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An Explicit Formula for Bernoulli Polynomials with a q Parameter in Terms of r-Whitney Numbers

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Abstract: We define the Bernoulli polynomials with a q parameter in terms of r-Whitney numbers of the second kind. Some algebraic properties and combinatorial identities of these polynomials are given. Also, we obtain several relations between the Cauchy and Bernoulli polynomials with a q parameter in terms of r-Whitney numbers of both kinds.

Keywords: Bernoulli numbers and polynomials, r-Whitney numbers, Stirling numbers.

1 Introduction

The Bernoulli polynomials $B_n(z)$ are defined by the generating function [4]

$$\sum_{n=0}^{\infty} B_n(z)\frac{t^n}{n!} = \frac{t\,e^{zt}}{e^t-1}. \tag{1}$$

When $z=0$, $B_n = B_n(0)$ are called the Bernoulli numbers. Graham et al. [8, P. 560] represented B_n also as

$$B_n = \sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}S(n,k), \tag{2}$$

where $S(n,k)$ are the Stirling numbers of the second kind [4].

For more identities and explicit formulas for Bernoulli numbers in terms of Stirling numbers of the first and second kind, see [7,10,11], in addition to this some identities involving Bernoulli polynomials and r-Whitney numbers are given in [15,9].

For all integers $n,r \geq 0$, $B_n(r)$ can be written explicitly in terms of r-Stirling numbers of the second kind $S_r(n,k)$ [12]

$$B_n(r) = \sum_{k=0}^{n}\frac{(-1)^k k!}{k+1}S_r(n+r,k+r). \tag{3}$$

Cenkci et al. [2] introduced the poly-Bernoulli polynomials with a q parameter $B_{n,q}^{(k)}(z)$ as a generalization of the poly-Bernoulli polynomials $B_n^{(k)}(z)$. Let q be a real number with $q \neq 0$, [2] defined the poly-Bernoulli polynomials with a q parameter by

$$\sum_{n=0}^{\infty} B_{n,q}^{(k)}(z)\frac{t^n}{n!} = \frac{q\,e^{-zt}}{1-e^{-qt}}\sum_{n=1}^{\infty}\left(\frac{(1-e^{-qt})}{q}\right)^n\frac{1}{n^k}. \tag{4}$$

Remark 1. *If $z=0$, then $B_{n,q}^{(k)}(0) = B_{n,q}^{(k)}$ are the poly-Bernoulli numbers with a q parameter [2].*

If $q=1$, then $B_{n,1}^{(k)}(z) = B_n^{(k)}(z)$ are the poly-Bernoulli polynomials [5].

If $q=1$, $z=0$, then $B_{n,1}^{(k)}(0) = B_n^{(k)}$ are the poly-Bernoulli numbers, defined in [13].

Setting $k=1$ in (4), and let $B_n^q(z) = B_{n,q}^{(1)}(z)$ denote the Bernoulli polynomials with q parameter, we obtain

$$\sum_{n=0}^{\infty} B_n^q(z)\frac{t^n}{n!} = \frac{q\,e^{-zt}}{1-e^{-qt}}\sum_{n=1}^{\infty}\left(\frac{(1-e^{-qt})}{q}\right)^n\frac{1}{n}. \tag{5}$$

If $z=0$, then $B_n^q(0) = B_n^q$ are the Bernoulli numbers with q parameter. Note that e^{zt} is replaced by e^{-zt} (see [2,1,5]).

Recently, Duran et al. [6] introduced the Hermite based poly-Bernoulli polynomials with a q parameter with $B_{n,q}^{(k)}(z)$ as a special case.

In this paper, we are interested in Bernoulli polynomials with a q parameter. Therefore, we begin with the following definitions of some tools which will be useful for sequel of this paper.

For non-negative integers n and k with $0 \leq k \leq n$, let $w(n,k) = w_{q,r}(n,k)$ and $W(n,k) = W_{q,r}(n,k)$ denote the

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r-Whitney numbers of the first and second kind, respectively, defined in [15] by

$$q^n(x)_n = \sum_{k=0}^{n} w(n,k)\,(qx+r)^k, \tag{6}$$

$$(qx+r)^n = \sum_{k=0}^{n} q^k W(n,k)\,(x)_k. \tag{7}$$

The exponential generating function of $W(n,k)$ is given by [15]

$$\sum_{n\geq k} W(n,k)\frac{t^n}{n!} = \frac{e^{rt}}{k!}\left(\frac{e^{qt}-1}{q}\right)^k. \tag{8}$$

The parameters $r \geq 0$ and $q > 0$ are usually taken to be integers, but both may also be considered as indeterminates [16].

