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Estimation of Population Product in the Presence of Non-Response and Measurement Error in Successive Sampling

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Abstract: This paper deal with the problem of estimating the finite population product in the presence of non-response and measurement error in successive sampling under different situations: (i) when there is non-response and measurement error on first occasion only; (ii) when there is non-response on first occasion only; (iii) when there is measurement error on first occasion only; (iv) when there is non-response and measurement error on second occasion only; (v) when there is non-response on second occasion only; (vi) when there is measurement error on second occasion only. The properties of the proposed estimators are studied and the gain in efficiency of the proposed estimators over the direct estimate using no information gathered on the first occasion is computed. The proposed strategy has been compared with other existing estimators and the efficiency conditions have been obtained. An empirical study is carried out to study the performance of the theoretical findings.

Keywords: Product estimator, successive sampling, non-response, measurement error and gain in efficiency.

1 Introduction

The theory and practice of surveying the same population at different points of time called repetitive or successive sampling. Usually, information collected on the same population from one period to the next. In such a sampling technique, we perform a partial replacement of units from one occasion to another. For example, labor force surveys are conducted monthly to estimate the employment status, monthly/weekly data on the price of goods are collected to determine Consumer Price Index (CPI), political opinion surveys are conducted at regular intervals to know the voter preference, etc. (Karna and Nath [1]). In such cases, the use of successive sampling may be a better alternative to provide an efficient and reliable estimate. Theory of Successive (Rotation) sampling started with the work of Jessen [2] by utilizing the entire information collected in the previous investigation. Further, this theory was extended by Patterson [3], Rao and Graham [4], Gupta [5], Das [6], Chaturvedi and Tripathi [7], Okafor and Arnab [8], Okafor [9], Feng and Zou [10], Birader and Singh [11], Singh and Singh [12], Singh and Vishwakarma [13], Singh and Vishwakarma [14], Singh and Kumar [15], Kumar [16], Singh and Pal [17,18], etc. In many situations, information on an auxiliary variable may be readily available on the first as well as second occasion. For example, the seating capacity of each vehicle or ship is known in survey sampling of transportation, the number of beds in different hospitals may be known in the hospital surveys, etc. (Singh [19]). In the case of successive sampling, the auxiliary information is considered advantageous to utilize the entire information collected in the previous investigation. Sen [20,21,22] used the auxiliary information on the first occasion for estimating the population mean on the current occasion. Further, Singh et al. [23], and Singh and Pal [24] have also used auxiliary information on both occasions in successive sampling. It is very common to experience in most of the sample surveys that the data cannot be collected from all the units that are selected in the sample survey. For example, the selected families (units) may not be at home on the first attempt and some may refuse to cooperate with the interviewer, even if contacted. This is particularly happening in mail surveys, who are requested to send back their response (Singh and Kumari Priyanka

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[25]). In that case, many respondents do not reply even after some callbacks. The resulting incompleteness of sample units is called non-response. Hansen and Hurwitz [26] suggested a technique of handling non-response while estimating the population mean by taking a sub-sample from the non-respondent group. Further, this technique was studied by Cochran [27], Rao [28], Khare and Srivastava [29], Kumar et al. [30], Singh and Kumar [31], etc. It is also experienced that in sample survey, the researchers also face the problem of measurement error while collecting the data from the respondents (individuals). Measurement error is the difference between the value that is observed and the true value of the variable in the study. For example, in the survey regarding household consumption or expenditure, there may be a great possibility that the respondent may fail to recall that how much they spend on various items over time (Kumar et al. [32]). Many researchers have studied measurement error like Cochran [33], Cochran [34], Fuller [35], Shalabh [36], Manisha and Singh [37,38], Wang [39], Allen et al. [40], Singh and Karpe [41,42,43], Shukla, Pathak and Thakur [44], Sharma and Singh [45], etc. Further, it is also possible that the problem of measurement error and non-response face at the same time. Jackman [46] dealt with both non-response and measurement error simultaneously in the case of voter turnout. Dixon [47] also studied the estimation of non-response bias and measurement error on the data from Consumer Expenditure Quarterly Interview Survey (CEQ), Current Population Survey (CPS), National Health Interview Survey (NHIS), etc. As we know that very few numbers of studies are available where the interaction of both non-response and measurement error are studied together. Azeem [48], Kumar et al. [49], proposed a class of estimators for estimating the population mean in the presence of both non-response and measurement error in the case of stratified random sampling by utilizing two auxiliary variables which are highly correlated variable with the variable under study. Azeem and Hanif [50] were also studied the together effect of non-response and measurement error for estimating the population mean in a simple random sampling scheme. Furthermore, Zahid and Shabbir [51] have also studied the effect of both non-response and measurement error for estimating the population mean in the case of stratified random sampling by using single auxiliary information. Recently, El-Din et al. [52], Zhao et al. [53], Almongy et al. [54,55] have studied the application of different statistical distribution. In the present paper, we made an attempt to estimate the population product in presence of non-response and measurement error in successive sampling over two occasions. To support the theoretical findings, we investigate an empirical study.

2 Sample Selection Procedure

Consider the size of the population contains N units. Assume that the sample size on both occasions is of size n. Here we use a simple random sampling and ignored the factor of correction for the population size N is adequately large (large enough). A simple random sampling of size n be drawn on the first occasion from a population of size N. Let y and x are the characteristics variable on the first and second occasion, respectively. Here we assume that there is the presence of some non-response for estimating the population product in successive sampling. We can divide the population into two classes, in the first class those who will respond and in the second class, those who will not respond and the size of these two classes are N_1 and N_2, respectively. By using simple random sampling n units are selected on the first occasion and the questionnaire is mailed to the selected sample units. Out of these n units, a random sub-sample of size $m = np$ $(0 < p < 1)$ units are retained (matched) in the second occasion and also an additional independent (unmatched with t he first occasion) sample of size $u = nq = n - m, (q = 1 - p)$ is drawn on the second occasion from the remaining population $(N - n)$ units, so that the sample size on the second occasion is also n. Also, let us assume that in the unmatched portion of the sample for both occasions u_1 units respond and u_2 units do not respond. Similarly, on the matched portion m_1 units respond and m_2 units do not respond. Again, a sub-sample of size m_{h2} units are drawn from the non-respondent class of the matched portion of the sample for both occasions for collecting information through personal interviews. Similarly, a sub-sample of size u_{h2} units is drawn from the non-respondent class of the unmatched portion of the sample on both occasions. Also, there is a presence of measurement error associated with these sample units i.e. $V_i = x_i$ - X_i , $U_i = y_i$ - Y_i ; for both occasion, which are random in nature with mean zero and population variance S_U^2 and S_V^2. S_x^2 and S_y^2 are population variances of X and Y, respectively. C_U^2 , C_V^2 , C_x^2 and C_y^2 are the coefficient of variation for variables, respectively. Also, ρ_{yx} , ρ_{uv} , ρ_{yu} and ρ_{vx} are coefficient of correlation between the variable y and x. In our study, both non-response and measurement error are present simultaneously. Following notation are uesd:

x_i (y_i) : the variable $x(y)$ for the i^{th} occasion, $i = 1,2$;

$T_1 = \mu_{y_1}\mu_{x_1}$ $(T_2 = \mu_{y_2}\mu_{x_2})$: the population product on the first (second) occasion;

$\hat{T}_1 = \hat{\mu}_{y_1}\hat{\mu}_{x_1}$ $(\hat{T}_2 = \hat{\mu}_{y_2}\hat{\mu}_{x_2})$: the estimator of the population product on the first (second)occasion;

$R_1 = \frac{\mu_{y_1}}{\mu_{x_1}}$ $(R_2 = \frac{\mu_{y_2}}{\mu_{x_2}})$: the population ratio on the first (second) occasion;

$\hat{T}^{**}_{1m} = \hat{\mu}^{**}_{y_{1m}}\hat{\mu}^{**}_{x_{1m}}$ $(\hat{T}^{**}_{2m} = \hat{\mu}^{**}_{y_{2m}}\hat{\mu}^{**}_{x_{2m}})$: the estimator of the population product on the first (second) occasion based on both non-response and measurement error for the matched portion.

$\hat{T}^{**}_{1u} = \hat{\mu}^{**}_{y_{1u}}\hat{\mu}^{**}_{x_{1u}}$ $(\hat{T}^{**}_{2u} = \hat{\mu}^{**}_{y_{2u}}\hat{\mu}^{**}_{x_{2u}})$: the estimator of the

population product on the first (second) occasion based on both non-response and measurement error for the unmatched portion.
Thus, we have the following set up

Ist occasion	$\hat{T}_{1u}^{**}$	$\hat{T}_{1m}^{**}$	
2nd occasion		$\hat{T}_{2m}^{**}$	$\hat{T}_{2u}^{**}$

where

$\hat{\mu}_{y_{1m}^{**}}=\dfrac{m_1\hat{\mu}_{y_{1m_1}}+m_2\hat{\mu}_{y_{1m_{h2}}}}{m}$, $\hat{\mu}_{y_{2m}^{**}}=\dfrac{m_1\hat{\mu}_{y_{2m_1}}+m_2\hat{\mu}_{y_{2m_{h2}}}}{m}$,

$\hat{\mu}_{y_{1u}^{**}}=\dfrac{u_1\hat{\mu}_{y_{1u_1}}+u_2\hat{\mu}_{y_{1u_{h2}}}}{u}$, $\hat{\mu}_{y_{2u}^{**}}=\dfrac{u_1\hat{\mu}_{y_{2u_1}}+u_2\hat{\mu}_{y_{2u_{h2}}}}{u}$,

$\hat{\mu}_{x_{1m}^{**}}=\dfrac{m_1\hat{\mu}_{x_{1m_1}}+m_2\hat{\mu}_{x_{1m_{h2}}}}{m}$, $\hat{\mu}_{x_{2m}^{**}}=\dfrac{m_1\hat{\mu}_{x_{2m_1}}+m_2\hat{\mu}_{x_{2m_{h2}}}}{m}$,

$\hat{\mu}_{x_{1u}^{**}}=\dfrac{u_1\hat{\mu}_{x_{1u_1}}+u_2\hat{\mu}_{x_{1u_{h2}}}}{u}$, $\hat{\mu}_{x_{2u}^{**}}=\dfrac{u_1\hat{\mu}_{x_{2u_1}}+u_2\hat{\mu}_{x_{2u_{h2}}}}{u}$,

$\hat{\mu}_{y_{jm_1}}=\frac{1}{m_1}\sum_{i=1}^{m_1}y_{ji}$, $\hat{\mu}_{x_{jm_1}}=\frac{1}{m_1}\sum_{i=1}^{m_1}x_{ji}$, $\hat{\mu}_{y_{jm_{h2}}}=\frac{1}{m_{h2}}\sum_{l=1}^{m_{h2}}y_{jl}$,

$\hat{\mu}_{x_{jm_{h2}}}=\frac{1}{m_{h2}}\sum_{l=1}^{m_{h2}}x_{jl}$, $\hat{\mu}_{y_{ju_1}}=\frac{1}{u_1}\sum_{k=1}^{u_1}y_{jk}$, $\hat{\mu}_{x_{ju_1}}=\frac{1}{u_1}\sum_{k=1}^{u_1}x_{jk}$,

$\hat{\mu}_{y_{ju_{h2}}}=\frac{1}{u_{h2}}\sum_{r=1}^{u_{h2}}y_{jr}$, $\hat{\mu}_{x_{ju_{h2}}}=\frac{1}{u_{h2}}\sum_{r=1}^{u_{h2}}x_{jr}$; $j=1,2$.

