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A New Generalization of Fuzzy Ideals of Ternary Semigroups

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Abstract: Our aim in this paper is to introduce and study a new sort of fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, quasi-ideal, bi-ideal) of a ternary semigroup, called interval-valued (α,β)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, quasi-ideal, bi-ideal). We also characterize regular ternary semigroups in terms of these interval-valued fuzzy ideals.

Keywords: Ternary semigroup, fuzzy point, interval valued $(\in,\in\vee q)$-fuzzy ideals, interval valued $(\in,\in\vee q)$-fuzzy bi-ideals

1 Introduction

The study of ternary algebraic systems was initiated by Lehmer [1] in 1932. He investigated certain algebraic systems which turn out to be ternary groups. Banach [2], showed by an example that a ternary semigroup does not necessarily reduced to an ordinary semigroup. Sioson [3], studied ternary semigroups with a special reference of ideals and radicals. He also introduced the notion of regular ternary semigroup and characterized them by using the notion of quasi-ideals. Dixit and Dewan studied quasi-ideals and bi-ideals of ternary semigroup in his paper [4]. The algebraic structures play a prominent role in mathematics with wide range of applications in variety of disciplines such as computer science, information science, control engineering, pattern recognition etc.

The theory of fuzzy sets was first developed by Zadeh [5] and has been applied to many branches in mathematics. Rosenfeld started the fuzzification of algebraic structures [6]. Kuroki is responsible for much of ideal theory of fuzzy semigroups (see [7,8,9,10,11]). Later, in 1975 Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, that is, a fuzzy set with an interval-valued membership function [12]. The interval-valued fuzzy subgroups were first defined and studied by Biswas [13] which are the fuzzy subgroups of the same nature of the the fuzzy subgroups defined by Rosenfeld. A new type of fuzzy subgroup, that is, $(\in,\in\vee q)$-fuzzy subgroup was introduced in an earlier paper of Bhakat and Das [14] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy point and fuzzy set, which was introduced by Pu and Liu [15]. In fact, the $(\in,\in\vee q)$-fuzzy subgroup is a significant generalization of Rosenfeld's fuzzy subgroup. The present authors in [16], defined and discussed (α,β)-fuzzy ideals in ternary semigroups. For further studies on this topic the reader is referred to [17,18,19, 20]. Akram et al. in [21] introduced the notion of interval-valued (α,β)-fuzzy k-algebras. Zhan et al. put forth the idea of interval-valued $(\in,\in\vee q)$-fuzzy fuzzy filters in pseudo BL-algebras [22]. Ma et al. discussed interval-valued fuzzy ideals of pseudo MV-agebras (see [23]). Recently Shabir and Mahmood studied interval-valued (α,β)-fuzzy ideals of hemirings.

In this paper we initiate the study of interaval-valued $(\in,\in\vee q)$-fuzzy ternary subsemigroups (left ideals, right ideals), interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideals and interval-valued $(\in,\in\vee q)$-fuzzy bi-ideals in teranary semigroups and several related properties are studied. We also characterize regular ternary semigroups in terms of these ideals.

2 Preliminaries

A ternary semigroup is an algebraic structure $(S,[\;\;])$ with one ternary operation satisfying the associative law:

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$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$ for all $x_1, x_2, x_3, x_4, x_5 \in S$. For the sake of convenience, $[x_1x_2x_3]$ will be written as '$x_1x_2x_3$' and consider the ternary operation as multiplication. A non-empty subset A of a ternary semigroup S is called a ternary subsemigroup of S if $AAA \subseteq A$. By a left (right, lateral) ideal of a ternary semigroup S we mean a non-empty subset A of S such that $SSA \subseteq A$ ($ASS \subseteq A, SAS \subseteq A$). If a non-empty subset A of S is a left and right ideal of S, then it is called a two sided ideal of S. If a non-empty subset A of a ternary semigroup S is a left, right and lateral ideal of S, then it is called an ideal of S. A non-empty subset A of a ternary semigroup S is called a quasi-ideal of S if $ASS \cap SAS \cap SSA \subseteq A$ and $ASS \cap SSASS \cap SSA \subseteq A$. A is called a bi-ideal of S if it is a ternary subsemigroup of S and $ASASA \subseteq A$. A non-empty subset A of a ternary semigroup S is called a generalized bi-ideal of S if $ASASA \subseteq A$. It is clear that every left (right, lateral) ideal of S is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal of S. An element a of a ternary semigroup S is called regular if there exists an element $x \in S$ such that $axa = a$. A ternary semigroup S is called regular if every element of S is regular.

2.1 ***Example***

Let $\mathbb{Z}^-$ be the set of all negative integers. Then with usual ternary multiplication, $\mathbb{Z}^-$ forms a ternary semigroup.

2.2 Example

The set of all odd permutations of a non-empty set X, under ternary composition forms a ternary semigroup.

2.3 Example

Let $S = \{-i, 0, i\}$. Then S is a ternary semigroup under the ternary multiplication of complex numbers.

2.4 Theorem [3]

A ternary semigroup S is regular if and only if $R \cap M \cap L = RML$ for every right ideal R, every lateral ideal M and every left ideal L of S.

2.5 Theorem [24]

The following conditions on a ternary semigroup S are equivalent:

(1) S is regular;
(2) $BSBSB = B$ for every bi-ideal B of S;
(3) $QSQSQ = Q$ for every quasi-ideal Q of S.

2.6 Theorem [24]

The following conditions on a ternary semigroup S are equivalent:

(1) S is regular;
(2) $R \cap L = RSL$ for every right ideal R and every left ideal L of S.

By an interval number $\widetilde{a}$ we mean a closed subinterval $[a^-, a^+]$ where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers is denoted by $D[0,1]$. The interval $[a,a]$ can be identified by the number $a \in [0,1]$. For the interval numbers $\widetilde{a}_i = [a_i^-, a_i^+]$ and $\widetilde{b}_i = [b_i^-, b_i^+] \in D[0,1]$, $i \in \Lambda$, we define

$$\widetilde{a}_i \vee \widetilde{b}_i = rmax\{\widetilde{a}_i, \widetilde{b}_i\} = [\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)],$$
$$\widetilde{a}_i \wedge \widetilde{b}_i = r\min\{\widetilde{a}_i, \widetilde{b}_i\} = [\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)],$$
$$rinf\widetilde{a}_i = \left[\wedge_{i\in\Lambda} a_i^-, \wedge_{i\in\Lambda} a_i^+\right], \ rsup\widetilde{a}_i = \left[\vee_{i\in\Lambda} a_i^-, \vee_{i\in\Lambda} a_i^+\right].$$

Define on $D[0,1]$ an order relation "$\leq$" by

(1) $\widetilde{a}_1 \leq \widetilde{a}_2 \Longleftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$
(2) $\widetilde{a}_1 = \widetilde{a}_2 \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$
(3) $\widetilde{a}_1 < \widetilde{a}_2 \Longleftrightarrow \widetilde{a}_1 \leq \widetilde{a}_2$ and $\widetilde{a}_1 \neq \widetilde{a}_2$
(4) $k\widetilde{a}_i = [ka_i^-, ka_i^+]$, whenever $0 \leq k \leq 1$.

It is clear that the set $(D[0,1], \leq, \vee, \wedge)$ forms a complete bounded lattice with $\wedge = r\min$, $\vee = r\max$, $\widetilde{0} = [0,0]$ as the bottom element and $\widetilde{1} = [1,1]$ as the top element. For any two interval-valued fuzzy subsets $\widetilde{\lambda}$ and $\widetilde{\mu}$ of S, $\widetilde{\lambda} \leqslant \widetilde{\mu}$ means that, for all $x \in S$, $\widetilde{\lambda}(x) \leqslant \widetilde{\mu}(x)$.

A fuzzy subset λ of a universe S is a function from S into the unit closed interval $[0,1]$, that is, $\lambda : S \to [0,1]$.

Let S be a set. A mapping $\widetilde{\lambda} : S \to D[0,1]$ is called an interval-valued fuzzy set of X.

For three interval-valued fuzzy subsets $\widetilde{\lambda}, \widetilde{\mu}$ and $\widetilde{\nu}$ of a ternary semigroup S we define the multiplication of these interval-valued fuzzy subsets of S by:

$$\left(\widetilde{\lambda} \circ \widetilde{\mu} \circ \widetilde{\nu}\right)(a) = \begin{cases} \vee_{a=xyz}\left\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y) \wedge \widetilde{\nu}(z)\right\} \\ \text{if there exist } x,y,z \in S \text{ such that } a = xyz, \\ [0,0] \quad \text{otherwise.} \end{cases}$$

for all $a \in S$.

The symbols $\widetilde{\lambda} \wedge \widetilde{\mu}$ and $\widetilde{\lambda} \vee \widetilde{\mu}$ will mean the following interval-valued fuzzy subsets of S:

$$\left(\widetilde{\lambda} \wedge \widetilde{\mu}\right)(x) = \widetilde{\lambda}(x) \wedge \widetilde{\mu}(x)$$
$$\left(\widetilde{\lambda} \vee \widetilde{\mu}\right)(x) = \widetilde{\lambda}(x) \vee \widetilde{\mu}(x) \ \text{ for all } x \in S.$$

If $A \subseteq S$, then the interval-valued characteristic function of A is a function $\widetilde{C}_A$ of S defined by:

$$\widetilde{C}_A(x) = \begin{cases} [1,1] & \text{if } x \in A \\ [0,0] & \text{if } x \notin A. \end{cases}$$

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of a ternary semigroup S. Then the crisp set

$$U\left(\widetilde{\lambda};\widetilde{t}\right)=\left\{x\in S:\widetilde{\lambda}\left(x\right)\geq\widetilde{t}\right\},$$

where $\widetilde{t}\in D[0,1]$, is called the level subset of $\widetilde{\lambda}$.