Komatsu [14] defined the Cauchy polynomials with a q parameter of the first and second kind, denoted by $c_n^q(z)$, $\hat{c}_n^q(z)$, respectively

$$c_n^q(z) = \int_0^1 (x-z|q)_n\,dx \qquad q\neq 0, \tag{9}$$

$$\hat{c}_n^q(z) = \int_0^1 (-x+z|q)_n\,dx \qquad q\neq 0. \tag{10}$$

Recently, Shiha [17] gave explicit formulae for $c_n^q(r)$ and $\hat{c}_n^q(r)$ in terms of r-Whitney numbers of the first kind

$$c_n^q(r) = \sum_{k=0}^{n} w(n,k)\,\frac{1}{k+1} \qquad q\neq 0, \tag{11}$$

$$\hat{c}_n^q(-r) = \sum_{k=0}^{n} (-1)^k w(n,k)\,\frac{1}{k+1} \qquad q\neq 0. \tag{12}$$

The relations between $c_n^q(r)$, $\hat{c}_n^q(r)$ and $W(n,k)$ are (see [17])

$$\sum_{k=0}^{n} W(n,k)\,c_k^q(r) = \frac{1}{n+1} \tag{13}$$

$$\sum_{k=0}^{n} W(n,k)\,\hat{c}_k^q(-r) = \frac{(-1)^n}{n+1} \tag{14}$$

The aim of this paper is to establish an explicit formula for computing $B_n^q(z)$ in terms of r-Whitney numbers of the second kind $W(n,k)$. Several relations between $B_n^q(z)$ and $c_n^q(z)$ are obtained in terms of r-Whitney numbers of both kinds.

2 The main results

The main result may be formulated as the following theorem.

Theorem 1. *We define the Bernoulli polynomials with a q parameter in terms of $W(n,k)$ by*

$$B_n^q(r) = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1}\,W(n,k). \tag{15}$$

Proof. Using (8), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}W(n,k)\,\frac{t^n}{n!}\\
&=\sum_{k=0}^{\infty}(-1)^k\frac{k!}{k+1}\sum_{n=k}^{\infty}W(n,k)\,\frac{t^n}{n!}\\
&=\sum_{k=0}^{\infty}(-1)^k\frac{k!}{k+1}\,\frac{e^{rt}}{k!}\left(\frac{e^{qt}-1}{q}\right)^k\\
&=\frac{q\,e^{rt}}{1-e^{qt}}\sum_{k=0}^{\infty}\left(\frac{1-e^{qt}}{q}\right)^{k+1}\frac{1}{k+1}\\
&=\frac{q\,e^{rt}}{1-e^{qt}}\sum_{k=1}^{\infty}\left(\frac{1-e^{qt}}{q}\right)^{k}\frac{1}{k}\\
&=\sum_{n=0}^{\infty}B_n^q(r)\,\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of both sides, we get (15) (notice that e^{rt} is replaced by e^{-rt} in [2]).

On the other hand,

$$\begin{aligned}
\sum_{n=0}^{\infty}B_n^q(r)\,\frac{t^n}{n!} &= \frac{q\,e^{rt}}{1-e^{qt}}\sum_{k=1}^{\infty}\left(\frac{1-e^{qt}}{q}\right)^{k}\frac{1}{k}\\
&=\frac{q\,e^{rt}}{1-e^{qt}}\left(-\ln(1-(\frac{1-e^{qt}}{q}))\right)\\
&=\frac{q\,e^{rt}}{e^{qt}-1}\left(\ln\frac{q-1+e^{qt}}{q}\right).
\end{aligned} \tag{16}$$

The first few polynomials are

$B_0^q(r) = 1,$

$B_1^q(r) = r - \frac{1}{2},$

$B_2^q(r) = r^2 - r - \frac{1}{2}q + \frac{2}{3},$

$B_3^q(r) = r^3 - \frac{3}{2}r^2 + (2-\frac{3}{2}q)r - \frac{1}{2}q^2 + 2q - \frac{3}{2},$

$B_4^q(r) = r^4 - 2r^3 + (4-3q)r^2 - 2(q^2-4q+3)r$
$\quad -\frac{1}{2}q^3 + \frac{14}{3}q^2 - 9q + \frac{24}{5}.$

Corollary 1. *If $r = 0$, $B_n^q(0) = B_n^q$ are the Bernoulli numbers with q parameter, using (16), we get*

$$\sum_{n=0}^{\infty}B_n^q\,\frac{t^n}{n!} = \frac{q}{e^{qt}-1}\left(\ln\frac{q-1+e^{qt}}{q}\right), \tag{17}$$

$W_{q,0}(n,k) = q^{n-k}S(n,k)$, then B_n^q can be expressed in terms of $S(n,k)$ as

$$B_n^q = \sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}\,q^{n-k}S(n,k). \tag{18}$$

If $q=1$, $B_n^1(r)=B_n(r)$ and $W_{1,r}(n,k)$ are reduced to $S_r(n+r,k+r)$ then, we get the explicit formula (3), and

$$\sum_{n=0}^{\infty} B_n(r)\frac{t^n}{n!}=\frac{t\,e^{rt}}{e^t-1}. \tag{19}$$

If $q=1$ and $r=0$, then $B_n^1(0)=B_n$ and $W_{1,0}(n,k)$ are reduced to $S(n,k)$ then, we get the explicit formula (2).