Now, consider the estimator for population product T_2 on current (second) occasion as:

$$\hat{T}_2^{**}=a\hat{T}_{1u}^{**}+b\hat{T}_{1m}^{**}+c\hat{T}_{2m}^{**}+d\hat{T}_{2u}^{**} \tag{1}$$

where (a,b,c,d) are constants.

We have

$E(\hat{T}_{1u}^{**}) = E(\hat{T}_{1m}^{**}) = T_1$ and $E(\hat{T}_{2u}^{**}) = E(\hat{T}_{2m}^{**}) = T_2$

We find that

$$E(\hat{T}_2^{**})=(a+b)T_1+(c+d)T_2. \tag{2}$$

The estimator $\hat{T}_2^{**}$ be an unbiased estimator of T_2, then we must have

$(a+b)=0$ and $(c+d)=1$.

Substituting the values of b and d , we have

$$\hat{T}_2^{**}=a(\hat{T}_{1u}^{**}-\hat{T}_{1m}^{**})+c\hat{T}_{2m}^{**}+(1-c)\hat{T}_{2u}^{**} \tag{3}$$

To obtain the variance of $\hat{T}_2^{**}$ in the presence of non-response and measurement error, we write

$\hat{\mu}_{y_{1u}}^{**}=\mu_{y_1}+\omega_{y1uo}^{*}, \hat{\mu}_{y_{2u}}^{**}=\mu_{y_2}+\omega_{y2uo}^{*}$,
$\hat{\mu}_{y_{1m}}^{**}=\mu_{y_1}+\omega_{y1mo}^{*}, \hat{\mu}_{y_{2m}}^{**}=\mu_{y_2}+\omega_{y2mo}^{*}$,
$\hat{\mu}_{x_{1u}}^{**}=\mu_{x_1}+\omega_{x1u1}^{*}, \hat{\mu}_{x_{2u}}^{**}=\mu_{x_2}+\omega_{x2u1}^{*}$,
$\hat{\mu}_{x_{1m}}^{**}=\mu_{x_1}+\omega_{x1m1}^{*}, \hat{\mu}_{x_{2m}}^{**}=\mu_{x_2}+\omega_{x2m1}^{*}$,

where

$\omega_{y1uo}^{*}=\frac{1}{\sqrt{u}}\left(\omega_{y1u}^{*}+\omega_{U1u}^{*}\right)$, $\omega_{y2uo}^{*}=\frac{1}{\sqrt{u}}\left(\omega_{y2u}^{*}+\omega_{U2u}^{*}\right)$,
$\omega_{y1mo}^{*}=\frac{1}{\sqrt{m}}\left(\omega_{y1m}^{*}+\omega_{U1m}^{*}\right)$, $\omega_{y2mo}^{*}=\frac{1}{\sqrt{m}}\left(\omega_{y2m}^{*}+\omega_{U2m}^{*}\right)$,
$\omega_{x1u1}^{*}=\frac{1}{\sqrt{u}}\left(\omega_{x1u}^{*}+\omega_{V1u}^{*}\right)$, $\omega_{x2u1}^{*}=\frac{1}{\sqrt{u}}\left(\omega_{x2u}^{*}+\omega_{V2u}^{*}\right)$,
$\omega_{x1m1}^{*}=\frac{1}{\sqrt{m}}\left(\omega_{x1m}^{*}+\omega_{V1m}^{*}\right)$, $\omega_{x2m1}^{*}=\frac{1}{\sqrt{m}}\left(\omega_{x2m}^{*}+\omega_{V2m}^{*}\right)$,

$\omega_{y1u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}\left(Y_{1i}^{*}-\mu_{y_1}\right)$, $\omega_{y2u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}\left(Y_{2i}^{*}-\mu_{y_2}\right)$,
$\omega_{y1m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\left(Y_{1i}^{*}-\mu_{y_1}\right)$, $\omega_{y2m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\left(Y_{2i}^{*}-\mu_{y_2}\right)$,
$\omega_{x1u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}\left(X_{1i}^{*}-\mu_{x_1}\right)$, $\omega_{x2u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}\left(X_{2i}^{*}-\mu_{x_2}\right)$,
$\omega_{x1m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\left(X_{1i}^{*}-\mu_{x_1}\right)$, $\omega_{x2m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\left(X_{2i}^{*}-\mu_{x_2}\right)$,
$\omega_{U1u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}U_{1i}^{*}$, $\omega_{U2u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}U_{2i}^{*}$,
$\omega_{U1m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}U_{1i}^{*}$, $\omega_{U2m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}U_{2i}^{*}$,
$\omega_{V1u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}V_{1i}^{*}$, $\omega_{V2u}^{*}=\frac{1}{\sqrt{u}}\sum_{i=1}^{u}V_{2i}^{*}$,
$\omega_{V1m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}V_{1i}^{*}$, $\omega_{V2m}^{*}=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}V_{2i}^{*}$,
$U_{1i}^{*}=y_{1i}^{*}-Y_{1i}^{*}, V_{1i}^{*}=x_{1i}^{*}-X_{1i}^{*}$,
$U_{2i}^{*}=y_{2i}^{*}-Y_{2i}^{*}, V_{2i}^{*}=x_{2i}^{*}-X_{2i}^{*}$,

such that

$E\left(\omega_{yjuo}^{*}\right)=E\left(\omega_{xju1}^{*}\right)=E\left(\omega_{yjmo}^{*}\right)=E\left(\omega_{xjm1}^{*}\right)=0;(j=1,2)$.
$E\left(\omega_{y1uo}^{*}\right)^2=\frac{1}{u}\left(S_{y1}^2+S_{U1}^2\right)+\frac{\theta}{u}\left(S_{y1(2)}^2+S_{U1(2)}^2\right)$,
$E\left(\omega_{y2uo}^{*}\right)^2=\frac{1}{u}\left(S_{y2}^2+S_{U2}^2\right)+\frac{\theta}{u}\left(S_{y2(2)}^2+S_{U2(2)}^2\right)$,
$E\left(\omega_{y1mo}^{*}\right)^2=\frac{1}{m}\left(S_{y1}^2+S_{U1}^2\right)+\frac{\theta}{m}\left(S_{y1(2)}^2+S_{U1(2)}^2\right)$,
$E\left(\omega_{y2mo}^{*}\right)^2=\frac{1}{m}\left(S_{y2}^2+S_{U2}^2\right)+\frac{\theta}{m}\left(S_{y2(2)}^2+S_{U2(2)}^2\right)$,
$E\left(\omega_{x1u1}^{*}\right)^2=\frac{1}{u}\left(S_{x1}^2+S_{V1}^2\right)+\frac{\theta}{u}\left(S_{x1(2)}^2+S_{V1(2)}^2\right)$,
$E\left(\omega_{x2u1}^{*}\right)^2=\frac{1}{u}\left(S_{x2}^2+S_{V2}^2\right)+\frac{\theta}{u}\left(S_{x2(2)}^2+S_{V2(2)}^2\right)$,
$E\left(\omega_{x1m1}^{*}\right)^2=\frac{1}{m}\left(S_{x1}^2+S_{V1}^2\right)+\frac{\theta}{m}\left(S_{x1(2)}^2+S_{V1(2)}^2\right)$,
$E\left(\omega_{x2m1}^{*}\right)^2=\frac{1}{m}\left(S_{x2}^2+S_{V2}^2\right)+\frac{\theta}{m}\left(S_{x2(2)}^2+S_{V2(2)}^2\right)$,
$E\left(\omega_{y1uo}^{*}\omega_{x1u1}^{*}\right)=\frac{1}{u}\rho_{y1x1}S_{y1}S_{x1}+\frac{\theta}{u}\rho_{y1x1(2)}S_{y1(2)}S_{x1(2)}$,
$E\left(\omega_{y2uo}^{*}\omega_{x2u1}^{*}\right)=\frac{1}{u}\rho_{y2x2}S_{y1}S_{x1}+\frac{\theta}{u}\rho_{y2x2(2)}S_{y2(2)}S_{x2(2)}$,
$E\left(\omega_{y1mo}^{*}\omega_{x1m1}^{*}\right)=\frac{1}{m}\rho_{y1x1}S_{y1}S_{x1}+\frac{\theta}{m}\rho_{y1x1(2)}S_{y1(2)}S_{x1(2)}$,
$E\left(\omega_{y2mo}^{*}\omega_{x2m1}^{*}\right)=\frac{1}{m}\rho_{y2x2}S_{y2}S_{x2}+\frac{\theta}{m}\rho_{y2x2(2)}S_{y2(2)}S_{x2(2)}$,
$E\left(\omega_{y1mo}^{*}\omega_{y2mo}^{*}\right)=\frac{1}{m}\rho_{y1y2}S_{y1}S_{y2}+\frac{\theta}{m}\rho_{y1y2(2)}S_{y1(2)}S_{y2(2)}$,
$E\left(\omega_{y1mo}^{*}\omega_{x2m1}^{*}\right)=\frac{1}{m}\rho_{y1x2}S_{y1}S_{x2}+\frac{\theta}{m}\rho_{y1x2(2)}S_{y1(2)}S_{x2(2)}$,
$E\left(\omega_{x1m1}^{*}\omega_{y2mo}^{*}\right)=\frac{1}{m}\rho_{x1y2}S_{x1}S_{y2}+\frac{\theta}{m}\rho_{x1y2(2)}S_{x1(2)}S_{y2(2)}$,
$E\left(\omega_{x1m1}^{*}\omega_{x2m1}^{*}\right)=\frac{1}{m}\rho_{x1x2}S_{x1}S_{x2}+\frac{\theta}{m}\rho_{x1x2(2)}S_{x1(2)}S_{x2(2)}$.