2.7 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued fuzzy ternary subsemigroup of S if $\widetilde{\lambda}\left(xyz\right)\geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z)\right\}$ for all $x, y, z\in S$.

2.8 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued fuzzy left (right, lateral) ideal of S if

$\widetilde{\lambda}\left(xyz\right)\geq\widetilde{\lambda}(z)\left(\widetilde{\lambda}(x),\ \widetilde{\lambda}(y)\right)$ for all $x,y,z\in S$.

2.9 Definition

An interval-valued fuzzy subset (ternary subsemigroup) $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued fuzzy generalized bi-ideal (bi-ideal) of S if $\widetilde{\lambda}\left(xuyvz\right)\geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z)\right\}$ for all $x,\ y,\ z,\ u,\ v\in S$.

3 Interval-valued (α,β)-fuzzy ideals

Throughout this paper S will denote a ternary semigroup and $\alpha,\ \beta\in\{\in,\ q,\ \in\vee q,\ \in\wedge q\}$ unless otherwise specified.

The concept of quasi-coincidence of a fuzzy point can be extended to the concept of quasi-coincidence of an interval-valued fuzzy set.

An interval-valued fuzzy set $\widetilde{\lambda}$ of a ternary semigroup S of the form

$$\widetilde{\lambda}(y)=\begin{cases}\widetilde{t}(\neq[0,0]) & \text{if } y=x,\\ [0,0] & \text{if } y\neq x,\end{cases}$$

is called an interval-valued fuzzy point with support x and interval-value $\widetilde{t}$ and is denoted by $x_{\widetilde{t}}$. An interval-valued fuzzy point $x_{\widetilde{t}}$ is said to belongs to (resp. be quasi-coincident with) an interval-valued fuzzy set $\widetilde{\lambda}$, written as $x_{\widetilde{t}}\in\widetilde{\lambda}$ (resp. $x_{\widetilde{t}}q\widetilde{\lambda}$) if $\widetilde{\lambda}(x)\geq\widetilde{t}$(resp. $\widetilde{\lambda}(x)+\widetilde{t}>[1,1]$). If $x_{\widetilde{t}}\in\widetilde{\lambda}$ or $x_{\widetilde{t}}q\widetilde{\lambda}$, then we write $x_{\widetilde{t}}\in\vee q\widetilde{\lambda}$. If $x_{\widetilde{t}}\in\widetilde{\lambda}$ and $x_{\widetilde{t}}q\widetilde{\lambda}$, then we write $x_{\widetilde{t}}\in\wedge q\widetilde{\lambda}$. The symbol $\overline{\in\vee q}$ means the statement $\in\vee q$ does not hold.

We also emphasis that $\widetilde{a}=[a^-,a^+]$ must satisfy the following properties:

$[a^-,a^+]<[0.5,0.5]$ or $[0.5,0.5]\leq[a^-,a^+]$ for all $x\in S$.

3.1 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued (α,β)-fuzzy ternary subsemigroup of S, if:

(T1) $x_{\widetilde{t}}\alpha\ \widetilde{\lambda}$, $y_{\widetilde{r}}\alpha\ \widetilde{\lambda}$ and $z_{\widetilde{s}}\alpha\ \widetilde{\lambda}$ implies $(xyz)_{r\min\{\widetilde{t},\widetilde{r},\widetilde{s}\}}\ \beta\ \widetilde{\lambda}$ for all $x, y, z\in S$, where $\widetilde{t}, \widetilde{r}, \widetilde{s}\in D[0,1]$.

Let $\widetilde{\lambda}$ be an interval-valued fuzzy set of S such that $\widetilde{\lambda}(x)\leq[0.5,0.5]$, for all $x\in S$. Suppose that $x\in S$ and $[0,0]<\widetilde{t}\leq[1,1]$ be such that $x_{\widetilde{t}}\in\wedge q\widetilde{\lambda}$. Then $\widetilde{\lambda}(x)\geq\widetilde{t}$ and $\widetilde{\lambda}(x)+\widetilde{t}>[1,1]$. It follows that $[1,1]<\widetilde{\lambda}(x)+\widetilde{t}\leq\widetilde{\lambda}(x)+\widetilde{\lambda}(x)=2\widetilde{\lambda}(x)$, which implies $\widetilde{\lambda}(x)>[0.5,0.5]$. This means that $\left\{x_{\widetilde{t}}\mid x_{\widetilde{t}}\in\wedge q\widetilde{\lambda}\right\}=\emptyset$. Therefore, the case $\alpha=\in\wedge q$ in above definition is omitted.

3.2 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued (α,β)-fuzzy left (resp. right, lateral) ideal of S, where $\alpha\neq\in\wedge q$, if it satisfy:

(T2) $z_{\widetilde{t}}\alpha\ \widetilde{\lambda}$ and $x,y\in S$ implies $(xyz)_{\widetilde{t}}\beta\widetilde{\lambda}$ (resp. $(zxy)_{\widetilde{t}}\beta\widetilde{\lambda}$, $(xzy)_{\widetilde{t}}\beta\widetilde{\lambda}$).

3.3 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued (α,β)-fuzzy two sided ideal of S if it is both an interval-valued (α,β)-fuzzy left ideal and interval-valued (α,β)-fuzzy right ideal of S. An interval-valued fuzzy subset of a ternary semigroup S is called an interval-valued (α,β)-fuzzy ideal of S if it is an interval-valued (α,β)-fuzzy left ideal, interval-valued (α,β)-fuzzy right ideal and interval-valued (α,β)-fuzzy lateral ideal of S.

3.4 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued (α,β)-fuzzy bi-ideal of S, where $\alpha\neq\in\wedge q$, if it satisfies the conditions (T1) and (T3), where,

(T3) For all $u,\ v,\ x,\ y,\ z\in S$ and $[0,0]<\widetilde{t}_4,\ \widetilde{t}_5,\ \widetilde{t}_6\in D[0,1]$, $x_{\widetilde{t}_4}\alpha\widetilde{\lambda}$, $y_{\widetilde{t}_5}\alpha\widetilde{\lambda}$ and $z_{\widetilde{t}_6}\alpha\widetilde{\lambda}$ implies $(xuyvz)_{r\min\{\widetilde{t}_4,\widetilde{t}_5,\widetilde{t}_6\}}\beta\widetilde{\lambda}$.

3.5 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued (α,β)-fuzzy generalized bi-ideal of S, where $\alpha\neq\in\wedge q$, if it satisfies the condition (T3).

3.6 Lemma

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is an interval-valued fuzzy ternary subsemigroup of S, if and only if $\widetilde{\lambda}$ is an interval-valued $(\in,\in)$-fuzzy ternary subsemigroup of S.

*Proof.*Let $\widetilde{\lambda}$ be an interval-valued fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $[0,0] < \widetilde{t}_1, \widetilde{t}_2, \widetilde{t}_3 \in D[0,1]$ be such that $x_{\widetilde{t}_1} \in \widetilde{\lambda}, y_{\widetilde{t}_2} \in \widetilde{\lambda}$ and $z_{\widetilde{t}_3} \in \widetilde{\lambda}$. Then $\widetilde{\lambda}(x) \geqslant \widetilde{t}_1$, $\widetilde{\lambda}(y) \geqslant \widetilde{t}_2$, $\widetilde{\lambda}(z) \geqslant \widetilde{t}_3$. Since $\widetilde{\lambda}$ is an interval-valued fuzzy ternary subsemigroup of S, so $\widetilde{\lambda}(xyz) \geqslant rmin\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\lambda}(z)\right\} \geqslant rmin\{\widetilde{t}_1, \widetilde{t}_2, \widetilde{t}_3\}$. Hence $(xyz)_{rmin\{\widetilde{t}_1,\widetilde{t}_2,\widetilde{t}_3\}} \in \widetilde{\lambda}$.

Conversely, assume that $\widetilde{\lambda}$ is an interval-valued $(\in,\in)$-fuzzy ternary subsemigroup of S. Suppose on the contrary that there exist $x, y, z \in S$ such that $\widetilde{\lambda}(xyz) < \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{\lambda}(z)$. Let $[0,0] < \widetilde{t} \leq [1,1]$ be such that $\widetilde{\lambda}(xyz) < \widetilde{t} \leq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{\lambda}(z)$. Then $x_{\widetilde{t}} \in \widetilde{\lambda}$, $y_{\widetilde{t}} \in \widetilde{\lambda}$ and $z_{\widetilde{t}} \in \widetilde{\lambda}$ but $(xyz)_{\widetilde{t}} \overline{\in} \widetilde{\lambda}$, which contradicts our hypothesis. Hence $\widetilde{\lambda}(xyz) \geqslant \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{\lambda}(z)$.

3.7 Lemma

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is an interval-valued fuzzy left (right, lateral) ideal of S if and only if $\widetilde{\lambda}$ is an interval-valued $(\in,\in)$-fuzzy left (right, lateral) ideal of S.

*Proof.*The proof is similar to the proof of Lemma 3.6.

In a similar fashion we can prove the following lemma.

3.8 Lemma

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is an interval-valued fuzzy generalized bi-ideal (bi-ideal) of S if and only if $\widetilde{\lambda}$ is an interval-valued $(\in,\in)$-fuzzy generalized bi-ideal (bi-ideal) of S.

3.9 Theorem

Let $\widetilde{\lambda}$ be a non-zero interval-valued (α,β)-fuzzy ternary subsemigroup (resp. left ideal, right ideal, lateral ideal) of a ternary semigroup S. Then the set $\widetilde{\lambda}_0 = \left\{x \in S : \widetilde{\lambda}(x) > [0,0]\right\}$ is a (crisp) ternary subsemigroup (resp. left ideal, right ideal, lateral ideal) of S.

