 1$

$$\begin{aligned} \min_{|z|=1} |P(Q(Rz)) - \beta P(Q(rz))| \\ \geq |R^{nm} - r^{nm} \beta| |z|^{nm} \min_{|z|=1} |P(Q(z))|, \end{aligned} \quad \text{for } |z| \geq 1. \quad (1.17)$$

The result is best possible and equality holds in inequality (1.17), for $P(Q(z)) = te^{i\alpha} z^{nm}$, $t > 0$, where $Q(z) = z^m$.

Dividing both side inequality (1.17) by $R - 1$ and take the limit as $R \rightarrow 1$ along with $\beta = r = 1$, we get the following result:

Corollary 5: If $P(Q(z))$ is a polynomial of degree nm having all its zeros in $|z| \leq 1$, then for $|z| \geq 1$

$$\min_{|z|=1} |P'(Q(z))| \geq \frac{nm|z|^{nm}}{|Q'(z)|} \min_{|z|=1} |P(Q(z))|. \quad (1.18)$$

For $|Q(z)| = z$, inequality (1.18) will reduce to inequality (1.6).

Substituting $\beta = 0$ in inequality (1.17), we get the following:

Corollary 6: If $P(Q(z))$ is a polynomial of degree nm having all its zeros in $|z| \leq 1$, then for $R > 1$, we have

$$\min_{|z|=1} |P(Q(Rz))| \geq |R^{nm}| |z|^{nm} \min_{|z|=1} |P(Q(z))|, \quad \text{for } |z| \geq 1. \quad (1.19)$$

For $Q(z) = z$, inequality (1.19) will reduce to inequality (1.7).

We now prove the following interesting result, which provides the compact generalization of inequality (1.8).

Theorem 4: If $P(S(z))$ is a polynomial of degree nm which does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R > r \geq 1$

$$\begin{aligned} & |P(S(Rz)) - \beta P(S(rz))| \\ & \leq \left\{ \frac{|R^{nm} - r^{nm}\beta||z|^{nm} + |1 - \beta|}{2} \right\} \max_{|z|=1} |P(S(z))| \\ & - \left\{ \frac{|R^{nm} - \beta||z|^{nm} - |1 - \beta|}{2} \right\} \min_{|z|=1} |P(S(z))|, \\ & \quad \text{for } |z| \geq 1. \quad (1.20) \end{aligned}$$

The result is best possible and equality holds in inequality (1.20), for $P(Q(z)) = az^{nm} + b$, $|a| \leq |b|$, where $Q(z) = z^n$.

Dividing both side inequality (1.20) by $R - 1$ and take the limit as $R \rightarrow 1$ along with $\beta = r = 1$, we get the following result:

Corollary 7: If $P(S(z))$ is a polynomial of degree nm which does not vanish in $|z| < 1$, then for $R > 1$, we have for $|z| \geq 1$

$$\begin{aligned} |P'(S(z))| & \leq \left\{ \frac{nm|z|^{nm} + 1}{2|S'(z)|} \right\} \max_{|z|=1} |P(S(z))| \\ & - \left\{ \frac{nm|z|^{nm} - 1}{2|S'(z)|} \right\} \min_{|z|=1} |P(S(z))| \quad (1.21) \end{aligned}$$

For $S(z) = z$, inequality (1.21) will reduce to

$$|P'(z)| \leq \left\{ \frac{n|z|^{n-1} + 1}{2} \right\} \max_{|z|=1} |P(z)| - \left\{ \frac{n|z|^{n-1} - 1}{2} \right\} \min_{|z|=1} |P(z)|. \quad (1.22)$$

Substituting $\beta = 0$ in inequality (1.20), we get the following result:

Corollary 8: If $P(S(z))$ is a polynomial of degree nm which does not vanish in $|z| < 1$, then for $R > 1$ and $|z| \geq 1$

$$\begin{aligned} |P(S(Rz))| & \leq \left\{ \frac{|R^{nm}| |z|^{nm} + 1}{2} \right\} \max_{|z|=1} |P(S(z))| \\ & - \left\{ \frac{|R^{nm}| |z|^{nm} - 1}{2} \right\} \min_{|z|=1} |P(S(z))|. \quad (1.23) \end{aligned}$$

For $S(z) = z$ inequality (1.23), reduces to

$$\begin{aligned} |P(Rz)| & \leq \left\{ \frac{|R^n| |z|^n + 1}{2} \right\} \max_{|z|=1} |P(z)| \\ & - \left\{ \frac{|R^n| |z|^n - 1}{2} \right\} \min_{|z|=1} |P(z)|, \quad (1.24) \end{aligned}$$

for $|z| \geq 1$, which is the compact generalization of inequality (1.8).