It can be easily seen that

$$Cov\left(\hat{T}_{1u}^{**},\hat{T}_{1m}^{**}\right)=Cov\left(\hat{T}_{1u}^{**},\hat{T}_{2m}^{**}\right)=Cov\left(\hat{T}_{1u}^{**},\hat{T}_{2u}^{**}\right)$$
$$=Cov\left(\hat{T}_{1m}^{**},\hat{T}_{2u}^{**}\right)=Cov\left(\hat{T}_{2m}^{**},\hat{T}_{2u}^{**}\right)=0$$

where
$\theta=W_2(k-1), W_2=(N_2/N), k=(u_2/u_{h2})=(m_2/m_{h2})$,
$S_{yj}=(\mu_{y_j}C_{yj}), S_{yj(2)}=(\mu_{y_j}C_{yj(2)}); j=1,2,$
$S_{xj}=(\mu_{x_j}C_{xj}), S_{xj(2)}=(\mu_{xj}C_{xj(2)}); j=1,2,$

$(C_{yj}^2, C_{xj}^2; j=1,2)$ and $(C_{yj(2)}^2, C_{xj(2)}^2; j=1,2,)$, respectively denote the coefficient of variation to response and non-response class of $(x_j, y_j; j=1,2)$. $(\rho_{yjxj}, \rho_{yjxj(2)}; j=1,2,)$ and $(\rho_{yjxj}, \rho_{yjxj(2)}; i\neq j=1,2,)$, are the correlation coefficient between y_j and x_j, for respondent and non-respondents respectively.
Then, the variance of $\hat{T}_2^{**}$ is

$$Var\left(\hat{T}_2^{**}\right)=\left(\frac{a^2\mu_{x_1}^2}{npq}\right)B^*+\left(\frac{c^2}{np}+\frac{(1-c)^2}{nq}\right)\mu_{x_2}^2D^* - \left(\frac{2ac\mu_{x_1}\mu_{x_2}}{np}\right)K^* \quad (4)$$

where

$$B^*=\left\{A+\theta A_{(2)}\right\}, D^*=\left\{C+\theta C_{(2)}\right\},$$

$$K^*=\left[\left\{\rho_{y1y2}S_{y1}S_{y2}+R_2\rho_{y1x2}S_{y1}S_{x2}+R_1\rho_{x1y2}S_{x1}S_{y2} + R_1R_2\rho_{x1x2}S_{x1}S_{x2}\right\}+\theta\left\{\rho_{y1y2(2)}S_{y1(2)}S_{y2(2)} + R_2\rho_{y1x2(2)}S_{y1(2)}S_{x2(2)}+R_1\rho_{x1y2(2)}S_{x1(2)}S_{y2(2)} + R_1R_2\rho_{x1x2(2)}S_{x1(2)}S_{x2(2)}\right\}\right],$$

$$A=\left\{S_{y1}^2+S_{U1}^2+R_1^2S_{x1}^2+R_1^2S_{V1}^2+2R_1\rho_{y1x1}S_{y1}S_{x1}\right\},$$

$$A_{(2)}=\left\{S_{y1(2)}^2+S_{U1(2)}^2+R_1^2S_{x1(2)}^2+R_1^2S_{V1(2)}^2 + 2R_1\rho_{y1x1(2)}S_{y1(2)}S_{x1(2)}\right\},$$

$$C=\left\{S_{y2}^2+S_{U2}^2+R_2^2S_{x2}^2+R_2^2S_{V2}^2+2R_2\rho_{y2x2}S_{y2}S_{x2}\right\},$$

$$C_{(2)}=\left\{S_{y2(2)}^2+S_{U2(2)}^2+R_2^2S_{x2(2)}^2+R_2^2S_{V2(2)}^2 + 2R_2\rho_{y2x2(2)}S_{y2(2)}S_{x2(2)}\right\},$$

$$R_1^2=\left(\mu_{y_1}^2/\mu_{x_1}^2\right), R_2^2=\left(\mu_{y_2}^2/\mu_{x_2}^2\right).$$

The variance of $\hat{T}_2^{**}$ is minimum when

$$a=\frac{pq\mu_{x_2}D^*K^*}{\mu_{x_1}(D^*B^*-q^2K^{*2})}=a_{opt}^{(0)} \quad (5)$$

and

$$c=\frac{pD^*B^*}{(D^*B^*-q^2K^{*2})}=c_{opt}^{(0)} \quad (6)$$

Substitute the optimum value of a and c from (5) and (6) in (3), we get the optimum estimator $\hat{T}_2^{**}$ as

$$\hat{T}_2^{***}=\left[\frac{pq\mu_{x_2}D^*K^*}{\mu_{x_1}(D^*B^*-q^2K^{*2})}\left(\hat{T}_{1u}^{**}-\hat{T}_{1m}^{**}\right) + \frac{pD^*B^*}{(D^*B^*-q^2K^{*2})}\hat{T}_{2m}^{**} + \left\{1-\frac{pD^*B^*}{(D^*B^*-q^2K^{*2})}\right\}\hat{T}_{2u}^{**}\right] \quad (7)$$

The variance of $\hat{T}_2^{***}$ is obtained as

$$Var\left(\hat{T}_2^{***}\right)=\frac{D^*\mu_{x_2}^2}{n}\left(\frac{(D^*B^*-qK^{*2})}{(D^*B^*-q^2K^{*2})}\right) \quad (8)$$

Note that if $q=0, p=1$, complete matching or $p=0, q=1$, no matching, thus the variance of $\hat{T}_2^{***}$ given at (8) has the same value as

$$Var\left(\hat{T}_2^{***}\right)=\frac{D^*\mu_{x_2}^2}{n} \quad (9)$$

Thus, for current estimates, equal precision is obtained either by keeping the same sample or by changing it on a every occasion.

If $\mu_{x_1}=\mu_{x_2}$, thus estimator given in (7) is simplified as

$$\hat{T}_2^{***}=\left[\frac{pqD^*K^*}{(D^*B^*-q^2K^{*2})}\left(\hat{T}_{1u}^{**}-\hat{T}_{1m}^{**}\right) + \frac{pD^*B^*}{(D^*B^*-q^2K^{*2})}\hat{T}_{2m}^{**} + \left\{1-\frac{pD^*B^*}{(D^*B^*-q^2K^{*2})}\right\}\hat{T}_{2u}^{**}\right] \quad (10)$$

then the variance is unchanged i.e. same as in (8).

To minimize the variance of $\hat{T}_2^{***}$, differentiate $Var\left(\hat{T}_2^{***}\right)$ with respect to q and equating to zero, we have

$$q=\frac{D^*B^*-\sqrt{D^{*2}B^{*2}-D^*B^*K^{*2}}}{K^{*2}}=q_{opt}^{(0)} \quad (11)$$

Thus, the minimum variance of $\hat{T}_2^{***}$ is given as

$$min.Var\left(\hat{T}_2^{***}\right)=\frac{D^*\mu_{x_2}^2}{n}\frac{D^*B^*+\sqrt{D^{*2}B^{*2}-D^*B^*K^{*2}}}{2D^*B^*} \quad (12)$$

However, if only the estimate using information gathered on the second occasion is considered, then the estimator of the population product is

$$\hat{T}^{**}=p\hat{T}_{2m}^{**}+q\hat{T}_{2u}^{**} \quad (13)$$

The variance of $\hat{T}^{**}$ is given as:

$$Var\left(\hat{T}^{**}\right)=\frac{D^*\mu_{x_2}^2}{n} \quad (14)$$

Remark 1: When there is non-response and measurement error on the first occasion only, then minimum variance linear unbiased estimator (MVLUE) for the population product on current occasion can be obtained as follow

$$\hat{T}_{21}^{**} = a\left(\hat{T}_{1u}^{**} - \hat{T}_{1m}^{**}\right) + c\hat{T}_{2m} + (1-c)\hat{T}_{2u} \tag{15}$$

where

$\hat{T}_{2m} = \hat{\mu}_{y_2m}\hat{\mu}_{x_2m}$, $\hat{T}_{2u} = \hat{\mu}_{y_2u}\hat{\mu}_{x_2u}$,
$\hat{\mu}_{y_2m} = \mu_{y_2}(1+\varepsilon_{2m0})$, $\hat{\mu}_{x_2m} = \mu_{x_2}(1+\varepsilon_{2m1})$,
$\hat{\mu}_{y_2u} = \mu_{y_2}(1+\varepsilon_{2u0})$, $\hat{\mu}_{x_2u} = \mu_{x_2}(1+\varepsilon_{2u1})$,
$E(\varepsilon_{2m0}) = E(\varepsilon_{2m1}) = E(\varepsilon_{2u0}) = E(\varepsilon_{2u1}) = 0$,
$E(\varepsilon_{2m0})^2 = \frac{1}{m}C_{y2}^2, E(\varepsilon_{2m1})^2 = \frac{1}{m}C_{x2}^2$,
$E(\varepsilon_{2u0})^2 = \frac{1}{u}C_{y2}^2, E(\varepsilon_{2u1})^2 = \frac{1}{u}C_{x2}^2$,
$E(\varepsilon_{2m0}\varepsilon_{2m1}) = \frac{1}{m}\rho_{y2x2}C_{y2}C_{x2}$,
$E(\varepsilon_{2u0}\varepsilon_{2u1}) = \frac{1}{u}\rho_{y2x2}C_{y2}C_{x2}$.

The variance of $\hat{T}_{21}^{**}$ is given as

$$Var\left(\hat{T}_{21}^{**}\right) = \left(\frac{a^2\mu_{x_1}^2}{npq}\right)B^* + \left(\frac{c^2}{np} + \frac{(1-c)^2}{nq}\right)\mu_{x_2}^2 H - \left(\frac{2ac\mu_{x_1}\mu_{x_2}}{np}\right)k_1 \tag{16}$$

where

$H_1 = \left\{S_{y2}^2 + R_2^2 S_{x2}^2 + 2R_2 S_{y2x2}\right\}$,
$k_1 = \left\{S_{y1y2} + R_2 S_{y1x2} + R_1 S_{x1y2} + R_1 R_2 S_{x1x2}\right\}$

The variance of $\hat{T}_{21}^{**}$ is minimum when

$a = \frac{pq\mu_{x_2}k_1 H}{\mu_{x_1}(HB^* - q^2k_1^2)} = a_{opt}^{(1)}$ and $c = \frac{pHB^*}{(HB^* - q^2k_1^2)} = c_{opt}^{(1)}$

Substitute the optimum values of a and c in (15). Thus, the estimator $\hat{T}_{21}^{**}$ turns out to be:

$$\hat{T}_{21}^{***} = \left[\frac{pq\mu_{x_2}k_1 H}{\mu_{x_1}(HB^* - q^2k_1^2)}\left(\hat{T}_{1u}^{**} - \hat{T}_{1m}^{**}\right) + \frac{pHB^*}{(HB^* - q^2k_1^2)}\hat{T}_{2m} + \left\{1 - \frac{pHB^*}{(HB^* - q^2k_1^2)}\right\}\hat{T}_{2u}\right] \tag{17}$$