*Proof.*Let $x, y, z \in \widetilde{\lambda}_0$. Then $\widetilde{\lambda}(x) > [0,0]$, $\widetilde{\lambda}(y) > [0,0]$, $\widetilde{\lambda}(z) > [0,0]$. Suppose $\widetilde{\lambda}(xyz) = [0,0]$. If $\alpha \in \{\in, \in\vee q\}$, then $x_{\widetilde{\lambda}(x)} \alpha \widetilde{\lambda}$, $y_{\widetilde{\lambda}(y)} \alpha \widetilde{\lambda}$, $z_{\widetilde{\lambda}(z)} \alpha \widetilde{\lambda}$ but $\widetilde{\lambda}(xyz) = [0,0] < r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\lambda}(z)\right\}$ and $\widetilde{\lambda}(xyz) + rmin\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\lambda}(z)\right\} \leq [0,0] + [1,1] = [1,1]$. This implies that $(xyz)_{rmin\left\{\widetilde{\lambda}(x),\widetilde{\lambda}(y),\widetilde{\lambda}(z)\right\}} \overline{\beta} \widetilde{\lambda}$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, which is a contradiction. Hence $\widetilde{\lambda}(xyz) > [0,0]$, that is, $xyz \in \widetilde{\lambda}_0$.

Also $x_{[1,1]} q \widetilde{\lambda}, y_{[1,1]} q \widetilde{\lambda}, z_{[1,1]} q \widetilde{\lambda}$ but $(xyz)_{[1,1]} \overline{\beta} \widetilde{\lambda}$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$. Hence $\widetilde{\lambda}(xyz) > [0,0]$. Thus, $xyz \in \widetilde{\lambda}_0$. This implies that $\widetilde{\lambda}_0$ is a ternary subsemigroup of S.

3.10 Theorem

Let $\widetilde{\lambda}$ be a non-zero interval-valued (α,β)-fuzzy generalized bi-ideal (bi-ideal) of a ternary semigroup S. Then the set $\widetilde{\lambda}_0 = \left\{x \in S : \widetilde{\lambda}(x) > [0,0]\right\}$ is a (crisp) generalized bi-ideal (bi-ideal) of S.

*Proof.*The proof is similar to the proof of Theorem 3.9.

3.11 Theorem

Let L be a non-empty subset of a ternary semigroup S and $\alpha \in \{\in, q, \in\vee q\}$. Then L is a left (resp. right, lateral) ideal of S if and only if the interval-valued fuzzy subset $\widetilde{\lambda}$ of S defined by:

$$\widetilde{\lambda}(x) = \begin{cases} \geq [0.5, 0.5] \text{ if } x \in L \\ [0,0] \quad \text{if } x \in S - L \end{cases}$$

is an interval-valued $(\alpha, \in\vee q)$-fuzzy left (resp. right, lateral) ideal of S.

*Proof.*Suppose L is a left ideal of S.

(1) Let $x, y, z \in S$ and $[0,0] < \widetilde{t} \leq [1,1]$ be such that $z_{\widetilde{t}} \in \widetilde{\lambda}$. Then $\widetilde{\lambda}(z) \geq t$ and so $z \in L$. This implies that $xyz \in L$. Thus, if $\widetilde{t} \leq [0.5, 0.5]$, then $\widetilde{\lambda}(xyz) \geq [0.5, 0.5] \geq \widetilde{t}$ implies $\widetilde{\lambda}(xyz) \geq \widetilde{t}$, that is, $(xyz)_{\widetilde{t}} \in \widetilde{\lambda}$. If $\widetilde{t} > [0.5, 0.5]$, then $\widetilde{\lambda}(xyz) + \widetilde{t} > [0.5, 0.5] + [0.5, 0.5] = [1,1]$. This implies that $(xyz)_{\widetilde{t}} q \widetilde{\lambda}$. Therefore, $(xyz)_{\widetilde{t}} \in \vee q \widetilde{\lambda}$. Thus, $\widetilde{\lambda}$ is an interval-valued $(\in, \in\vee q)$-fuzzy left ideal of S.

(2) Let $x, y, z \in S$ and $[0,0] < \widetilde{t} \leq [1,1]$ be such that $z_{\widetilde{t}} q \widetilde{\lambda}$. Then $\widetilde{\lambda}(z) + \widetilde{t} > [1,1]$, implies $z \in L$. This implies that $xyz \in L$. Thus, if $\widetilde{t} \leq [0.5, 0.5]$, then $\widetilde{\lambda}(xyz) \geq [0.5, 0.5] \geq \widetilde{t}$. This implies that $(xyz)_{\widetilde{t}} \in \widetilde{\lambda}$. If $\widetilde{t} > [0.5, 0.5]$, then $\widetilde{\lambda}(xyz) + \widetilde{t} > [0.5, 0.5] + [0.5, 0.5] = [1,1]$. This implies that $\widetilde{\lambda}(xyz) + \widetilde{t} > [1,1]$, that is, $(xyz)_{\widetilde{t}} q \widetilde{\lambda}$. Thus,

$(xyz)_{\widetilde{t}} \in \vee q\widetilde{\lambda}$. This shows that $\widetilde{\lambda}$ is an interval-valued $(q, \in \vee q)$-fuzzy left ideal of S.

(3) Let $x, y, z \in S$ and $[0,0] < \widetilde{t} \leq [1,1]$ be such that $z_{\widetilde{t}} \in \vee q\widetilde{\lambda}$. This implies that $z_{\widetilde{t}} \in \widetilde{\lambda}$ or $z_{\widetilde{t}} q\widetilde{\lambda}$. Then $\widetilde{\lambda}(z) \geq \widetilde{t}$ or $\widetilde{\lambda}(z) + \widetilde{t} > [1,1]$. Thus, if $\widetilde{\lambda}(z) \geq \widetilde{t}$, then $z \in L$. This implies that $xyz \in L$. If $\widetilde{\lambda}(z) + \widetilde{t} > [1,1]$, then $z \in L$. This implies that $xyz \in L$. Analogous to (1) and (2) we obtain $(xyz)_{\widetilde{t}} \in \vee q\widetilde{\lambda}$.

Hence $\widetilde{\lambda}$ is an interval-valued $(\in \vee q, \in \vee q)$-fuzzy left ideal of S.

Conversely, assume that $\widetilde{\lambda}$ is an interval-valued $(\alpha, \in \vee q)$-fuzzy left ideal of S. Then $L = \widetilde{\lambda}_0$. Thus, by Theorem 3.9, L is a left ideal of S.

In a similar fashion we can prove the following theorems.

3.12 Theorem

Let A be a non-empty subset of a ternary semigroup S and $\alpha \in \{\in, q, \in \vee q\}$. Then A is a ternary subsemigroup (bi-ideal, generalized bi-ideal) of S if and only if the interval-valued fuzzy subset $\widetilde{\lambda}$ of S defined by:

$$\widetilde{\lambda}(x) = \begin{cases} \geq [0.5, 0.5] & \text{if } x \in A \\ [0,0] & \text{if } x \in S - A \end{cases}$$

is an interval-valued $(\alpha, \in \vee q)$-fuzzy ternary subsemigroup (bi-ideal, generalized bi-ideal) of S.

3.13 Proposition

Every interval-valued $(q, \in \vee q)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal) of a ternary semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal) of S.

3.14 Remark

Every interval-valued (α, β)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, bi-ideal, generalized bi-ideal) of a ternary semigroup S is an interval-valued $(\alpha, \in \vee q)$-fuzzy (left ideal, right ideal, lateral ideal, bi-ideal, generalized bi-ideal) of S.

3.15 Remark

Every interval-valued $(\alpha, \in \vee q)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, bi-ideal, generalized bi-ideal) of a ternary semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, bi-ideal, generalized bi-ideal) of S.

The above remarks show the importance of interval-valued $(\in, \in \vee q)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, bi-ideal, generalized bi-ideal) in interval-valued (α, β)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, bi-ideal, generalized bi-ideal) of a ternary semigroup S. Therefore, in next section we study interval-valued $(\in, \in \vee q)$-fuzzy ideals of a ternary semigroup S.

4 Interval-valued $(\in, \in \vee q)$-fuzzy ideals

Recall that an interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is called an interval-valued $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal of S if $z_{\widetilde{t}} \in \widetilde{\lambda}$ and $x, y \in S$ implies $(xyz)_{\widetilde{t}} \in \vee q\widetilde{\lambda}$ $\left((zxy)_{\widetilde{t}} \in \vee q\widetilde{\lambda},\ (xzy)_{\widetilde{t}} \in \vee q\widetilde{\lambda}\right)$ for all $[0,0] < \widetilde{t} \leq [1,1]$.

4.1 Theorem

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of a ternary semigroup S. Then $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal of S if and only if $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(z),\ [0.5, 0.5]\right\}$

$(\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x),\ [0.5, 0.5]\right\}$, $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(y),\ [0.5, 0.5]\right\})$.

*Proof.*Let $\widetilde{\lambda}$ be an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S. Assume on the contrary that there exist $x, y, z \in S$ such that $\widetilde{\lambda}(xyz) < r\min\left\{\widetilde{\lambda}(z),\ [0.5, 0.5]\right\}$. Choose $[0,0] < \widetilde{t} \leq [1,1]$ such that $\widetilde{\lambda}(xyz) < \widetilde{t} \leq r\min\left\{\widetilde{\lambda}(z),\ [0.5, 0.5]\right\}$. Then $z_{\widetilde{t}} \in \widetilde{\lambda}$ but $(xyz)_{\widetilde{t}} \overline{\in \vee q} \widetilde{\lambda}$, which is a contradiction. Hence $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(z),\ [0.5, 0.5]\right\}$.