Letting $R \rightarrow 1$ in inequality (1.24), we have

$$|P(z)| \leq \left\{ \frac{|z|^n + 1}{2} \right\} \max_{|z|=1} |P(z)| - \left\{ \frac{|z|^n - 1}{2} \right\} \min_{|z|=1} |P(z)|, \quad (1.25)$$

for $|z| \geq 1$.

2 Lemmas

For the proof of these Theorems we need the following lemmas.

The first lemma is a special case of the result due to Govil et al. [5].

Lemma 1: If $G(z)$ is a polynomial of degree at most n having all its zeros in $|z| < k$, where $k \leq 1$, then $|G(Rz)| > |G(rz)|$, for $|z| \geq 1$ and $R > r \geq 1$.

Lemma 2: If $P(S(z))$ is a polynomial of degree nm which does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$

$$\begin{aligned} & |P(S(Rz)) - \beta P(S(rz))| \\ & \leq |Q(S(Rz)) - \beta Q(S(rz))|, \quad \text{for } |z| \geq 1. \end{aligned}$$

Proof: Clearly the result is true for $R = 1$. Hence we suppose that $R > r \geq 1$. Since all the zeros of $P(z)$ lie in $|z| \geq 1$, therefore for every real or complex number λ with $|\lambda| > 1$, the polynomial $G(z) = P(S(z)) - \lambda Q(S(z))$, where $Q(S(z)) = z^{nm} P(S(\frac{1}{z}))$ has all its zeros in $|z| \leq 1$. Now applying Lemma 1 to the polynomial $G(z)$, we get

$$|G(Rz)| > |G(rz)|, \quad \text{for } |z| = 1 \text{ and } R > r \geq 1.$$

Since all the zeros of $G(z)$ lie in $|z| < 1$, therefore all the zeros of $G(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$. Hence, for any real or

complex number β with $|\beta| \leq 1$, we have by Rouché's theorem all the zeros of the polynomial

$$H(z) = G(Rz) - \beta G(rz) = (P(S(Rz)) - \beta P(S(rz))) - \lambda(Q(S(Rz)) - \beta Q(S(rz))). \quad (1.26)$$

lie in $|z| < 1$ for $|\lambda| > 1$ and $R > r \geq 1$. Inequality (1.26) implies

$$|P(S(Rz)) - \beta P(S(rz))| \leq |Q(S(Rz)) - \beta Q(S(rz))|,$$

$$\text{for } |z| \geq 1 \text{ and } R > r \geq 1. \quad (1.27)$$

For if this is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$|P(S(Rz_0)) - \beta P(S(rz_0))| > |Q(S(Rz_0)) - \beta Q(S(rz_0))|, \quad \text{for } R > 1.$$

We take

$$\lambda = \frac{P(S(Rz_0)) - \beta P(S(rz_0))}{Q(S(Rz_0)) - \beta Q(S(rz_0))}.$$

So that $|\lambda| > 1$, for this choice of λ , we have $H(z_0) = 0$ for $|z_0| \geq 1$, which is contradiction to the fact that all the zeros of $H(z)$ lie in $|z| < 1$. Thus

$$|P(S(Rz)) - \beta P(S(rz))| \leq |Q(S(Rz)) - \beta Q(S(rz))|, \quad \text{for } |z| \geq 1 \text{ and } R > r \geq 1.$$

3 Proof of Theorems

Proof of Theorem 1: For $R = 1$ there is nothing to prove. Henceforth we assume $R > 1$. If $\max_{|z|=1} |P(Q(z))| = M$, then $|P(Q(z))| \leq M$, for $|z| = 1$. Therefore, by Rouché's theorem it follows that all the zeros of $G(z) = P(Q(z)) + \lambda z^{nm}M$ lie in $|z| < 1$, for every λ with $|\lambda| > 1$.

Now, by lemma 1 we have $|G(z)| \leq |G(R(z))|$, for $|z| = 1$ and $R > r \geq 1$. Since all the zeros of the polynomial $G(R(z))$ lie in $|z| < (\frac{1}{R}) < 1$, therefore, for any real or complex number β with $|\beta| < 1$, we have by Rouché's theorem, all the zeros of the polynomial

$$G(Rz) - G(rz) = (P(Q(Rz)) - \beta P(Q(rz))) + \lambda(R^{nm} - r^{nm}\beta)z^{nm}M$$

also lie in $|z| < 1$ for every $R > r \geq 1$ and $|\lambda| > 1$.