The variance of $\hat{T}_{21}^{***}$ is obtained as

$$Var\left(\hat{T}_{21}^{***}\right) = \frac{H\mu_{x_2}^2}{n}\left(\frac{(HB^* - qk_1^2)}{(HB^* - q^2k_1^2)}\right) \tag{18}$$

The optimum value of q is obtained as

$$q = \frac{HB^* - \sqrt{H^2B^{*2} - HB^*k_1^2}}{k_1^2} = q_{opt}^{(1)} \tag{19}$$

Now put the optimum value of q from (19) in (18), we get the minimum variance of $\hat{T}_{21}^{***}$ as

$$min.Var\left(\hat{T}_{21}^{***}\right) = \frac{H\mu_{x_2}^2}{n}\,\frac{HB^* + \sqrt{H^2B^{*2} - HB^*k_1^2}}{2HB^*} \tag{20}$$

Remark 2: When there is non-response on the first occasion only, then minimum variance linear unbiased estimator (MVLUE) for the population product on current occasion can be obtained as follows

$$\hat{T}_{22}^{*} = a\left(\hat{T}_{1u}^{*} - \hat{T}_{1m}^{*}\right) + c\hat{T}_{2m} + (1-c)\hat{T}_{2u} \tag{21}$$

where

$\hat{T}_{1u}^* = \hat{\mu}_{y_1u}^*\hat{\mu}_{x_1u}^*$, $\hat{T}_{1m}^* = \hat{\mu}_{y_1m}^*\hat{\mu}_{x_1m}^*$,
$\hat{\mu}_{y_1u}^* = \mu_{y_1}(1+e_{1u0})$, $\hat{\mu}_{x_1u}^* = \mu_{x_1}(1+e_{1u1})$,
$\hat{\mu}_{y_1m}^* = \mu_{y_1}(1+e_{1m0})$, $\mu_{x_1m}^* = \mu_{x_1}(1+e_{1m1})$,
$E(e_{1u0}) = E(e_{1u1}) = E(e_{1m0}) = E(e_{1m1}) = 0$,
$E(e_{1u0})^2 = \frac{1}{u}C_{y1}^2 + \frac{\theta}{u}C_{y1(2)}^2$, $E(e_{1u1})^2 = \frac{1}{u}C_{x1}^2 + \frac{\theta}{u}C_{x1(2)}^2$,
$E(e_{1u0}e_{1u1}) = \frac{1}{u}\rho_{y1x1}C_{y1}C_{x1} + \frac{\theta}{u}\rho_{y1x1(2)}C_{y1(2)}C_{x1(2)}$,
$E(e_{1m0})^2 = \frac{1}{m}C_{y1}^2 + \frac{\theta}{m}C_{y1(2)}^2$, $E(e_{1m1})^2 = \frac{1}{m}C_{x1}^2 + \frac{\theta}{m}C_{x1(2)}^2$,
$E(e_{1m0}e_{1m1}) = \frac{1}{m}\rho_{y1x1}C_{y1}C_{x1} + \frac{\theta}{m}\rho_{y1x1(2)}C_{y1(2)}C_{x1(2)}$.

The variance of $\hat{T}_{22}^{*}$ is obtained as

$$Var\left(\hat{T}_{22}^{*}\right) = \left(\frac{a^2\mu_{x_1}^2}{npq}\right)E^* + \left(\frac{c^2}{np} + \frac{(1-c)^2}{nq}\right)\mu_{x_2}^2 H - \left(\frac{2ac\mu_{x_1}\mu_{x_2}}{np}\right)k_1 \tag{22}$$

where

$E^* = \left\{G + \theta G_{(2)}\right\}$, $G = S_{y1}^2 + R_1^2 S_{x1}^2 + 2R_1 S_{y1x1}$,
$G_{(2)} = S_{y1(2)}^2 + R_1^2 S_{x1(2)}^2 + 2R_1 S_{y1x1(2)}$.

The variance of $\hat{T}_{22}^{*}$ is minimum when

$a = \frac{pq\mu_{x_2}k_1 H}{\mu_{x_1}(HE^* - q^2k_1^2)} = a_{opt}^{(2)}$ and $c = \frac{pHE^*}{(HE^* - q^2k_1^2)} = c_{opt}^{(2)}$

Substitute the optimum values of a and c in (21), the optimum estimator $\hat{T}_{22}^{*}$ can be written as

$$\hat{T}_{22}^{**} = \left[\frac{pq\mu_{x_2}k_1 H}{\mu_{x_1}(HE^* - q^2k_1^2)}\left(\hat{T}_{1u}^{*} - \hat{T}_{1m}^{*}\right) + \frac{pHE^*}{(HE^* - q^2k_1^2)}\hat{T}_{2m} + \left\{1 - \frac{pHE^*}{(HE^* - q^2k_1^2)}\right\}\hat{T}_{2u}\right] \tag{23}$$

The variance of $\hat{T}_{22}^{**}$ is obtained as

$$Var\left(\hat{T}_{22}^{**}\right) = \frac{H\mu_{x_2}^2}{n}\left(\frac{(HE^* - qk_1^2)}{(HE^* - q^2k_1^2)}\right) \tag{24}$$

which is minimum, when

$$q=\frac{HE^{*}-\sqrt{H^{2}E^{*2}-HE^{*}k_{1}^{2}}}{k_{1}^{2}}=q_{opt}^{(2)} \tag{25}$$

After putting the optimum value of q from (25) in (24), we get the minimum variance of $\hat{T}_{22}^{**}$ as

$$min.Var\left(\hat{T}_{22}^{**}\right)=\frac{H\mu_{x_2}^{2}}{n}\frac{HE^{*}+\sqrt{H^{2}E^{*2}-HE^{*}k_{1}^{2}}}{2HE^{*}} \tag{26}$$

Remark 3: When there is measurement error on the first occasion only, then minimum variance linear unbiased estimator (MVLUE) for the population product on current occasion can be obtained as follows

$$\hat{T}_{23}^{'}=a\left(\hat{T}_{1u}^{'}-\hat{T}_{1m}^{'}\right)+c\hat{T}_{2m}+\left(1-c\right)\hat{T}_{2u} \tag{27}$$

where

$\hat{T}_{1u}^{'}=\hat{\mu}_{y_1u}^{'}\hat{\mu}_{x_1u}^{'}$, $\hat{T}_{1m}^{'}=\hat{\mu}_{y_1m}^{'}\hat{\mu}_{x_1m}^{'}$,
$\hat{\mu}_{y_1u}^{'}=\mu_{y_1}+\omega_{y1u0}$, $\hat{\mu}_{x_1u}^{'}=\mu_{x_1}+\omega_{x1u1}$,
$\hat{\mu}_{y_1m}^{'}=\mu_{y_1}+\omega_{y1m0}$, $\hat{\mu}_{x_1m}^{'}=\mu_{X_1}+\omega_{x1m1}$,
$\omega_{y1uo}=\frac{1}{\sqrt{u}}\left(\omega_{y1u}+\omega_{U1u}\right)$, $\omega_{x1u1}=\frac{1}{\sqrt{u}}\left(\omega_{x1u}+\omega_{V1u}\right)$,
$\omega_{y1mo}=\frac{1}{\sqrt{m}}\left(\omega_{y1m}+\omega_{U1m}\right)$, $\omega_{x1m1}=\frac{1}{\sqrt{m}}\left(\omega_{x1m}+\omega_{V1m}\right)$,
$E\left(\omega_{y1uo}\right)^{2}=\frac{1}{u}\left(S_{y1}^{2}+S_{U1}^{2}\right)$, $E\left(\omega_{x1u1}\right)^{2}=\frac{1}{u}\left(S_{x1}^{2}+S_{V1}^{2}\right)$,
$E\left(\omega_{y1uo}\omega_{x1u1}\right)=\frac{1}{u}\rho_{y1x1}S_{y1}S_{x1}$,
$E\left(\omega_{y1mo}\right)^{2}=\frac{1}{m}\left(S_{y1}^{2}+S_{U1}^{2}\right)$, $E\left(\omega_{x1m1}\right)^{2}=\frac{1}{m}\left(S_{x1}^{2}+S_{V1}^{2}\right)$,
$E\left(\omega_{y1mo}\omega_{x1m1}\right)=\frac{1}{m}\rho_{y1x1}S_{y1}S_{x1}$.

The variance of $\hat{T}_{23}^{'}$ is given as

$$Var\left(\hat{T}_{23}^{'}\right)=\left(\frac{a^{2}\mu_{x_1}^{2}}{npq}\right)A+\left(\frac{c^{2}}{np}+\frac{(1-c)^{2}}{nq}\right)\mu_{x_2}^{2}H-\left(\frac{2ac\mu_{x_1}\mu_{x_2}}{np}\right)k_{1} \tag{28}$$

The above variance of $\hat{T}_{23}^{'}$ is minimum when

$a=\frac{pq\mu_{x_2}k_1H}{\mu_{x_1}(HA-q^2k_1^2)}=a_{opt}^{(3)}$ and $c=\frac{pHA}{(HA-q^2k_1^2)}=c_{opt}^{(3)}$

Substitute the optimum values of a and c in (27),we get

$$\hat{T}_{23}^{''}=\left[\frac{pq\mu_{x_2}k_1H}{\mu_{x_1}(HA-q^2k_1^2)}\left(\hat{T}_{1u}^{'}-\hat{T}_{1m}^{'}\right)+\frac{pHA}{(HA-q^2k_1^2)}\hat{T}_{2m}+\left\{1-\frac{pHA}{(HA-q^2k_1^2)}\right\}\hat{T}_{2u}\right] \tag{29}$$

The variance of $\hat{T}_{23}^{''}$ is obtained as

$$Var\left(\hat{T}_{23}^{''}\right)=\frac{H\mu_{x_2}^{2}}{n}\left(\frac{(HA-qk_1^2)}{(HA-q^2k_1^2)}\right) \tag{30}$$

which is minimum, when

$$q=\frac{HA-\sqrt{H^{2}A^{2}-HAk_{1}^{2}}}{k_{1}^{2}}=q_{opt}^{(3)} \tag{31}$$

After putting the optimum value of q from (31) in (30), we get the minimum variance of $\hat{T}_{23}^{''}$ as

$$min.Var\left(\hat{T}_{23}^{''}\right)=\frac{H\mu_{x_2}^{2}}{n}\frac{HA+\sqrt{H^{2}A^{2}-HAk_{1}^{2}}}{2HA} \tag{32}$$