Conversely, assume that $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(z),\ [0.5,\ 0.5]\right\}$. Let $z_{\widetilde{t}} \in \widetilde{\lambda}$. Then $\widetilde{\lambda}(z) \geq \widetilde{t}$. Now, $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(z),\ [0.5, 0.5]\right\} \geq r\min\{\widetilde{t},\ [0.5, 0.5]\}$. If $\widetilde{t} \leq [0.5, 0.5]$, then $\widetilde{\lambda}(xyz) \geq \widetilde{t}$ so $(xyz)_{\widetilde{t}} \in \widetilde{\lambda}$. If $\widetilde{t} > [0.5, 0.5]$, then $\widetilde{\lambda}(xyz) > [0.5, 0.5]$. Thus, $\widetilde{\lambda}(xyz) + \widetilde{t} > [0.5, 0.5] + [0.5, 0.5] = [1,1]$. This implies that $(xyz)_{\widetilde{t}} q\widetilde{\lambda}$. Hence $(xyz)_{\widetilde{t}} \in \vee q\widetilde{\lambda}$. Therefore, $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S.

4.2 Corollary

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of a ternary semigroup S. Then $\widetilde{\lambda}$ is an interval-valued

$(\in,\in\vee q)$-fuzzy two sided ideal of S if and only if $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(z),\ [0.5,0.5]\right\}$ and $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x),\ [0.5,0.5]\right\}$.

4.3 Theorem

The intersection of interval-valued $(\in,\in\vee q)$-fuzzy left (right, lateral) ideals of a ternary semigroup S is an interval-valued $(\in,\in\vee q)$-fuzzy left (right, lateral) ideal of S.

*Proof.*Let $\left\{\widetilde{\lambda}_i\right\}_{i\in\Lambda}$ be a family of interval-valued $(\in,\in\vee q)$-fuzzy left ideals of a ternary semigroup S and $x, y, z \in S$.

Since each $\widetilde{\lambda}_i$ is an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S so $\widetilde{\lambda}_i(xyz) \geq \widetilde{\lambda}_i(z) \wedge [0.5,0.5]$ for each $i \in \Lambda$. Thus,

$$\begin{aligned}\left(\wedge_{i\in\Lambda}\widetilde{\lambda}_i\right)(xyz) &= \wedge_{i\in\Lambda}\widetilde{\lambda}_i(xyz) \geq \wedge_{i\in\Lambda}\left(\widetilde{\lambda}_i(z)\wedge[0.5,0.5]\right)\\ &= \left(\wedge_{i\in\Lambda}\widetilde{\lambda}_i(z)\right)\wedge[0.5,0.5]\\ &= \left(\wedge_{i\in\Lambda}\widetilde{\lambda}_i\right)(z)\wedge[0.5,0.5].\end{aligned}$$

Thus, $\left(\wedge_{i\in\Lambda}\widetilde{\lambda}_i\right)(xyz) \geq \left(\wedge_{i\in\Lambda}\widetilde{\lambda}_i\right)(z)\wedge[0.5,0.5]$.

Hence $\wedge_{i\in\Lambda}\widetilde{\lambda}_i$ is an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S.

4.4 Theorem

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is an interval-valued $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S if and only if

$$\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}.$$

*Proof.*The proof follows from the Theorem 4.1.

4.5 Remark

Every $(\in,\in)$-fuzzy ternary subsemigroup of a ternary semigroup S is an interval-valued $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S but the converse need not be true.

4.6 Example

Let $S = \{-i,\ 0,\ i\}$. Then S is a ternary semigroup under the ternary multiplication of complex numbers. Define an interval-valued fuzzy subset $\widetilde{\lambda}$ of S by:

$\widetilde{\lambda}(i) = [0.6,0.8]$, $\widetilde{\lambda}(-i) = [0.7,0.75]$, $\widetilde{\lambda}(0) = [0.2,0.35]$. Then it is simple to verify that $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy ternary subsemigroup, but not an interval-valued $(\in,\in)$-fuzzy ternary subsemigroup of S.

4.7 Theorem

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is an interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S if and only if

$\widetilde{\lambda}(xuyvz) \geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}$ for all $u, v, x, y, z \in S$.

*Proof.*Assume that $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S. Suppose on the contrary that there exist $u, v, x, y, z \in S$ such that

$\widetilde{\lambda}(xuyvz) < r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}$.

Choose $[0,0] < \widetilde{t} \leq [1,1]$ such that

$\widetilde{\lambda}(xuyvz) < \widetilde{t} \leq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}$. Then $x_{\widetilde{t}} \in \widetilde{\lambda}$, $y_{\widetilde{t}} \in \widetilde{\lambda}$ and $z_{\widetilde{t}} \in \widetilde{\lambda}$ but $(xuyvz)_{r\min\{\widetilde{t},\widetilde{t},\widetilde{t}\}}\overline{\in\vee q}\widetilde{\lambda} = (xuyvz)_{\widetilde{t}}\overline{\in\vee q}\widetilde{\lambda}$, which is a contradiction. Hence, $\widetilde{\lambda}(xuyvz) \geq r\min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\lambda}(z), [0.5,0.5]\right\}$.

Conversely, assume that $\widetilde{\lambda}(xuyvz) \geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z), [0.5,0.5]\right\}$ for all $u, v, x, y, z \in S$. Let $x_{\widetilde{t}_1} \in \widetilde{\lambda}$, $y_{\widetilde{t}_2} \in \widetilde{\lambda}$ and $z_{\widetilde{t}_3} \in \widetilde{\lambda}$ for all $[0,0] < \widetilde{t}_1 \leq [1,1]$, $[0,0] < \widetilde{t}_2 \leq [1,1]$, $[0,0] < \widetilde{t}_3 \leq [1,1]$. Then $\widetilde{\lambda}(x) \geq \widetilde{t}_1$, $\widetilde{\lambda}(y) \geq \widetilde{t}_2$, $\widetilde{\lambda}(z) \geq \widetilde{t}_3$. Hence,

$$\begin{aligned}\widetilde{\lambda}(xuyvz) &\geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}\\ &\geq r\min\left\{\widetilde{t}_1,\ \widetilde{t}_2,\ \widetilde{t}_3,\ [0.5,0.5]\right\}.\end{aligned}$$

If $r\min\{\widetilde{t}_1,\widetilde{t}_2,\widetilde{t}_3\} \leq [0.5,0.5]$, then $\widetilde{\lambda}(xuyvz) \geq r\min\{\widetilde{t}_1,\ \widetilde{t}_2,\ \widetilde{t}_3\}$. This implies that $(xuyvz)_{r\min\{\widetilde{t}_1,\widetilde{t}_2,\widetilde{t}_3\}} \in \widetilde{\lambda}$. If $r\min\{\widetilde{t}_1,\ \widetilde{t}_2,\ \widetilde{t}_3\} > [0.5,0.5]$, then

$$\begin{aligned}\widetilde{\lambda}(xuyvz) + r\min\{\widetilde{t}_1,\ \widetilde{t}_2,\ \widetilde{t}_3\} &> [0.5,0.5]+[0.5,0.5]\\ &= [1,1],\end{aligned}$$

so $(xuyvz)_{rmin\{\widetilde{t}_1,\widetilde{t}_2,\widetilde{t}_3\}}\, q\widetilde{\lambda}$. Thus, $(xuyvz)_{rmin\{\widetilde{t}_1,\widetilde{t}_2,\widetilde{t}_3\}} \in \vee q\widetilde{\lambda}$. Therefore $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S.

4.8 Theorem

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal of S if and only if it satisfies:

(1) $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}$;

(2) $\widetilde{\lambda}(xuyvz) \geq r\min\left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}$

for all $u, v, x, y, z \in S$.

*Proof.*The proof follows from Theorem 4.4 and Theorem 4.7.

Now, we characterize the interval-valued $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal of S by their level subsets.

4.9 Theorem

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of a ternary semigroup S. Then $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal of S if and only if $U\left(\widetilde{\lambda};\widetilde{t}\right)(\neq \emptyset)$ is a left (right, lateral) ideal of S for all $[0,0] < \widetilde{t} \leq [1,1]$.

*Proof.*Suppose that $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S. Let $x, y \in S$ and $z \in U\left(\widetilde{\lambda};\widetilde{t}\right)$ for some $[0,0] < \widetilde{t} \leq [0.5,0.5]$. Then $\widetilde{\lambda}(z) \geq \widetilde{t}$. Since $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(z),\ [0.5,0.5]\right\} \geq r\min\left\{\widetilde{t}, [0.5,\ 0.5]\right\} = \widetilde{t}$. This implies that $\widetilde{\lambda}(xyz) \geq \widetilde{t}$. This implies that $xyz \in U\left(\widetilde{\lambda};\widetilde{t}\right)$. Hence $U\left(\widetilde{\lambda};\widetilde{t}\right)$ is a left ideal of S.

Conversely, assume that $U\left(\widetilde{\lambda};\widetilde{t}\right)(\neq \emptyset)$ is a left ideal of S for all $[0,0] < \widetilde{t} \leq [0.5,0.5]$. We show that $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S. Suppose on the contrary that there exist x, y, $z \in S$ such that $\widetilde{\lambda}(xyz) < r\min\left\{\widetilde{\lambda}(z),\ [0.5,0.5]\right\}$. Choose $[0,0] < \widetilde{t} \leq [0.5,0.5]$ such that $\widetilde{\lambda}(xyz) < \widetilde{t} \leq r\min\left\{\widetilde{\lambda}(z),\ [0.5,0.5]\right\}$. Then $z \in U\left(\widetilde{\lambda};\widetilde{t}\right)$ but $xyz \notin U\left(\widetilde{\lambda};\widetilde{t}\right)$, which is a contradiction. Thus, $\widetilde{\lambda}(xyz) \geq r\min\left\{\widetilde{\lambda}(z),\ [0.5,0.5]\right\}$. Hence $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S.