This implies, for $|z| \geq 1$ and $R > r \geq 1$

$$|P(Q(Rz)) - \beta P(Q(rz))| \leq |R^{nm} - \beta||z|^{nm} \max_{|z|=1} |P(Q(z))|.$$

For if this is not true, then there is some point $z = z_0$ with $|z_0| \geq 1$, such that

$$|P(Q(Rz_0)) - \beta P(Q(rz_0))| > |R^{nm} - r^{nm}\beta||z_0|^{nm}M,$$

we take

$$\lambda = \frac{P(Q(Rz_0)) - \beta P(Q(rz_0))}{(R^{nm} - r^{nm}\beta)z_0^{nm}M},$$

then $|\lambda| > 1$ and for this choice of λ we have $G(Rz_0) - \beta G(rz_0) = 0$ with $|z| \geq 1$, which is a contradiction. Hence

$$|P(Q(Rz)) - \beta P(Q(rz))| \leq |R^{nm} - r^{nm}\beta||z|^{nm} \max_{|z|=1} |P(Q(z))|.$$

for $|z| \geq 1$ and $R > r \geq 1$. That proves the Theorem 1.

Proof of Theorem 2: If $\max_{|z|=1} |P(Q(z))| = M$, then $|P(Q(z))| \leq M$, for $|z| = 1$. Therefore, for any real or complex number α with $|\alpha| > 1$ we have by Rouché's theorem, that the polynomial $G(z) = P(S(z)) + \alpha M$ does not vanish in $|z| < 1$. Now, applying Lemma 2 to the polynomial $G(z)$ we get for every real or complex number β with $|\beta| \leq 1$,

$$\begin{aligned} &|P(S(Rz)) - \beta P(S(rz)) + \alpha(1 - \beta)M| \\ &\leq |Q(S(Rz)) - \beta Q(S(rz)) + \overline{\alpha}(R^{nm} - r^{nm}\beta)z^{nm}M|, \end{aligned} \quad (3.1)$$

for $|z| \geq 1$ and $R > r \geq 1$, where $Q(S(z)) = z^{nm}P(S(\frac{1}{z}))$. Choosing the argument of α in the right hand side of inequality (3.1), we have

$$\begin{aligned} &|Q(S(Rz)) - \beta Q(S(rz)) + \overline{\alpha}(R^{nm} - r^{nm}\beta)z^{nm}M| \\ &= |\alpha||R^{nm} - r^{nm}\beta||z|^{nm}M - |Q(S(Rz)) - \beta Q(S(rz))|. \end{aligned}$$

Therefore, from inequality (3.1), we have

$$\begin{aligned} &|P(S(Rz)) - \beta P(S(rz))| - |\alpha||1 - \beta|M \\ &\leq |\alpha||R^{nm} - r^{nm}\beta||z|^{nm}M - |Q(S(Rz)) - \beta Q(S(rz))|, \end{aligned}$$

for $|z| \geq 1$ and $R > r \geq 1$. Equivalently, for $|z| \geq 1$ and $R > r \geq 1$, we have

$$\begin{aligned} &|P(S(Rz)) - \beta P(S(rz))| + |Q(S(Rz)) - \beta Q(S(rz))| \\ &\leq |\alpha|\{|R^{nm} - r^{nm}\beta||z|^{nm} + |1 - \beta|\} \max_{|z|=1} |P(S(z))|. \end{aligned}$$

Let $|\alpha| \rightarrow 1$, we get for $R > r \geq 1$

$$\begin{aligned} &|P(S(Rz)) - \beta P(S(rz))| + |Q(S(Rz)) - \beta Q(S(rz))| \\ &\leq \{|R^{nm} - r^{nm}\beta||z|^{nm} + |1 - \beta|\} \max_{|z|=1} |P(S(z))|, \end{aligned}$$

$|z| \geq 1$. That proves the Theorem 2.

Proof of Theorem 3: If $P(Q(z))$ has a zero on $|z| = 1$, then the result is trivial. So we assume all the zeros of $P(Q(z))$ lie in $|z| < 1$. If $m = \min_{|z|=1} |P(Q(z))|$, then $m > 0$ and $m \leq |P(Q(z))|$ for $|z| = 1$. Suppose that $P(Q(z))$ has all its zeros in $|z| \leq 1$. Therefore if α is any complex number with $|\alpha| < 1$, then it follows by Rouché's theorem that the polynomial

$F(z) = P(Q(z)) - \alpha m z^{nm}$ has all its zeros in $|z| < 1$. Further, we have by lemma 1

$$|F(rz)| < |F(Rz)|, \text{ for } |z| = 1 \text{ and } R > r \geq 1.$$

Since all the zeros of $F(Rz)$ lie in $|z| < \frac{1}{R} < 1$, it follows from Rouché's theorem for every real or complex number β with $|\beta| \leq 1$ and $R > r \geq 1$, that the polynomial $F(Rz) - \beta F(rz)$ has all its zeros in $|z| < 1$. That is

$$F(Rz) - \beta F(rz) \neq 0 \text{ for } |z| \geq 1 \text{ and } R > r \geq 1.$$

or equivalently

$$G(z) = (P(Q(Rz)) - \beta P(Q(rz))) - \alpha m(R^{nm} - r^{nm}\beta)z^n \neq 0.$$