It is known that the r-Whitney numbers satisfy the following orthogonality relation [15]:

$$\sum_{l=k}^{n} w(n,l)\,W(l,k)=\sum_{l=k}^{n} W(n,l)\,w(l,k)=\delta_{kn}, \tag{20}$$

where δ_{kn} is the Kronecker delta.

Corollary 2. *The relation between $w(n,k)$ and $B_n^q(r)$ is given by*

$$\sum_{j=0}^{n}(-1)^n w(n,j)\,B_j^q(r)=\frac{n!}{n+1}. \tag{21}$$

Proof.

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^n w(n,j)\,B_j^q(r)\\
&=\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{n+k}\frac{k!}{k+1}\,w(n,j)\,W(j,k)\\
&=\sum_{k=0}^{n}(-1)^{n+k}\frac{k!}{k+1}\sum_{j=k}^{n} w(n,j)W(j,k)\\
&=\sum_{k=0}^{n}(-1)^{n+k}\frac{k!}{k+1}\,\delta_{kn}=\frac{n!}{n+1}.
\end{aligned}$$

The numbers $W(n,k)$ are determined by (see [15]).

$$W(n,k)=\frac{1}{q^k\,k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(r+jq)^n, \tag{22}$$

then, we obtain

Theorem 2. *For $n\geq 0$, we have*

$$B_n^q(r)=\sum_{k=0}^{n}\frac{1}{q^k\,(k+1)}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(r+jq)^n \tag{23}$$

Theorem 3. *For $n\geq 0$, we have*

$$c_n^q(r)=\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(-1)^k}{k!}\,w(n,k)\,w(k,j)\,B_j^q(r). \tag{24}$$

$$\hat{c}_n^q(-r)=\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{1}{k!}\,w(n,k)\,w(k,j)\,B_j^q(r). \tag{25}$$

$$B_n^q(r)=\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^k\,k!\,W(n,k)\,W(k,j)\,c_j^q(r). \tag{26}$$

$$B_n^q(r)=\sum_{k=0}^{n}\sum_{j=0}^{k}k!\,W(n,k)\,W(k,j)\,\hat{c}_j^q(-r). \tag{27}$$

Proof. We shall prove the first and the third identities. Other identities are proven similarly. By (21), and using (11), we have

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(-1)^k}{k!}\,w(n,k)\,w(k,j)\,B_j^q(r)\\
&=\sum_{k=0}^{n}\frac{1}{k!}\,w(n,k)\sum_{j=0}^{k}(-1)^k\,w(k,j)\,B_j^q(r)\\
&=\sum_{k=0}^{n}\frac{1}{k+1}\,w(n,k)=c_n^q(r).
\end{aligned}$$

By (13), and using (15), we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^k\,k!\,W(n,k)\,W(k,j)\,c_j^q(r)\\
&=\sum_{k=0}^{n}(-1)^k\,k!\,W(n,k)\sum_{j=0}^{k}W(k,j)\,c_j^q(r)\\
&=\sum_{k=0}^{n}(-1)^k\,k!\,W(n,k)\,\frac{1}{k+1}=B_n^q(r).
\end{aligned}$$

It is known that (see [3])

$$W_{q,r+s}(n,k)=\sum_{j=k}^{n}\binom{n}{j}r^{n-j}\,W_{q,s}(j,k). \tag{28}$$

Therefore, we get the following properties, which are similar to those for classical Bernoulli polynomials and numbers.

Theorem 4. *For $n\geq 0$, we have*

$$B_n^q(r+s)=\sum_{j=0}^{n}\binom{n}{j}r^{n-j}B_j^q(s). \tag{29}$$

$$B_n^q(r)=\sum_{j=0}^{n}\binom{n}{j}r^{n-j}B_j^q. \tag{30}$$

Proof.

$$\begin{aligned}
B_n^q(r+s)&=\sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}\,W_{q,r+s}(n,k)\\
&=\sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^k\frac{k!}{k+1}\binom{n}{j}r^{n-j}\,W_{q,s}(j,k)\\
&=\sum_{j=0}^{n}\binom{n}{j}r^{n-j}\sum_{k=0}^{j}(-1)^k\frac{k!}{k+1}\,W_{q,s}(j,k)\\
&=\sum_{j=0}^{n}\binom{n}{j}r^{n-j}\,B_j^q(s).
\end{aligned}$$

Setting $s=0$ in (29), we get the second relation (30)

Corollary 3. *Setting $q=1$ in (29) and (30), we obtain the following properties of $B_n(r)$ [18]*

$$B_n(r+s)=\sum_{j=0}^{n}\binom{n}{j}r^{n-j}B_j(s),$$

$$B_n(r)=\sum_{j=0}^{n}\binom{n}{j}r^{n-j}B_j.$$

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