4.10 Definition

An interval-valued fuzzy subset $\widetilde{\lambda}$ of a ternary semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S if and only if it satisfies:

(1)

$$\widetilde{\lambda}(x) \geq r\min\left\{\begin{array}{c}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(x), \left(\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\right)(x),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(x), [0.5,0.5]\end{array}\right\};$$

(2)

$$\widetilde{\lambda}(x) \geq r\min\left\{\begin{array}{c}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(x),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(x),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(x), [0.5,0.5]\end{array}\right\};$$

where $\widetilde{\mathscr{S}}$ is the interval-valued fuzzy subset of S mapping every element of S on $[1,1]$.

4.11 Proposition

Every interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of a ternary semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.

*Proof.*Straightforward.

4.12 Theorem

Let $\widetilde{\lambda}$ be an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of a ternary semigroup S. Then the set $\widetilde{\lambda}_0 = \left\{x \in S : \widetilde{\lambda}(x) > [0,0]\right\}$ is a quasi-ideal of S.

*Proof.*To show that $\widetilde{\lambda}_0$ is a quasi-ideal of S we show that $SS\widetilde{\lambda}_0 \cap S\widetilde{\lambda}_0 S \cap \widetilde{\lambda}_0 SS \subseteq \widetilde{\lambda}_0$ and $SS\widetilde{\lambda}_0 \cap SS\widetilde{\lambda}_0 SS \cup \widetilde{\lambda}_0 SS \subseteq \widetilde{\lambda}_0$.

Let $a \in SS\widetilde{\lambda}_0 \cap S\widetilde{\lambda}_0 S \cap \widetilde{\lambda}_0 SS$. Then $a \in SS\widetilde{\lambda}_0$, $a \in S\widetilde{\lambda}_0 S$ and $a \in \widetilde{\lambda}_0 SS$. This implies that there exist x, y, $z \in \widetilde{\lambda}_0$ and s_1, s_2, s_3, t_1, t_2, $t_3 \in S$ such that $a = s_1 t_1 x$, $a = s_2 y t_2$, $a = z s_3 t_3$. Now,

$$\widetilde{\lambda}(a) \geq r\min\left\{\begin{array}{c}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a), \left(\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a), [0.5,0.5]\end{array}\right\}.$$

Since

$$\begin{aligned}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a) &= \vee_{a=pqr}\left\{\widetilde{\mathscr{S}}(p)\wedge\widetilde{\mathscr{S}}(q)\wedge\widetilde{\lambda}(r)\right\}\\ &\geq \widetilde{\mathscr{S}}(s_1)\wedge\widetilde{\mathscr{S}}(t_1)\wedge\widetilde{\lambda}(x)\\ &= [1,1]\wedge[1,1]\wedge\widetilde{\lambda}(x) = \widetilde{\lambda}(x).\end{aligned}$$

Similarly, $\left(\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\right)(a) \geq \widetilde{\lambda}(y)$ and $\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a) \geq \widetilde{\lambda}(z)$. Thus,

$$\begin{aligned}\widetilde{\lambda}(a) &\geq r\min\left\{\begin{array}{c}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a), \left(\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a), [0.5,0.5]\end{array}\right\}\\ &\geq r\min\left\{\widetilde{\lambda}(z), \widetilde{\lambda}(y), \widetilde{\lambda}(x), [0.5,0.5]\right\}\\ &> [0,0] \quad (\text{since } \widetilde{\lambda}(x) > [0,0],\ \widetilde{\lambda}(y)\\ &> [0,0],\ \widetilde{\lambda}(z) > [0,0]).\end{aligned}$$

This implies that $a \in \widetilde{\lambda}_0$. Thus, $SS\widetilde{\lambda}_0 \cap S\widetilde{\lambda}_0 S \cap \widetilde{\lambda}_0 SS \subseteq \widetilde{\lambda}_0$. Next we show $SS\widetilde{\lambda}_0 \cap SS\widetilde{\lambda}_0 SS \cap \widetilde{\lambda}_0 SS \subseteq \widetilde{\lambda}_0$. Let $a \in SS\widetilde{\lambda}_0 \cap SS\widetilde{\lambda}_0 SS \cap \widetilde{\lambda}_0 SS$. Then $a \in SS\widetilde{\lambda}_0$ and $a \in SS\widetilde{\lambda}_0 SS$ and $a \in \widetilde{\lambda}_0 SS$. Thus, $a = s_1 t_1 x$, $a = z s_3 t_3$, $a = s_1 t_2 y s_4 t_4$ for $x, y, z \in \widetilde{\lambda}_0$ and $s_1, s_2, s_3, s_4, \in S$.

Now,

$$\widetilde{\lambda}_0(a) \geq r\min\left\{\begin{array}{c}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a),[0.5,0.5]\end{array}\right\}$$

and by above arguments $\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a) \geq \widetilde{\lambda}(x)$, $\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a) \geq \widetilde{\lambda}(z)$ and

$$\begin{aligned}&\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a)\\ &= \vee_{a=rst}\left\{\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(r)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\mathscr{S}}(t)\right\}\\ &= \vee_{a=rst}\left\{\left(\begin{array}{c}\vee_{r=mnk}\widetilde{\mathscr{S}}(m)\wedge\widetilde{\mathscr{S}}(n)\\ \wedge\widetilde{\lambda}(k)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\mathscr{S}}(t)\end{array}\right)\right\},\end{aligned}$$

$$\begin{aligned}&\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a)\\ &= \vee_{a=(mnk)st}\left\{\begin{array}{c}\widetilde{\mathscr{S}}(p)\wedge\widetilde{\mathscr{S}}(q)\\ \wedge\widetilde{\lambda}(r)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\mathscr{S}}(t)\end{array}\right\}\\ &\geq \widetilde{\mathscr{S}}(s_1)\wedge\widetilde{\mathscr{S}}(s_2)\wedge\widetilde{\lambda}(y)\\ &\quad\wedge\widetilde{\mathscr{S}}(s_3)\wedge\widetilde{\mathscr{S}}(s_4)\\ &= [1,1]\wedge[1,1]\wedge\widetilde{\lambda}(y)\wedge[1,1]\wedge[1,1]\\ &= \widetilde{\lambda}(y).\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{\lambda}(a) &\geq r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),\ [0.5,0.5]\end{array}\right\}\\ &\geq \left\{\widetilde{\lambda}(x),\ \widetilde{\lambda}(y),\ \widetilde{\lambda}(z),\ [0.5,0.5]\right\}\\ &> [0,0] \quad (\text{since } \widetilde{\lambda}(x) > [0,0],\end{aligned}$$

$\widetilde{\lambda}(y) > [0,0]$, $\widetilde{\lambda}(z) > [0,0]$).

Thus, $a \in \widetilde{\lambda}_0$ and hence $SS\widetilde{\lambda}_0 \cap SS\widetilde{\lambda}_0SS \cap \widetilde{\lambda}_0SS \subseteq \widetilde{\lambda}_0$. Therefore, $\widetilde{\lambda}_0$ is a quasi-ideal of S.

4.13 Lemma

A non-empty subset Q of a ternary semigroup S is a quasi-ideal of S if and only if the interval-valued characteristic function $\widetilde{C}_Q$, of Q, is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S.

*Proof.*Let Q be a quasi-ideal of S and $\widetilde{C}_Q$ the interval-valued characteristic function of Q and $x \in S$. If $x \notin Q$, then $x \notin SSQ$ or $x \notin QSS$ or $x \notin SQS$. If $x \notin SSQ$, then $\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\right)(x) = [0,0]$ so,

$$\begin{aligned}&r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\right)(x),\left(\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\right)(x),\\ \left(\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(x),[0.5,0.5]\end{array}\right\}\\ &= [0,0] = \widetilde{C}_Q(x).\end{aligned}$$

Similarly, for other cases. If $x \in Q$, then

$$\begin{aligned}\widetilde{C}_Q(x) &= [1,1]\\ &\geq r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\right)(x),\left(\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\right)(x),\\ \left(\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(x),[0.5,0.5]\end{array}\right\}.\end{aligned}$$

Similarly,

$$\widetilde{C}_Q(x) \geq r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\right)(x),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(x),\\ \left(\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(x),[0.5,0.5]\end{array}\right\}.$$

Thus, $\widetilde{C}_Q$ is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S.

Conversely, assume that $\widetilde{C}_Q$ is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S. We show that Q is a quasi-ideal of S. Let $a \in SSQ\cap SQS\cap QSS$. Then $a \in SSQ$ and $a \in SQS$ and $a \in QSS$. Thus, there exist $x,y,z \in Q$ and $s_1,s_2,s_3,t_1,t_2,t_3 \in S$ such that $a = s_1t_1x$ and $a = s_2yt_2$ and $a = zs_3t_3$. Now,

$$\begin{aligned}\left(\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a) &= \vee_{a=pqr}\left\{\widetilde{C}_Q(p)\wedge\widetilde{\mathscr{S}}(q)\wedge\widetilde{\mathscr{S}}(r)\right\}\\ &\geq \widetilde{C}_Q(z)\wedge\widetilde{\mathscr{S}}(s_3)\wedge\widetilde{\mathscr{S}}(t_3)\\ &= [1,1]\wedge[1,1]\wedge[1,1] = [1,1].\end{aligned}$$

So $\left(\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a) = [1,1]$. Similarly, $\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\right)(a) = [1,1]$ and $\left(\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\right)(a) = [1,1]$. Hence

$$\begin{aligned}\widetilde{C}_Q(a) &\geq r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\right)(a),\left(\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),[0.5,0.5]\end{array}\right\}\\ &= [0.5,0.5].\end{aligned}$$

Thus, $C_Q(a) \geq [0.5,0.5]$. This implies that $C_Q(a) = [1,1]$. Hence $a \in Q$ so $SSQ\cap SQS\cap QSS \subseteq Q$.