This give for every $|\beta| \leq 1, |z| \geq 1$ and $R > r \geq 1$

$$|P(Q(Rz)) - \beta P(Q(rz))| \geq |\alpha| m |R^{nm} - r^{nm}\beta| |z|^{nm}.$$

For if this is not true, there is a point $z = z_o$ with $|z_o| \geq 1$ such that

$$|P(Q(Rz_o)) - \beta P(Q(rz_o))| < |\alpha| m |R^{nm} - r^{nm}\beta| |z_o|^{nm}.$$

We take

$$\alpha = \frac{P(Q(Rz_o)) - \beta P(Q(rz_o))}{m(R^{nm} - r^{nm}\beta)z_o^{nm}},$$

then $|\alpha| < 1$ and with this choice of α , we have for inequality $G(z_o) = 0$ for $|z_o| \geq 1$. But this is contradiction to the fact that $G(z) \neq 0$ for $|z| \geq 1$. Thus

$$|P(Q(Rz)) - \beta P(Q(rz))| \geq |\alpha| m |R^{nm} - r^{nm}\beta| |z|^{nm}$$

Letting $|\alpha| \rightarrow 1$, we get for $|\beta| \leq 1, |z| \geq 1$ and $R > r \geq 1$

$$|P(Q(Rz)) - \beta P(Q(rz))| \geq m |R^{nm} - r^{nm}\beta| |z|^{nm}.$$

That proves the Theorem 3.

Proof of Theorem 4: If $m = \min_{|z|=1} |P(S(z))|$ then $m \leq |P(S(z))|$ for $|z| \leq 1$. If $P(S(z))$ has all its zeros in $|z| \geq 1$ then for any given complex number α with $|\alpha| \leq 1$ the polynomial $G(z) = P(S(z)) + \alpha m z^{nm}$ does not vanish in $|z| < 1$. Now applying Lemma 2 to the polynomial $G(z)$, we get

$$\begin{aligned} & |P(S(Rz)) - \beta P(S(rz)) + \alpha m(R^{nm} - r^{nm}\beta)z^{nm}| \\ & \leq |Q(S(Rz)) - \beta Q(S(rz)) + \bar{\alpha} m(1 - \beta)|, \end{aligned}$$

for $|z| \geq 1$ and $R > r \geq 1$ where $Q(S(z)) = z^{nm} \overline{P(\frac{1}{z})}$.

Choosing the argument of α in the left hand side of above inequality such that

$$\begin{aligned} & |P(S(Rz)) - \beta P(S(rz)) + \alpha m(R^{nm} - r^{nm}\beta)z^{nm}| \\ & = |P(S(Rz)) - \beta P(S(rz))| + |\alpha| m |R^{nm} - r^{nm}\beta| |z|^{nm} \end{aligned}$$

for $|z| \geq 1$, we get

$$\begin{aligned} & |P(S(Rz)) - \beta P(S(rz))| + |\alpha| m |R^{nm} - r^{nm}\beta| |z|^{nm} \\ & \leq |Q(S(Rz)) - \beta Q(S(rz))| + |\alpha| m |1 - \beta|, \end{aligned}$$

for $|z| \geq 1$ and $R > r \geq 1$. Let $|\alpha| \rightarrow 1$, we obtain for every β with $|\beta| \leq 1$,

$$\begin{aligned} & |P(S(Rz)) - \beta P(S(rz))| \\ & \leq |Q(S(Rz)) - \beta Q(S(rz))| \\ & \quad + m \{ |R^{nm} - r^{nm}\beta| |z|^{nm} - |1 - \beta| \}. \end{aligned} \quad (3.2)$$

for $|z| \geq 1$ and $R > r \geq 1$. The inequality (3.2), with the help of Theorem 2 gives

$$\begin{aligned} & |P(S(Rz)) - \beta P(S(rz))| \leq \\ & \left\{ \frac{|R^{nm} - r^{nm}\beta| |z|^{nm} + |1 - \beta|}{2} \right\} \max_{|z|=1} |P(S(z))| \\ & \quad - \left\{ \frac{|R^{nm} - \beta| |z|^{nm} - |1 - \beta|}{2} \right\} \min_{|z|=1} |P(S(z))|, \\ & \text{for } |z| \geq 1. \end{aligned}$$

This completes the proof of the Theorem 4.

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