Remark 4: When there is non-response and measurement error on the second occasion only, then minimum variance linear unbiased estimator (MVLUE) for the population product on current occasion can be obtained as follow

$$\hat{T}_{24}^{**}=a\left(\hat{T}_{1u}-\hat{T}_{1m}\right)+c\hat{T}_{2m}^{**}+\left(1-c\right)\hat{T}_{2u}^{**} \tag{33}$$

where

$\hat{T}_{1u}=\hat{\mu}_{y_1u}\hat{\mu}_{x_1u}$, $\hat{T}_{1m}=\hat{\mu}_{y_1m}\hat{\mu}_{x_1m}$,
$\hat{\mu}_{y_1u}=\mu_{y_1}\left(1+\varepsilon_{1u0}\right)$, $\hat{\mu}_{x_1u}=\mu_{x_1}\left(1+\varepsilon_{1u1}\right)$,
$\hat{\mu}_{y_{1m}}=\mu_{y_1}\left(1+\varepsilon_{1m0}\right)$, $\hat{\mu}_{x_1m}=\mu_{x_1}\left(1+\varepsilon_{1m1}\right)$,
$E\left(\varepsilon_{1u0}\right)=E\left(\varepsilon_{1u1}\right)=E\left(\varepsilon_{1m0}\right)=E\left(\varepsilon_{1m1}\right)=0$,
$E\left(\varepsilon_{1u0}\right)^{2}=\frac{1}{u}C_{y1}^{2}$, $E\left(\varepsilon_{1u1}\right)^{2}=\frac{1}{u}C_{x1}^{2}$,
$E\left(\varepsilon_{1m0}\right)^{2}=\frac{1}{m}C_{y1}^{2}$, $E\left(\varepsilon_{1m1}\right)^{2}=\frac{1}{m}C_{x1}^{2}$,
$E\left(\varepsilon_{1u0}\varepsilon_{1u1}\right)=\frac{1}{u}\rho_{y1x1}C_{y1}C_{x1}$,
$E\left(\varepsilon_{1m0}\varepsilon_{1m1}\right)=\frac{1}{m}\rho_{y1x1}C_{y1}C_{x1}$.

The variance of $\hat{T}_{24}^{**}$ is given as

$$Var\left(\hat{T}_{24}^{**}\right)=\left(\frac{a^{2}\mu_{x_1}^{2}}{npq}\right)G+\left(\frac{c^{2}}{np}+\frac{(1-c)^{2}}{nq}\right)\mu_{x_2}^{2}D^{*}-\left(\frac{2ac\mu_{x_1}\mu_{x_2}}{np}\right)k_{1} \tag{34}$$

The variance of $\hat{T}_{24}^{**}$ is minimum when

$a=\frac{pq\mu_{x_2}k_1D*}{\mu_{x_1}(D^*G-q^2k_1^2)}=a_{opt}^{(4)}$ and $c=\frac{pD^*G}{(D^*G-q^2k_1^2)}=c_{opt}^{(4)}$

Substitute the optimum values of a and c in (33). Thus the estimator $\hat{T}_{24}^{**}$ becomes

$$\hat{T}_{24}^{***}=\left[\frac{pq\mu_{x_2}k_1D^{*}}{\mu_{x_1}(D^{*}G^{*}-q^2k_1^2)}\left(\hat{T}_{1u}-\hat{T}_{1m}\right)+\frac{pD^{*}G}{(D^{*}G-q^2k_1^2)}\hat{T}_{2m}^{**}+\left\{1-\frac{pD^{*}G}{(D^{*}G-q^2k_1^2)}\right\}\hat{T}_{2u}^{**}\right] \tag{35}$$

The variance of $\hat{T}_{24}^{***}$ is obtained as

$$Var\left(\hat{T}_{24}^{***}\right)=\frac{\mu_{x_2}^{2}D^{*}}{n}\left(\frac{(D^{*}G-qk_1^2)}{(D^{*}G-q^2k_1^2)}\right) \tag{36}$$

which is minimum, when

$$q = \frac{D^*G - \sqrt{D^{*2}G^2 - D^*Gk_1^2}}{k_1^2} = q_{opt}^{(4)} \quad (37)$$

Substitute the optimum value of q from (37) in (36), we get the minimum variance of $\hat{T}_{24}^{***}$ as

$$min.Var\left(\hat{T}_{24}^{***}\right) = \frac{D^*\mu_{x_2}^2}{n}\frac{D^*G + \sqrt{D^{*2}G^2 - D^*Gk_1^2}}{2D^*G} \quad (38)$$

Remark 5: When there is non-response on the second occasion only, then minimum variance linear unbiased estimator (MVLUE) for the population product on current occasion can be obtained as follows

$$\hat{T}_{25}^* = a\left(\hat{T}_{1u} - \hat{T}_{1m}\right) + c\hat{T}_{2m}^* + (1-c)\hat{T}_{2u}^* \quad (39)$$

where

$\hat{T}_{2m}^* = \hat{\mu}_{y_2m}^*\hat{\mu}_{x_2m}^*$, $\hat{T}_{2u}^* = \hat{\mu}_{y_2u}^*\hat{\mu}_{x_2u}^*$,
$\hat{\mu}_{y_2m}^* = \mu_{y_2}(1+e_{2m0})$, $\hat{\mu}_{x_2m}^* = \mu_{x_2}(1+e_{2m1})$,
$\hat{\mu}_{y_2u}^* = \mu_{y_2}(1+e_{2u0})$, $\hat{\mu}_{x_2u}^* = \mu_{x_2}(1+e_{2u1})$,
$E(e_{2m0}) = E(e_{2m1}) = E(e_{2u0}) = E(e_{2u1}) = 0$,
$E(e_{2m0})^2 = \frac{1}{m}C_{y2}^2 + \frac{\theta}{m}C_{y2(2)}^2, E(e_{2m1})^2 = \frac{1}{m}C_{x2}^2 + \frac{\theta}{m}C_{x2(2)}^2$,
$E(e_{2m0}e_{2m1}) = \frac{1}{m}\rho_{y2x2}C_{y2}C_{x2} + \frac{\theta}{m}\rho_{y2x2(2)}C_{y2(2)}C_{x2(2)}$,
$E(e_{2u0})^2 = \frac{1}{u}C_{y2}^2 + \frac{\theta}{u}C_{y2(2)}^2, E(e_{2u1})^2 = \frac{1}{u}C_{x2}^2 + \frac{\theta}{u}C_{x2(2)}^2$,
$E(e_{2u0}e_{2u1}) = \frac{1}{u}\rho_{y2x2}C_{y2}C_{x2} + \frac{\theta}{u}\rho_{y2x2(2)}C_{y2(2)}C_{x2(2)}$.

The variance of $\hat{T}_{25}^*$ is given as

$$Var\left(\hat{T}_{25}^*\right) = \left(\frac{a^2\mu_{x_1}^2}{npq}\right)G + \left(\frac{c^2}{np} + \frac{(1-c)^2}{nq}\right)\mu_{x_2}^2F^* - \left(\frac{2ac\mu_{x_1}\mu_{x_2}}{np}\right)k_1 \quad (40)$$

where

$F^* = \{H + \theta H_{(2)}\}$,

The above variance of $\hat{T}_{25}^*$ is minimum for

$a = \frac{pq\mu_{x_2}k_1F^*}{\mu_{x_1}(F^*G - q^2k_1^2)} = a_{opt}^{(5)}$ and $c = \frac{pF^*G}{(F^*G - q^2k_1^2)} = c_{opt}^{(5)}$

Substitute the optimum values of a and c in (39), the estimator $\hat{T}_{25}^*$ becomes

$$\hat{T}_{25}^{**} = \left[\frac{pq\mu_{x_2}k_1F^*}{\mu_{x_1}(F^*G - q^2k_1^2)}\left(\hat{T}_{1u} - \hat{T}_{1m}\right) + \frac{pF^*G}{(F^*G - q^2k_1^2)}\hat{T}_{2m}^* + \left\{1 - \frac{pF^*G}{(F^*G - q^2k_1^2)}\right\}\hat{T}_{2u}^*\right] \quad (41)$$

The variance of $\hat{T}_{25}^{**}$ is obtained as

$$Var\left(\hat{T}_{25}^{**}\right) = \frac{\mu_{x_2}^2F^*}{n}\left(\frac{(F^*G - qk_1^2)}{(F^*G - q^2k_1^2)}\right) \quad (42)$$

which is minimum, when

$$q = \frac{F^*G - \sqrt{F^{*2}G^2 - F^*Gk_1^2}}{k_1^2} = q_{opt}^{(5)} \quad (43)$$

After substituting the optimum value of q from (43) in (42), we get the minimum variance of $\hat{T}_{25}^{**}$ as

$$min.Var\left(\hat{T}_{25}^{**}\right) = \frac{F^*\mu_{x_2}^2}{n}\frac{F^*G + \sqrt{F^{*2}G^2 - F^*Gk_1^2}}{2F^*G} \quad (44)$$

Remark 6: When there is measurement error on the second occasion only, then minimum variance linear unbiased estimator (MVLUE) for the population product on current occasion can be obtained as follows

$$\hat{T}_{26}' = a\left(\hat{T}_{1u} - \hat{T}_{1m}\right) + c\hat{T}_{2m}' + (1-c)\hat{T}_{2u}' \quad (45)$$

where

$\hat{T}_{2m}' = \hat{\mu}_{y_2m}'\hat{\mu}_{x_2m}'$, $\hat{T}_{2u}' = \hat{\mu}_{y_2u}'\hat{\mu}_{x_2u}'$,
$\hat{\mu}_{y_2m}' = \mu_{y_2} + \omega_{y2m0}$, $\hat{\mu}_{x_2m}' = \mu_{x_2} + \omega_{x2m1}$,
$\hat{\mu}_{y_2u}' = \mu_{y_2} + \omega_{y2u0}$, $\hat{\mu}_{x_2u}' = \mu_{x_2} + \omega_{x2u1}$,
$\omega_{y2uo} = \frac{1}{\sqrt{u}}(\omega_{y2u} + \omega_{U2u})$, $\omega_{x2u1} = \frac{1}{\sqrt{u}}(\omega_{x2u} + \omega_{V2u})$,
$\omega_{y2mo} = \frac{1}{\sqrt{m}}(\omega_{y2m} + \omega_{U2m})$, $\omega_{x2m1} = \frac{1}{\sqrt{m}}(\omega_{x2m} + \omega_{V2m})$,
$E(\omega_{y2uo})^2 = \frac{1}{u}(S_{y2}^2 + S_{U2}^2)$, $E(\omega_{x2u1})^2 = \frac{1}{u}(S_{x2}^2 + S_{V2}^2)$,
$E(\omega_{y2mo})^2 = \frac{1}{m}(S_{y2}^2 + S_{U2}^2)$, $E(\omega_{x2m1})^2 = \frac{1}{m}(S_{x2}^2 + S_{V2}^2)$,
$E(\omega_{y2uo}\omega_{x2u1}) = \frac{1}{u}\rho_{y2x2}S_{y2}S_{x2}$,
$E(\omega_{y2mo}\omega_{x2m1}) = \frac{1}{m}\rho_{y2x2}S_{y2}S_{x2}$.