Next, let $a \in SSQ\cap SSQSS\cap QSS$. Then $a \in SSQ$ and $a \in SSQSS$ and $a \in QSS$. Thus, there exist $s_1,s_2,s_3,s_4,t_1,t_2,t_3,t_4 \in S, x,y,z \in Q$ such that $a = s_1t_1x$, $a = zs_3t_3$ and $a = s_2t_2ys_4t_4$. For $a = s_1t_1x$, and $a = zs_3t_3$ $\widetilde{C}_Q = [1,1]$ as discussed above. Consider the case if $a = s_2t_2ys_4t_4$:

$$\begin{aligned}&\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a)\\ &= \vee_{a=rst}\left\{\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\right)(r)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\mathscr{S}}(t)\right\}\\ &= \vee_{a=rst}\left\{\begin{array}{c}\left(\vee_{r=lmn}\widetilde{\mathscr{S}}(l)\wedge\widetilde{\mathscr{S}}(m)\wedge\widetilde{C}_Q(n)\right)\\ \wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\mathscr{S}}(t)\end{array}\right\}\end{aligned}$$

$$\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a)$$
$$= \vee_{a=(lmn)st}\left\{\widetilde{\mathscr{S}}(l)\wedge\widetilde{\mathscr{S}}(m)\wedge\widetilde{C}_Q(n)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\mathscr{S}}(t)\right\}$$
$$\geq \widetilde{\mathscr{S}}(s_2)\wedge\widetilde{\mathscr{S}}(t_2)\wedge\widetilde{C}_Q(y)\wedge\widetilde{\mathscr{S}}(s_4)\wedge\widetilde{\mathscr{S}}(t_4)$$
$$= [1,1]\wedge[1,1]\wedge[1,1]\wedge[1,1]\wedge[1,1]$$
$$= [1,1].$$

This implies that

$$\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a) = [1,1].$$

Thus,

$$\widetilde{C}_Q(a) \geq r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ C_Q\right)(a),\\ \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{C}_Q\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),[0.5,0.5]\end{array}\right\}$$
$$= [0.5,0.5]$$
$$\widetilde{C}_Q(a) \geq [0.5,0.5].$$

This implies that $\widetilde{C}_Q(a) = [1,1]$, and so $a \in Q$. Thus $SSQ\cap SSQSS\cap QSS \subseteq Q$. Hence Q is a quasi-ideal of S.

By using similar arguments as in the proof of the above lemma we can prove the following lemmas.

4.14 Lemma

The interval-valued characteristic function $\widetilde{C}_L$, of L, is an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S if and only if L is a left ideal of S.

4.15 Lemma

The interval-valued characteristic function $\widetilde{C}_B$, of B, is an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal of S if and only if B is a bi-ideal of S.

4.16 Theorem

Every interval-valued $(\in,\in\vee q)$-fuzzy left (right, lateral) ideal of a ternary semigroup S is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S.

*Proof.*Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S and $a\in S$. Then,

$$\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a) = \vee_{a=xyz}\left\{\widetilde{\mathscr{S}}(x)\wedge\widetilde{\mathscr{S}}(y)\wedge\widetilde{\lambda}(z)\right\}$$
$$= \vee_{a=xyz}\widetilde{\lambda}(z).$$

This implies that

$$\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge[0.5,0.5] = \left(\vee_{a=xyz}\widetilde{\lambda}(z)\right)\wedge[0.5,0.5]$$
$$\leq \vee_{a=xyz}\widetilde{\lambda}(xyz) = \widetilde{\lambda}(a).$$

Thus,

$$\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge[0.5,0.5] \leq \widetilde{\lambda}(a). \tag{1}$$

Hence

$$\widetilde{\lambda}(a) \geq \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge[0.5,0.5]$$
$$\geq r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a),\left(\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),0.5\end{array}\right\}.$$

So

$$\widetilde{\lambda}(a) \geq r\min\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a),\left(\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),0.5\end{array}\right\}.$$

Again from (1)

$$\widetilde{\lambda}(a) \geq \left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge[0.5,0.5]$$
$$\geq rmin\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a),\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),[0.5,0.5]\end{array}\right\}.$$

Hence

$$\widetilde{\lambda}(a) \geq rmin\left\{\begin{array}{c}\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a),\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),\\ \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(a),[0.5,0.5]\end{array}\right\}.$$

Therefore, $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S.

4.17 Theorem

Every interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of a ternary semigroup S is an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal of S.

*Proof.*Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S. Then by Proposition 4.11, $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S.

Now,

$$\widetilde{\lambda}(xuyvz)$$
$$\geq \left(\begin{array}{c}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(xuyvz)\wedge\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\right)(xuyvz)\\ \wedge\left(\widetilde{\mathscr{S}}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(xuyvz)\end{array}\right)$$
$$\wedge[0.5,0.5]$$

$$= \left\{ \begin{array}{c} \left(\vee_{xuyvz=abc}\left\{\widetilde{\lambda}(a)\wedge\widetilde{\mathscr{S}}(b)\wedge\widetilde{\mathscr{S}}(c)\right\}\right) \\ \wedge\left(\vee_{xuyvz=rst}\left(\begin{array}{c}\vee_{r=s_1s_2s_3}\widetilde{\mathscr{S}}(s_1)\wedge\widetilde{\mathscr{S}}(s_2) \\ \wedge\widetilde{\lambda}(s_3)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\mathscr{S}}(t)\end{array}\right)\right) \\ \wedge\left(\vee_{xuyvz=lmn}\left\{\widetilde{\mathscr{S}}(l)\wedge\widetilde{\mathscr{S}}(m)\wedge\widetilde{\lambda}(n)\right\}\right) \end{array} \right\} \wedge[0.5,0.5]$$

$$\geq \left\{ \begin{array}{c} \left\{\widetilde{\lambda}(x)\wedge\widetilde{\mathscr{S}}(uyv)\wedge\widetilde{\mathscr{S}}(z)\right\} \\ \wedge\left(\left(\widetilde{\mathscr{S}}(x)\wedge\widetilde{\mathscr{S}}(u)\wedge\widetilde{\lambda}(y)\right)\wedge\widetilde{\mathscr{S}}(v)\wedge\widetilde{\mathscr{S}}(z)\right) \\ \wedge\left(\left\{\widetilde{\mathscr{S}}(x)\wedge\widetilde{\mathscr{S}}(uyv)\wedge\widetilde{\lambda}(z)\right\}\right) \end{array} \right\} \wedge[0.5,0.5]$$

$$= \widetilde{\lambda}(x)\wedge\widetilde{\lambda}(y)\wedge\widetilde{\lambda}(z)\wedge[0.5,0.5]$$

Thus, $\widetilde{\lambda}(xuyvz) \geq rmin\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\lambda}(z), [0.5,0.5]\right\}$. Hence $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.

The converse of the above theorem is not true in general.

4.18 Example

Let $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix} : a,b,c,d \in \mathbb{Z}_0^- \right\}$ and $B = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x,y \in \mathbb{Z}_0^- \right\}$, where $\mathbb{Z}_0^-$ is the set of non-positive integers. Then S is a ternary semigroup with respect to ternary matrix multiplication and B is a bi-ideal of S

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of S defined by:

$$\widetilde{\lambda}(x) = \begin{cases} [0.7, 0, 8] \text{ if } x = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \ p,q \in \mathbb{Z}_0^-, \\ [0.2, 0.4] \quad \text{otherwise.} \end{cases}$$

Then $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S, but not an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S.

4.19 Theorem

Every interval-valued $(\in, \in \vee q)$-fuzzy ideal of a ternary semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Proof. The proof follows from Theorem 4.16 and Theorem 4.17.

The converse of Theorem 4.19 is not true in general.

4.20 Example

Let $S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ p & q & r \end{pmatrix} : a,b,c,p,q,r \in \mathbb{Z}_0^- \right\}$ and $B = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & n \\ 0 & 0 & 0 \end{pmatrix} : m,n \in \mathbb{Z}_0^- \right\}$, where $\mathbb{Z}_0^-$ is the set of non-positive integers. Then S is a ternary semigroup with respect to ternary matrix multiplication and B is a bi-ideal of S

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of S defined by:

$$\widetilde{\lambda}(x) = \begin{cases} [0.6, 0.7] \text{ if } x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & n \\ 0 & 0 & 0 \end{pmatrix}; m,n \in \mathbb{Z}_0^- \\ [0.3, 0.4] \quad \text{otherwise.} \end{cases}$$

Then $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S, but not an interval-valued $(\in, \in \vee q)$-fuzzy ideal of S.

5 Lower and upper parts of interval-valued $(\in, \in \vee q)$-fuzzy ideals

5.1 Definition

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of a ternary semigroup S. Define the fuzzy subsets $\widetilde{\lambda}^+$ and $\widetilde{\lambda}^-$ of S as follows:

$\widetilde{\lambda}^+(x) = \widetilde{\lambda}(x) \vee [0.5, 0.5]$ and
$\widetilde{\lambda}^-(x) = \widetilde{\lambda}(x) \wedge [0.5, 0.5]$.

5.2 Lemma

Let $\widetilde{\lambda}, \widetilde{\mu}$ and $\widetilde{\nu}$ be interval-valued fuzzy subsets of a ternary semigroup S. Then the following hold:

(1) $\left(\widetilde{\lambda} \wedge \widetilde{\mu}\right)^- = \left(\widetilde{\lambda}^- \wedge \widetilde{\mu}^-\right)$;

(2) $\left(\widetilde{\lambda} \vee \widetilde{\mu}\right)^- = \left(\widetilde{\lambda}^- \vee \widetilde{\mu}^-\right)$;

(3) $\left(\widetilde{\lambda} \circ \widetilde{\mu} \circ \widetilde{\nu}\right)^- = \left(\widetilde{\lambda}^- \circ \widetilde{\mu}^- \circ \widetilde{\nu}^-\right)$.

Proof. The proofs of (1) and (2) are obvious.

(3) Let $a \in S$. If a is not expressible as $a = bcd$ for some $b,c,d \in S$, then $\left(\widetilde{\lambda} \circ \widetilde{\mu} \circ \widetilde{\nu}\right)(a) = 0$ and

$$\left(\widetilde{\lambda} \circ \widetilde{\mu} \circ \widetilde{\nu}\right)^-(a) = \left(\widetilde{\lambda} \circ \widetilde{\mu} \circ \widetilde{\nu}\right)(a) \wedge [0.5, 0.5] = 0 \wedge [0.5, 0.5] = 0.$$

Since a is not expressible as $a = bcd$ so $\left(\widetilde{\lambda}^- \circ \widetilde{\mu}^- \circ \widetilde{\nu}^-\right)(a) = 0$. Thus, in this case

$\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{-}=\left(\widetilde{\lambda}^{-}\circ\widetilde{\mu}^{-}\circ\widetilde{\nu}^{-}\right)$. If a is expressible as $a=xyz$, then

$$\begin{aligned}&\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{-}(a)\\ &=\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)(a)\wedge[0.5,0.5]\\ &=\left(\vee_{a=xyz}\left(\widetilde{\lambda}(x)\wedge\widetilde{\lambda}(y)\wedge\widetilde{\lambda}(z)\right)\right)\wedge[0.5,0.5]\\ &=\vee_{a=xyz}\left\{\begin{array}{c}\left(\widetilde{\lambda}(x)\wedge[0.5,0.5]\right)\wedge\left(\widetilde{\lambda}(y)\wedge[0.5,0.5]\right)\\ \wedge\left(\widetilde{\lambda}(z)\wedge[0.5,0.5]\right)\end{array}\right\}\\ &=\vee_{a=xyz}\left\{\widetilde{\lambda}^{-}(x)\wedge\widetilde{\mu}^{-}(y)\wedge\widetilde{\nu}^{-}(z)\right\}\\ &=\left(\widetilde{\lambda}^{-}\circ\widetilde{\mu}^{-}\circ\widetilde{\nu}^{-}\right)(a).\end{aligned}$$

Therefore,

$$\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{-}=\left(\widetilde{\lambda}^{-}\circ\widetilde{\mu}^{-}\circ\widetilde{\nu}^{-}\right).$$

5.3 Lemma

Let $\widetilde{\lambda},\widetilde{\mu}$ and $\widetilde{\nu}$ be interval-valued fuzzy subsets of a ternary semigroup S. Then the following hold:

(1) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{+}=\left(\widetilde{\lambda}^{+}\wedge\widetilde{\mu}^{+}\right)$;

(2) $\left(\widetilde{\lambda}\vee\widetilde{\mu}\right)^{+}=\left(\widetilde{\lambda}^{+}\vee\widetilde{\mu}^{+}\right)$;

(3) $\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{+}\geq\left(\widetilde{\lambda}^{+}\circ\widetilde{\mu}^{+}\circ\widetilde{\nu}^{+}\right)$.

If every element $x\in S$ is expressible as $x=abc$ for some $a,b,c\in S$, then $\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{+}=\left(\widetilde{\lambda}^{+}\circ\widetilde{\mu}^{+}\circ\widetilde{\nu}^{+}\right)$.

*Proof.*The proofs of (1) and (2) are obvious.

(3) Let $a\in S$. If a is not expressible as $a=bcd$ for some $b,c,d\in S$, then $\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)(a)=0$ and

$$\begin{aligned}\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{+}(a)&=\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)(a)\vee[0.5,0.5]\\ &=0\vee[0.5,0.5]=[0.5,0.5]\end{aligned}$$

and $\left(\widetilde{\lambda}^{+}\circ\widetilde{\mu}^{+}\circ\widetilde{\nu}^{+}\right)(a)=0$. So,

$$\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{+}\geq\left(\widetilde{\lambda}^{+}\circ\widetilde{\mu}^{+}\circ\widetilde{\nu}^{+}\right).$$

If a is expressible as $a=bcd$, then

$$\begin{aligned}\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{+}(a)&=\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)(a)\vee[0.5,0.5]\\ &=\left(\vee_{a=xyz}\{\widetilde{\lambda}(x)\wedge\widetilde{\mu}(y)\wedge\widetilde{\nu}(z)\}\right)\\ &\quad\vee[0.5,0.5]\\ &=\vee_{a=xyz}\left\{\begin{array}{c}\left(\widetilde{\lambda}(x)\vee[0.5,0.5]\right)\\ \wedge(\widetilde{\mu}(y)\vee[0.5,0.5])\\ \wedge(\widetilde{\nu}(z)\vee[0.5,0.5])\end{array}\right\}\\ &=\vee_{a=xyz}\left\{\widetilde{\lambda}^{+}(x)\wedge\widetilde{\mu}^{+}(y)\wedge\widetilde{\nu}^{+}(z)\right\}\\ &=\left(\widetilde{\lambda}^{+}\circ\widetilde{\mu}^{+}\circ\widetilde{\nu}^{+}\right)(a).\end{aligned}$$

Therefore,

$$\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^{+}=\left(\widetilde{\lambda}^{+}\circ\widetilde{\mu}^{+}\circ\widetilde{\nu}^{+}\right).$$

5.4 Definition

Let A be a non-empty subset of a ternary semigroup S. Then $\widetilde{C}_A^{-}$ and $\widetilde{C}_A^{+}$ are defined as:

$$\widetilde{C}_A^{-}(x)=\begin{cases}[0.5,0.5] & \text{if } x\in A\\ [0,0] & \text{if } x\notin A\end{cases}$$

and

$$\widetilde{C}_A^{+}(x)=\begin{cases}[1,1] & \text{if } x\in A\\ [0.5,0.5] & \text{if } x\notin A.\end{cases}$$

5.5 Lemma

The lower part of the interval-valued characteristic function, that is, $\widetilde{C}_L^{-}$ is an interval-valued $(\in,\in\vee q)$-fuzzy left (resp. right, lateral) ideal of a ternary semigroup S if and only if L is a left (resp. right, lateral) ideal of S.

*Proof.*Let L be a left ideal of S. Then by Theorem 3.11, $\widetilde{C}_L^{-}$ is an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S.

Conversely, assume that $\widetilde{C}_L^{-}$ is an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S. Let $z\in L$. Then $\widetilde{C}_L^{-}(z)=[0.5,0.5]$, so $z_{[0.5,0.5]}\in\widetilde{C}_L^{-}$. Since $\widetilde{C}_L^{-}$ is an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S, so $(xyz)_{[0.5,0.5]}\in\vee q\widetilde{C}_L^{-}$. This implies that $(xyz)_{[0.5,0.5]}\in\widetilde{C}_L^{-}$ or $(xyz)_{[0.5,0.5]}q\widetilde{C}_L^{-}$. Thus, $\widetilde{C}_L^{-}(xyz)\geq[0.5,0.5]$ or $\widetilde{C}_L^{-}(xyz)+[0.5,0.5]>[1,1]$. Now $\widetilde{C}_L^{-}(xyz)+[0.5,0.5]>[1,1]$ is impossible. Thus, $\widetilde{C}_L^{-}(xyz)\geq[0.5,0.5]$ which implies that $\widetilde{C}_L^{-}(xyz)=[0.5,0.5]$. This implies that $xyz\in L$. Therefore L is a left ideal of S.

Similarly, it can be seen that the lower part of the interval-valued characteristic function $\widetilde{C}_R^{-}$ $\left(\text{resp. }\widetilde{C}_M^{-}\right)$ is an interval-valued $(\in,\in\vee q)$-fuzzy right (resp. lateral) ideal of S if and only if R (resp. M) is right (resp. lateral) ideal of S. Thus, the lower part of the interval-valued characteristic function $\widetilde{C}_I^-$ is an interval-valued $(\in,\in\vee q)$-fuzzy (two sided) ideal of S if and only if I is (two sided) ideal of S.

5.6 Lemma

Let Q be a non-empty subset of a ternary semigroup S. Then Q is a quasi-ideal of S if and only if the lower part of the interval-valued characteristic function, that is, $\widetilde{C}_Q^-$ is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S.

5.7 Lemma

Let A,B and C be non-empty subsets of a ternary semigroup S. Then the following hold:

(1) $\widetilde{C}_A\wedge\widetilde{C}_B=\widetilde{C}_{A\cap B}$;
(2) $\widetilde{C}_A\vee\widetilde{C}_B=\widetilde{C}_{A\cup B}$;
(3) $\widetilde{C}_A\circ\widetilde{C}_B\circ\widetilde{C}_C=\widetilde{C}_{ABC}$.

5.8 Proposition

Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy left (resp. right, lateral) ideal of a ternary semigroup S. Then $\widetilde{\lambda}^-$ is an interval-valued fuzzy left (resp. right, lateral) ideal of S.