The variance of $\hat{T}_{26}'$ is obtained as

$$Var\left(\hat{T}_{26}'\right) = \left(\frac{a^2\mu_{x_1}^2}{npq}\right)G + \left(\frac{c^2}{np} + \frac{(1-c)^2}{nq}\right)\mu_{x_2}^2C - \left(\frac{2ac\mu_{x_1}\mu_{x_2}}{np}\right)k_1 \quad (46)$$

The above variance of $\hat{T}_{26}'$ is minimized when

$a = \frac{pq\mu_{x_2}k_1C}{\mu_{x_1}(CG - q^2k_1^2)} = a_{opt}^{(6)}$ and $c = \frac{pCG}{(CG - q^2k_1^2)} = c_{opt}^{(6)}$

Substitute the optimum values of a and c in (45), the estimator of $\hat{T}_{26}'$ turns out to be

$$\hat{T}_{26}'' = \left[\frac{pq\mu_{x_2}k_1C}{\mu_{x_1}(CG - q^2k_1^2)}\left(\hat{T}_{1u} - \hat{T}_{1m}\right) + \frac{pCG}{(CG - q^2k_1^2)}\hat{T}_{2m}' + \left\{1 - \frac{pCG}{(CG - q^2k_1^2)}\right\}\hat{T}_{2u}'\right] \quad (47)$$

The variance of $\hat{T}_{26}''$ is obtained as

$$Var\left(\hat{T}_{26}''\right) = \frac{C\mu_{x_2}^2}{n}\left(\frac{(CG - qk_1^2)}{(CG - q^2k_1^2)}\right) \quad (48)$$

which is minimum, when

$$q = \frac{CG - \sqrt{C^2G^2 - CGk_1^2}}{k_1^2} = q_{opt}^{(6)} \tag{49}$$

After putting the optimum value of q from (49) in (48), we get the minimum variance of $\hat{T}_{26}''$ as

$$min.Var\left(\hat{T}_{26}''\right) = \frac{C\mu_{x_2}^2}{n} \frac{CG + \sqrt{C^2G^2 - CGk_1^2}}{2CG} \tag{50}$$

3 Comparisons between Variance of the estimators

(i) Comparisons between Variance of $\hat{T}_2^{***}$ and $\hat{T}_{21}^{***}$, It can be seen that

$Var\left(\hat{T}_2^{***}\right) < Var\left(\hat{T}_{21}^{***}\right)$, if

$$\left(\frac{HB^* - qk_1^2}{HB^* - q^2k_1^2}\right) H > \left(\frac{D^*B^* - qK^{*2}}{D^*B^* - q^2K^{*2}}\right) D^* \tag{51}$$

(ii) Comparisons between Variance of $\hat{T}_2^{***}$ and $\hat{T}_{22}^{**}$, it can be seen that

$Var\left(\hat{T}_2^{***}\right) < Var\left(\hat{T}_{22}^{**}\right)$, if

$$\left(\frac{HE^* - qk_1^2}{HE^* - q^2k_1^2}\right) H > \left(\frac{D^*B^* - qK^{*2}}{D^*B^* - q^2K^{*2}}\right) D^* \tag{52}$$

(iii) Comparisons between Variance of $\hat{T}_2^{***}$ and $\hat{T}_{23}''$, it can be seen that

$Var\left(\hat{T}_2^{***}\right) < Var\left(\hat{T}_{23}''\right)$, if

$$\left(\frac{HA - qk_1^2}{HA - q^2k_1^2}\right) H > \left(\frac{D^*B^* - qK^{*2}}{D^*B^* - q^2K^{*2}}\right) D^* \tag{53}$$

(iv) Comparisons between Variance of $\hat{T}_2^{***}$ and $\hat{T}_{24}^{***}$, it can be seen that

$Var\left(\hat{T}_2^{***}\right) < Var\left(\hat{T}_{24}^{***}\right)$, if

$$\left(\frac{D^*G - qk_1^2}{D^*G - q^2k_1^2}\right) > \left(\frac{D^*B^* - qK^{*2}}{D^*B^* - q^2K^{*2}}\right) \tag{54}$$

(v) Comparisons between Variance of $\hat{T}_2^{***}$ and $\hat{T}_{25}^{**}$, it can be seen that

$Var\left(\hat{T}_2^{***}\right) < Var\left(\hat{T}_{25}^{**}\right)$, if

$$\left(\frac{F^*G - qk_1^2}{F^*G - q^2k_1^2}\right) F^* > \left(\frac{D^*B^* - qK^{*2}}{D^*B^* - q^2K^{*2}}\right) D^* \tag{55}$$

(vi) Comparisons between Variance of $\hat{T}_2^{***}$ and $\hat{T}_{26}''$, it can be seen that

$Var\left(\hat{T}_2^{***}\right) < Var\left(\hat{T}_{26}''\right)$, if

$$\left(\frac{CG - qk_1^2}{CG - q^2k_1^2}\right) C > \left(\frac{D^*B^* - qK^{*2}}{D^*B^* - q^2K^{*2}}\right) D^* \tag{56}$$

(vii) Comparisons between Variance of $\hat{T}_2^{***}$ and $\hat{T}^{**}$, it can be seen that

$Var\left(\hat{T}_2^{***}\right) < Var\left(\hat{T}^{**}\right)$, if

$$q < 1 \tag{57}$$

(viii) Comparisons between Variance of $\hat{T}_{21}^{***}$ and $\hat{T}_{22}^{**}$, it can be seen that

$Var\left(\hat{T}_{21}^{***}\right) < Var\left(\hat{T}_{22}^{**}\right)$, if

$$\left(\frac{HE^* - qk_1^2}{HE^* - q^2k_1^2}\right) > \left(\frac{HB^* - qk_1^2}{HB^* - q^2k_1^2}\right) \tag{58}$$

(ix) Comparisons between Variance of $\hat{T}_{21}^{***}$ and $\hat{T}_{23}''$, it can be seen that

$Var\left(\hat{T}_{21}^{***}\right) < Var\left(\hat{T}_{23}''\right)$, if

$$\left(\frac{HA - qk_1^2}{HA - q^2k_1^2}\right) > \left(\frac{HB^* - qk_1^2}{HB^* - q^2k_1^2}\right) \tag{59}$$

(x) Comparisons between Variance of $\hat{T}_{21}^{***}$ and $\hat{T}^{**}$, it can be seen that

$Var\left(\hat{T}_{21}^{***}\right) < Var\left(\hat{T}^{**}\right)$, only if

$$\left(\frac{D^*}{H}\right) > \left(\frac{HB^* - qk_1^2}{HB^* - q^2k_1^2}\right) \tag{60}$$

(xi) Comparisons between Variance of $\hat{T}_{24}^{***}$ and $\hat{T}_{25}^{**}$, it can be seen that

$Var\left(\hat{T}_{24}^{***}\right) < Var\left(\hat{T}_{25}^{**}\right)$, if

$$\left(\frac{F^*G - qk_1^2}{F^*G - q^2k_1^2}\right) F^* > \left(\frac{D^*G - qk_1^2}{D^*G - q^2k_1^2}\right) D^* \tag{61}$$

(xii) Comparisons between Variance of $\hat{T}_{24}^{***}$ and $\hat{T}_{26}''$, it can be seen that

$Var\left(\hat{T}_{24}^{***}\right) < Var\left(\hat{T}_{26}''\right)$, if

$$\left(\frac{CG - qk_1^2}{CG - q^2k_1^2}\right) C > \left(\frac{D^*G - qk_1^2}{D^*G - q^2k_1^2}\right) D^* \tag{62}$$

(xiii) Comparisons between Variance of $\hat{T}_{24}^{***}$ and $\hat{T}^{**}$, it can be seen that

$Var(\hat{T}_{24}^{***}) < Var(\hat{T}^{**})$, if

$$q < 1 \tag{63}$$

4 Gain in precision

Now we can compute gain in precision of the estimate obtained by using a linear estimate over the direct estimate using no information gathered on the first occasion.

$$G_{(1)} = \frac{Var(\hat{T}^{**})}{Var(\hat{T}_2^{***})} = \left(\frac{D^*B^* - q^2K^{*2}}{D^*B^* - qK^{*2}}\right) \tag{64}$$

$$G_{opt(1)} = \frac{Var(\hat{T}^{**})}{min.Var(\hat{T}_2^{***})} = \left(\frac{2D^*B^*}{D^*B^* + \sqrt{D^{*2}B^{*2} - D^*B^*K^{*2}}}\right) \tag{65}$$

$$G_{(2)} = \frac{Var(\hat{T}^{**})}{Var(\hat{T}_{21}^{***})} = \frac{D^*}{H}\left(\frac{HB^* - q^2k_1^2}{HB^* - qk_1^2}\right) \tag{66}$$

$$G_{opt(2)} = \frac{Var(\hat{T}^{**})}{min.Var(\hat{T}_{21}^{***})} = \left(\frac{2D^*B^*}{HB^* + \sqrt{H^2B^{*2} - HB^*k_1^2}}\right) \tag{67}$$

$$G_{(3)} = \frac{Var(\hat{T}^{**})}{Var(\hat{T}_{22}^{**})} = \frac{D^*}{H}\left(\frac{HE^* - q^2k_1^2}{HE^* - qk_1^2}\right) \tag{68}$$

$$G_{opt(3)} = \frac{Var(\hat{T}^{**})}{min.Var(\hat{T}_{22}^{**})} = \left(\frac{2D^*E^*}{HE^* + \sqrt{H^2E^{*2} - HE^*k_1^2}}\right) \tag{69}$$

$$G_{(4)} = \frac{Var(\hat{T}^{**})}{Var(\hat{T}_{23}'')} = \frac{D^*}{H}\left(\frac{HA - q^2k_1^2}{HA - qk_1^2}\right) \tag{70}$$

$$G_{opt(4)} = \frac{Var(\hat{T}^{**})}{min.Var(\hat{T}_{23}'')} = \left(\frac{2D^*A}{HA + \sqrt{H^2A^2 - HAk_1^2}}\right) \tag{71}$$

$$G_{(5)} = \frac{Var(\hat{T}^{**})}{Var(\hat{T}_{24}^{***})} = \left(\frac{D^*G - q^2k_1^2}{D^*G - qk_1^2}\right) \tag{72}$$

$$G_{opt(5)} = \frac{Var(\hat{T}^{**})}{min.Var(\hat{T}_{24}^{***})} = \left(\frac{2D^*G}{D^*G + \sqrt{D^{*2}G^2 - D^*Gk_1^2}}\right) \tag{73}$$

$$G_{(6)} = \frac{Var(\hat{T}^{**})}{Var(\hat{T}_{25}^{**})} = \frac{D^*}{F^*}\left(\frac{F^*G - q^2k_1^2}{F^*G - qk_1^2}\right) \tag{74}$$

$$G_{opt(6)} = \frac{Var(\hat{T}^{**})}{min.Var(\hat{T}_{25}^{**})} = \left(\frac{2D^*G}{F^*G + \sqrt{F^{*2}G^2 - F^*Gk_1^2}}\right) \tag{75}$$

$$G_{(7)} = \frac{Var(\hat{T}^{**})}{Var(\hat{T}_{26}'')} = \frac{D^*}{C}\left(\frac{CG - q^2k_1^2}{CG - qk_1^2}\right) \tag{76}$$

$$G_{opt(7)} = \frac{Var(\hat{T}^{**})}{min.Var(\hat{T}_{26}'')} = \left(\frac{2D^*G^*}{CG + \sqrt{C^2G^2 - CGk_1^2}}\right) \tag{77}$$

Now, we assume that;

$C_{y1} = C_{y2} = C_{x1} = C_{x2} = C_0$,
$C_{U1} = C_{U2} = C_{V1} = C_{V2} = C_1$,
$C_{y1(2)} = C_{y2(2)} = C_{x1(2)} = C_{x2(2)} = C_{0(2)}$,
$C_{U1(2)} = C_{U2(2)} = C_{V1(2)} = C_{V2(2)} = C_{1(2)}$,
$\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho$,
$\rho_{1(2)} = \rho_{2(2)} = \rho_{3(2)} = \rho_{4(2)} = \rho_{(2)}$,
$\rho_5 = \rho_6 = \rho_0, \rho_{5(2)} = \rho_{6(2)} = \rho_{0(2)}$,

where

$\rho_1(\rho_2)$: is the correlation coefficients between the variables y_1 and x_1 $(y_2$ and $x_2)$,
$\rho_3(\rho_4)$: is the correlation coefficients between the variables y_2 and x_1 $(y_1$ and $x_2)$,
$\rho_5(\rho_6)$: is the correlation coefficients between the variables x_1 and x_2 $(y_1$ and $y_2)$,
$\rho_{1(2)}(\rho_{2(2)})$: is the correlation coefficients between the variables $y_{1(2)}$ and $x_{1(2)}(y_{2(2)}$ and $x_{2(2)})$,
$\rho_{3(2)}(\rho_{4(2)})$: is the correlation coefficients between the variables $y_{2(2)}$ and $x_{1(2)}(y_{1(2)}$ and $x_{2(2)})$,
$\rho_{5(2)}(\rho_{6(2)})$: is the correlation coefficients between the variables $x_{1(2)}$ and $x_{2(2)}$ $(y_{1(2)}$ and $y_{2(2)})$.

Then, the expressions of B^*, D^*, K^*, E^* and F^* becomes

$B^* = 2\mu_{y_1}^2 d^*$, $D^* = 2\mu_{y_2}^2 d^*$, $K^* = 2\mu_{y_1}\mu_{y_2} t$, $E^* = 2\mu_{y_1}^2 d_1^*$,
$F^* = 2\mu_{y_2}^2 d_1^*$,
where
$d^* = \{\alpha_1 + \theta\alpha_{1(2)}\}$,
$t = \{(\rho + \rho_0)C_0^2 + \theta(\rho_{(2)} + \rho_{0(2)})C_{0(2)}^2\}$,
$d_1^* = \{(1+\rho)C_0^2 + \theta(1+\rho_{(2)})C_{0(2)}^2)\}$,

where
$\alpha_1 = (C_0^2 + C_1^2 + \rho C_0^2), \alpha_{1(2)} = (C_{0(2)}^2 + C_{1(2)}^2 + \rho_{(2)}C_{0(2)}^2)$

Considering the above assumption, the expressions of (11), (19), (25), (31), (37), (43), (49),(64)-(77) becomes

$$q_{opt}^{(0)} = \frac{d^{*2} - \sqrt{d^{*4} - d^{*2}t^2}}{t^2} \quad (78)$$

$$q_{opt}^{(1)} = q_{opt}^{(4)} = \frac{d^*(1+\rho) - \sqrt{d^{*2}(1+\rho)^2 - d^*(1+\rho)(\rho+\rho_0)^2C_0^2}}{(\rho+\rho_0)^2C_0^2} \quad (79)$$

$$q_{opt}^{(2)} = q_{opt}^{(5)} = \frac{d_1^*(1+\rho) - \sqrt{d_1^{*2}(1+\rho)^2 - d_1^*(1+\rho)(\rho+\rho_0)^2C_0^2}}{(\rho+\rho_0)^2C_0^2} \quad (80)$$

$$q_{opt}^{(3)} = q_{opt}^{(6)} = \frac{\alpha_1(1+\rho) - \sqrt{\alpha_1^2(1+\rho)^2 - \alpha_1(1+\rho)(\rho+\rho_0)^2C_0^2}}{(\rho+\rho_0)^2C_0^2} \quad (81)$$

$$G_{(1)} = \left(\frac{d^{*2} - q^2t^2}{d^{*2} - qt^2}\right) \quad (82)$$

$$G_{opt(1)} = \left(\frac{2d^*}{d^* + \sqrt{d^{*2} - t^2}}\right) \quad (83)$$

$$G_{(2)} = \left[\frac{d^*\{d^*(1+\rho) - (\rho+\rho_0)^2q^2C_0^2\}}{(1+\rho)\{d^*(1+\rho) - (\rho+\rho_0)^2qC_0^2\}C_0^2}\right] \quad (84)$$

$$G_{opt(2)} = \left[\frac{2d^{*2}}{\{d^*(1+\rho) + \sqrt{d^{*2}(1+\rho)^2 - d^*(1+\rho)(\rho+\rho_0)^2C_0^2}\}C_0^2}\right] \quad (85)$$

$$G_{(3)} = \left[\frac{d^*\{d_1^*(1+\rho) - (\rho+\rho_0)^2q^2C_0^2\}}{(1+\rho)\{d_1^*(1+\rho) - (\rho+\rho_0)^2qC_0^2\}C_0^2}\right] \quad (86)$$

$$G_{opt(3)} = \left[\frac{2d^*d_1^*}{\{d_1^*(1+\rho) + \sqrt{d^{*2}(1+\rho)^2 - d^*(1+\rho)(\rho+\rho_0)^2C_0^2}\}C_0^2}\right] \quad (87)$$

$$G_{(4)} = \left[\frac{d^*\{\alpha_1(1+\rho) - (\rho+\rho_0)^2q^2C_0^2\}}{(1+\rho)\{\alpha_1(1+\rho) - (\rho+\rho_0)^2qC_0^2\}C_0^2}\right] \quad (88)$$

$$G_{opt(4)} = \left[\frac{2d^*\alpha_1}{\{\alpha_1(1+\rho) + \sqrt{\alpha_1^2(1+\rho)^2 - \alpha_1(1+\rho)(\rho+\rho_0)^2C_0^2}\}C_0^2}\right] \quad (89)$$

$$G_{(5)} = \left[\frac{d^*(1+\rho) - (\rho+\rho_0)^2q^2C_0^2}{d^*(1+\rho) - (\rho+\rho_0)^2qC_0^2}\right] \quad (90)$$

$$G_{opt(5)} = \left[\frac{2d^*(1+\rho)}{\{d^*(1+\rho) + \sqrt{d^{*2}(1+\rho)^2 - d^*(1+\rho)(\rho+\rho_0)^2C_0^2}\}}\right] \quad (91)$$

$$G_{(6)} = \left[\frac{d^*\{d_1^*(1+\rho) - (\rho+\rho_0)^2q^2C_0^2\}}{d_1^*\{d_1^*(1+\rho) - (\rho+\rho_0)^2qC_0^2\}}\right] \quad (92)$$

$$G_{opt(6)} = \left[\frac{2d^*(1+\rho)C_0^2}{\{d_1^*(1+\rho) + \sqrt{d_1^{*2}(1+\rho)^2 - d_1^*(1+\rho)(\rho+\rho_0)^2C_0^2}\}}\right] \quad (93)$$

$$G_{(7)} = \left[\frac{d^*\{\alpha_1(1+\rho) - (\rho+\rho_0)^2q^2C_0^2\}}{\alpha_1\{\alpha_1(1+\rho) - (\rho+\rho_0)^2qC_0^2\}}\right] \quad (94)$$

$$G_{opt(7)} = \left[\frac{2d^*(1+\rho)}{\{\alpha_1(1+\rho) + \sqrt{\alpha_1^2(1+\rho)^2 - \alpha_1(1+\rho)(\rho+\rho_0)^2C_0^2}\}}\right] \quad (95)$$

5 Empirical Study

Further, we have calculated the gain in precision of proposed estimator in different situations with respect to $\hat{T}^{**}$ for different values of coefficient of variation, correlation coefficients, W_2, k and q.
Tables 1 to 5 shows the results and the following points are envisaged as

• for cases $C_0(>,=)C_{0(2)}$, $C_1(<,>)C_{1(2)}$, $\rho(<)\rho_0$, $\rho_{(2)}(<,>)\rho_{(2)}$, $W_2 = 0.3, 0.5, 0.8$, $k = 1.5, 2, 2.5$ and $q = 0.3, 0.5, 0.7$, the gain in precision is maximum over the direct estimator $\hat{T}^{**}$ in $\hat{T}_{22}^{**}$ i.e. in the situation when there is non-response on the first occasion only.

• for case $\rho(>)\rho_0$, the gain in precision is maximum over the direct estimator $\hat{T}^{**}$ in $\hat{T}_{25}^{**}$ i.e. in the situation when there is non-response on the second occasion only.

• for cases $C_0 < C_{0(2)}$, $C_1 = C_{1(2)}$, $\rho = \rho_0$, $\rho_{(2)} = \rho_{0(2)}$, the gain in precision is maximum over the direct estimator $\hat{T}^{**}$ in $\hat{T}''_{23}$ i.e. in the situation when there is measurement error on the first occasion only.

Table 1: Gain in precision, $G_{(1)}, G_{(2)}, G_{(3)}, G_{(4)}, G_{(5)}, G_{(6)}$ and $G_{(7)}$ of different estimators over $\hat{T}^{**}$ for different values of C_0 and $C_{0(2)}$.

$C_0 < C_{0(2)}$										
$\rho = 0.7, \rho_0 = 0.2, \rho_{(2)} = 0.5, \rho_{0(2)} = 0.3, C_0 = 0.4, C_1 = 0.5, C_{1(2)} = 1.5, W_2 = 0.8, q = 0.7$ and $k = 2.5$										
$C_{0(2)}$	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
1.00	5.02	2.02	1.10	1.01	18.52	18.61	19.09	1.00	2.44	9.95
1.50	7.27	4.32	2.31	1.02	26.79	26.84	27.65	1.00	1.69	14.41
2.00	10.42	7.47	3.98	1.03	38.36	38.40	39.63	1.00	1.40	20.65
$C_0 > C_{0(2)}$										
$\rho = 0.6, \rho_0 = 0.4, \rho_{(2)} = 0.4, \rho_{0(2)} = 0.2, C_0(2) = 0.2, C_1 = 1, C_{1(2)} = 2, W_2 = 0.6, q = 0.5$ and $k = 1.5$										
C_0	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.30	2.36	0.16	0.10	1.00	16.49	18.13	16.60	1.01	16.24	2.09
0.50	2.62	0.42	0.26	1.00	6.64	7.30	6.74	1.02	7.00	1.92
0.70	3.00	0.80	0.50	1.00	3.93	4.28	4.01	1.03	4.19	1.76
$C_0 = C_{0(2)}$										
$\rho = 0.6, \rho_0 = 0.3, \rho_{(2)} = 0.5, \rho_{0(2)} = 0.3, C_1 = 1.5, C_{1(2)} = 1.5, W_2 = 0.7, q = 0.3$ and $k = 2$										
$C_{0(2)}$	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.10	1.85	0.03	0.01	1.00	115.79	120.64	116.18	1.00	72.84	6.99
0.10	1.85	0.03	0.01	1.00	115.79	120.64	116.18	1.00	72.84	6.99
0.10	1.85	0.03	0.01	1.00	115.79	120.64	116.18	1.00	72.84	6.99

Table 2: Gain in precision, $G_{(1)}, G_{(2)}, G_{(3)}, G_{(4)}, G_{(5)}, G_{(6)}$ and $G_{(7)}$ of different estimators over $\hat{T}^{**}$ for different values of C_1 and $C_{1(2)}$.

$C_1 < C_{1(2)}$										
$\rho = 0.7, \rho_0 = 0.2, \rho_{(2)} = 0.5, \rho_{0(2)} = 0.3, C_0 = 0.4, C_{0(2)} = 0.2, C_1 = 0.5, W_2 = 0.8, q = 0.7$ and $k = 2.5$										
$C_{1(2)}$	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.70	1.18	0.34	0.18	1.01	4.41	4.58	4.49	1.01	3.63	2.34
1.00	1.79	0.34	0.18	1.00	6.66	6.96	6.82	1.01	5.50	3.55
1.50	3.29	0.34	0.18	1.00	12.17	12.78	12.52	1.00	10.10	6.53
$C_1 > C_{1(2)}$										
$\rho = 0.6, \rho_0 = 0.4, \rho_{(2)} = 0.4, \rho_{0(2)} = 0.2, C_0 = 0.5, C_0(2) = 0.3, C_{1(2)} = 0.6, W_2 = 0.6, q = 0.8$ and $k = 1.5$										
C_1	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.80	1.19	0.44	0.27	1.01	3.03	3.20	3.05	1.02	2.93	1.17
1.00	1.55	0.44	0.27	1.00	3.93	4.17	3.94	1.02	3.81	1.13
1.50	2.80	0.44	0.27	1.00	7.05	7.55	7.06	1.01	6.90	1.07
$C_1 = C_{1(2)}$										
$\rho = 0.5, \rho_0 = 0.3, \rho_{(2)} = 0.6, \rho_{0(2)} = 0.4, C_0 = 0.1, C_0(2) = 0.3, W_2 = 0.7, q = 0.6$ and $k = 2$										
$C_{1(2)}$	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.20	0.18	0.12	0.07	1.04	12.32	12.36	12.49	1.01	1.60	3.41
0.20	0.18	0.12	0.07	1.04	12.32	12.36	12.49	1.01	1.60	3.41
0.20	0.18	0.12	0.07	1.04	12.32	12.36	12.49	1.01	1.60	3.41

Table 3: Gain in precision, $G_{(1)}, G_{(2)}, G_{(3)}, G_{(4)}, G_{(5)}, G_{(6)}$ and $G_{(7)}$ of different estimators over $\hat{T}^{**}$ for different values of ρ and ρ_0.

$\rho > \rho_0$										
$\rho = 0.7, \rho_{(2)} = 0.6, \rho_{0(2)} = 0.7, C_0 = 0.5, C_{0(2)} = 0.2, C_1 = 0.8, C_{1(2)} = 0.2, W_2 = 0.2, q = 0.8$ and $k = 0.5$										
ρ_0	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.30	1.05	0.42	0.24	1.01	2.54	2.68	2.54	1.03	2.72	1.01
0.40	1.05	0.42	0.27	1.01	2.56	2.72	2.56	1.03	2.78	1.02
0.50	1.05	0.42	0.27	1.01	2.58	2.82	2.58	1.04	2.86	1.03
$\rho < \rho_0$										
$\rho_0 = 0.7, \rho_{(2)} = 0.2, \rho_{0(2)} = 0.3, C_0 = 0.7, C_0(2) = 0.3, C_1 = 1, C_{1(2)} = 2, W_2 = 0.4, q = 0.3$ and $k = 1.5$										
ρ	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.20	2.41	0.61	0.45	1.01	4.22	4.66	4.29	1.03	4.49	1.59
0.40	2.51	0.71	0.56	1.01	3.80	4.22	3.87	1.04	4.07	1.57
0.60	2.61	0.81	0.66	1.01	3.48	3.88	3.56	1.04	3.75	1.56
$\rho = \rho_0$										
$\rho_{(2)} = 0.5, \rho_{0(2)} = 0.1, C_0 = 0.4, C_0(2) = 0.6, C_1 = 0.5, C_{1(2)} = 1.5, W_2 = 0.5, q = 0.6$ and $k = 2.5$										
ρ	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.20	2.53	0.60	0.23	1.00	13.23	13.32	13.36	1.00	4.23	5.80
0.20	2.53	0.60	0.23	1.00	13.23	13.32	13.36	1.00	4.23	5.80
0.20	2.53	0.60	0.23	1.00	13.23	13.32	13.36	1.00	4.23	5.80

Table 4: Gain in precision, $G_{(1)}, G_{(2)}, G_{(3)}, G_{(4)}, G_{(5)}, G_{(6)}$ and $G_{(7)}$ of different estimators over $\hat{T}^{**}$ for different values of $\rho_{(2)}$ and $\rho_{0(2)}$.

$\rho_{(2)} > \rho_{0(2)}$										
$\rho = 0.7, \rho_0 = 0.4, \rho_{(2)} = 0.5, C_0 = 0.5, C_{0(2)} = 0.6, C_1 = 0.8, C_{1(2)} = 0.2, W_2 = 0.8, q = 0.8$ and $k = 2.5$										
$\rho_{0(2)}$	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.20	1.76	1.07	0.58	1.03	4.25	4.32	4.32	1.03	1.71	1.73
0.30	1.76	1.07	0.62	1.03	4.25	4.32	4.32	1.03	1.71	1.73
0.40	1.76	1.07	0.66	1.04	4.25	4.32	4.32	1.03	1.71	1.73
$\rho_{(2)} < \rho_{0(2)}$										
$\rho_0 = 0.5, \rho_0 = 0.6, \rho_{0(2)} = 0.4, C_0 = 0.5, C_0(2) = 0.6, C_1 = 1, C_{1(2)} = 2, W_2 = 0.6, q = 0.8$ and $k = 3.5$										
$\rho_{(2)}$	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.30	8.08	1.08	0.65	1.00	21.63	22.30	22.11	1.00	7.76	6.03
0.20	8.02	1.02	0.60	1.00	21.48	22.20	21.96	1.00	8.14	5.99
0.10	7.97	0.97	0.55	1.00	21.34	22.10	21.82	1.00	8.55	5.95
$\rho_{(2)} = \rho_{0(2)}$										
$\rho = 0.7, \rho_0 = 0.2, C_0 = 0.2, C_0(2) = 0.6, C_1 = 0.5, C_{1(2)} = 1.5, W_2 = 0.6, q = 0.5$ and $k = 2.5$										
$\rho_{(2)}$	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.50	2.83	0.55	0.36	1.00	41.67	41.97	42.25	1.00	5.15	9.03
0.50	2.83	0.55	0.36	1.00	41.67	41.97	42.25	1.00	5.15	9.03
0.50	2.83	0.55	0.36	1.00	41.67	41.97	42.25	1.00	5.15	9.03

Table 5: Gain in precision, $G_{(1)}, G_{(2)}, G_{(3)}, G_{(4)}, G_{(5)}, G_{(6)}$ and $G_{(7)}$ of different estimators over $\hat{T}^{**}$ for different values of W_2, k and q.

W_2										
$\rho = 0.7, \rho_0 = 0.2, \rho_{(2)} = 0.5, \rho_{0(2)} = 0.4, C_0 = 0.2, C_{0(2)} = 0.6, C_1 = 2, C_{1(2)} = 1, q = 0.7$ and $k = 2.5$										
W_2	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.30	4.76	0.31	0.18	1.00	70.07	70.96	70.08	1.00	15.51	1.17
0.50	5.22	0.47	0.28	1.00	76.87	77.48	76.88	1.00	11.14	1.29
0.80	5.92	0.72	0.42	1.00	87.06	87.50	87.09	1.00	8.31	1.46
k										
$\rho = 0.8, \rho_0 = 0.3, \rho_{(2)} = 0.4, \rho_{0(2)} = 0.6, C_0 = 0.2, C_0(2) = 0.6, C_1 = 2, C_{1(2)} = 1, W_2 = 0.8$, and $q = 0$										
k	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
1.50	4.67	0.27	0.19	1.00	65.00	66.59	65.02	1.00	17.52	1.15
2.00	5.28	0.48	0.33	1.00	73.36	74.33	73.39	1.00	11.26	1.30
2.50	5.88	0.68	0.48	1.00	81.72	82.45	81.76	1.00	8.77	1.45
q										
$\rho = 0.8, \rho_0 = 0.4, \rho_{(2)} = 0.5, \rho_{0(2)} = 0.7, C_0 = 0.2, C_{0(2)} = 0.6, C_1 = 2, C_{1(2)} = 1, W_2 = 0.8$, and $k = 2.5$										
q	d^*	d_1^*	t	$G_{(1)}$	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	$G_{(6)}$	$G_{(7)}$
0.30	5.92	0.72	0.57	1.00	82.32	83.00	82.36	1.00	8.30	1.46
0.50	5.92	0.72	0.57	1.00	82.33	83.16	82.38	1.00	8.32	1.46
0.70	5.92	0.72	0.57	1.00	82.32	83.01	82.36	1.00	8.30	1.46

6 Conclusion

From theoretical study, we may conclude that the proposed product type estimator contributes significantly to deal with different realistic situation of non-response and measurement errors, while estimating the population product on current (second) occasion in two-occasion successive sampling. The properties of the proposed estimators have been studied and efficiency conditions also developed. Numerical study also supports the theoretical results for different combination of values of the parameters. Hence, the proposed product type estimators may be recommended for their practical application.

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