*Proof.*Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S. Then for all $a,b,c\in S$, we have $\widetilde{\lambda}(abc)\geq\widetilde{\lambda}(c)\wedge[0.5,0.5]$. Thus,

$$\widetilde{\lambda}(abc)\wedge[0.5,0.5]\geq\left(\widetilde{\lambda}(c)\wedge[0.5,0.5]\right)\wedge[0.5,0.5].$$

This implies that $\widetilde{\lambda}^-(abc)\geq\widetilde{\lambda}^-(c)$. Hence $\widetilde{\lambda}^-$ is an interval-valued fuzzy left ideal of S.

We now characterize regular ternary semigroups by the properties of lower parts of interval-valued $(\in,\in\vee q)$-fuzzy ideals, interval-valued $(\in,\in\vee q)$-fuzzy left (right, lateral) ideals, interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideals, interval-valued $(\in,\in\vee q)$-fuzzy bi-ideals and interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideals.

5.9 Theorem

For a ternary semigroup S, the following conditions are equivalent:

(1) S is regular;

(2) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)^-=\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^-$ for every interval-valued $(\in,\in\vee q)$-fuzzy right ideal $\widetilde{\lambda}$, every interval-valued $(\in,\in\vee q)$-fuzzy lateral ideal $\widetilde{\mu}$ and every interval-valued $(\in,\in\vee q)$-fuzzy left ideal $\widetilde{\nu}$ of S.

Proof.(1)⇒(2): Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy right ideal, $\widetilde{\mu}$ an interval-valued $(\in,\in\vee q)$-fuzzy lateral ideal and $\widetilde{\nu}$ an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S and $a\in S$. Then

$$\begin{aligned}\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^-(a)&=\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)(a)\wedge[0.5,0.5]\\&=\left(\vee_{a=xyz}\left(\widetilde{\lambda}(x)\wedge\widetilde{\mu}(y)\wedge\widetilde{\nu}(z)\right)\right)\\&\quad\wedge[0.5,0.5]\\&\leq\vee_{a=xyz}\left(\widetilde{\lambda}(xyz)\wedge\widetilde{\mu}(xyz)\wedge\widetilde{\nu}(xyz)\right)\\&\quad\wedge[0.5,0.5]\\&=\left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\wedge\widetilde{\nu}(a)\right)\wedge[0.5,0.5]\\&=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)(a)\wedge[0.5,0.5]\\&=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)^-.\end{aligned}$$

Thus,

$$\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^-\leq\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)^-.$$

Since S is regular, so for any $a\in S$ there exists $x\in S$ such that $a=axa=a(xax)a$. Now

$$\begin{aligned}\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^-(a)&=\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)(a)\wedge[0.5,0.5]\\&=\left(\vee_{a=pqr}\left(\widetilde{\lambda}(p)\wedge\widetilde{\mu}(q)\wedge\widetilde{\nu}(r)\right)\right)\\&\quad\wedge[0.5,0.5]\\&\geq\left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(xax)\wedge\widetilde{\nu}(a)\right)\wedge[0.5,0.5]\\&\geq\left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\wedge\widetilde{\nu}(a)\right)\wedge[0.5,0.5]\\&=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)(a)\wedge[0.5,0.5]\\&=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)^-(a).\end{aligned}$$

Thus, $\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^-\geq\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)^-$. Hence, $\left(\widetilde{\lambda}\circ\widetilde{\mu}\circ\widetilde{\nu}\right)^-=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\wedge\widetilde{\nu}\right)^-$.

(2)⇒(1): Let R,M and L be the right, lateral and left ideals of S, respectively. Then, by Lemma 5.5, lower part of the interval-valued characteristic functions $\widetilde{C}_R^-$, $\widetilde{C}_M^-$, $\widetilde{C}_L^-$ are interval-valued $(\in,\in\vee q)$-fuzzy right ideal, interval-valued $(\in,\in\vee q)$-fuzzy lateral ideal and interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S, respectively. Thus, by hypothesis

$$\left(\widetilde{C}_R\wedge\widetilde{C}_M\wedge\widetilde{C}_L\right)^-=\left(\widetilde{C}_R\circ\widetilde{C}_M\circ\widetilde{C}_L\right)^-$$

$$\widetilde{C}_{R\cap M\cap L}^-=\widetilde{C}_{RML}^-.$$

Thus, $R\cap M\cap L=RML$. Hence by Theorem 2.4, S is regular.

5.10 Theorem

For a ternary semigroup S, the following assertions are equivalent:

(1) S is regular;

(2) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-} = \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy right ideal $\widetilde{\lambda}$ and every interval-valued $(\in,\in\vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.

Proof.(1)⇒(2): Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy right ideal and $\widetilde{\mu}$ an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S and $a\in S$. Then,

$$\begin{aligned}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}(a) &= \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)(a)\wedge[0.5,0.5]\\ &= \left(\vee_{a=xyz}\left(\widetilde{\lambda}(x)\wedge\widetilde{\mathscr{S}}(y)\wedge\widetilde{\mu}(z)\right)\right)\\ &\quad\wedge[0.5,0.5]\\ &= \vee_{a=xyz}\left(\widetilde{\lambda}(x)\wedge\widetilde{\mu}(z)\right)\wedge[0.5,0.5]\\ &\leq \vee_{a=xyz}\left(\widetilde{\lambda}(xyz)\wedge\widetilde{\mu}(xyz)\right)\wedge[0.5,0.5]\\ &= \left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\right)\wedge 0.5 = \left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}(a).\end{aligned}$$

Hence

$$\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}.$$

Since S is regular so for any $a\in S$ there exists $x\in S$ such that $a=axa=a(xax)a$. Then,

$$\begin{aligned}\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}(a) &= \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)(a)\wedge[0.5,0.5]\\ &= \left(\vee_{a=pqr}\left(\widetilde{\lambda}(p)\wedge\widetilde{\mathscr{S}}(q)\wedge\widetilde{\mu}(r)\right)\right)\\ &\quad\wedge[0.5,0.5]\\ &\geq \left(\widetilde{\lambda}(a)\wedge\widetilde{\mathscr{S}}(xax)\wedge\widetilde{\mu}(a)\right)\\ &\quad\wedge[0.5,0.5]\\ &= \left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\right)\wedge 0.5\\ &= \left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}(a).\end{aligned}$$

Thus,

$$\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}\geq\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}.$$

Therefore,

$$\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}.$$

(2)⇒(1): Let R and L be the right and left ideal of S, respectively. Then, by Lemma 5.5, lower part of the interval-valued characteristic functions $\widetilde{C}_R^{-}$ and $\widetilde{C}_L^{-}$ are interval-valued $(\in,\in\vee q)$-fuzzy right ideal and interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S, respectively. Thus, by hypothesis

$$\begin{aligned}\left(\widetilde{C}_R\wedge\widetilde{C}_L\right)^{-} &= \left(\widetilde{C}_R\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_L\right)^{-}\\ \widetilde{C}_{R\cap L}^{-} &= \left(\widetilde{C}_{RSL}\right)^{-}.\end{aligned}$$

Thus, $R\cap L=RSL$. Hence by Theorem 2.6, S is regular.

5.11 Theorem

For a ternary semigroup S, the following conditions are equivalent:

(1) S is regular;

(2) $\widetilde{\lambda}^{-} = \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal $\widetilde{\lambda}$ of S;

(3) $\widetilde{\lambda}^{-} = \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal $\widetilde{\lambda}$ of S;

(4) $\widetilde{\lambda}^{-} = \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal $\widetilde{\lambda}$ of S.

Proof.(1)⇒(2): Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S and $a\in S$. Since S is regular so there exists $x\in S$ such that $a=axa=axaxa$. Now,

$$\begin{aligned}&\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}(a)\\ &= \left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)\wedge[0.5,0.5]\\ &= \vee_{a=rst}\left\{\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(r)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\lambda}(t)\right\}\\ &\quad\wedge[0.5,0.5]\\ &\geq \left\{\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge\widetilde{\mathscr{S}}(x)\wedge\widetilde{\lambda}(a)\right\}\\ &\quad\wedge[0.5,0.5]\\ &= \left\{\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge\widetilde{\lambda}(a)\right\}\wedge[0.5,0.5]\\ &= \left\{\vee_{a=rst}\left(\widetilde{\lambda}(r)\wedge\widetilde{\mathscr{S}}(s)\wedge\widetilde{\lambda}(t)\right)\wedge\widetilde{\lambda}(t)\right\}\\ &\quad\wedge[0.5,0.5]\end{aligned}$$

$$\begin{aligned}&\geq \widetilde{\lambda}(a)\wedge\widetilde{\mathscr{S}}(x)\wedge\widetilde{\lambda}(a)\wedge[0.5,0.5]\\ &= \widetilde{\lambda}(a)\wedge[0.5,0.5]=\widetilde{\lambda}^{-}(a).\end{aligned}$$

Thus, $\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}\geq\widetilde{\lambda}^{-}$.

Since $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S, so

$$\begin{aligned}
&\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}(a)\\
&=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge[0.5,0.5]\\
&=\left(\vee_{a=xuyvz}\left\{\widetilde{\lambda}(x)\wedge\widetilde{\mathscr{S}}(u)\wedge\widetilde{\lambda}(y)\wedge\widetilde{\mathscr{S}}(v)\wedge\widetilde{\lambda}(z)\right\}\right)\\
&\quad\wedge[0.5,0.5]\\
&=\left(\vee_{a=xuyvz}\left\{\widetilde{\lambda}(x)\wedge\widetilde{\lambda}(y)\wedge\widetilde{\lambda}(z)\right\}\right)\wedge[0.5,0.5]\\
&\leq\vee_{a=xuyvz}\widetilde{\lambda}(xuyvz)\wedge[0.5,0.5]\\
&=\widetilde{\lambda}(a)\wedge[0.5,0.5]=\widetilde{\lambda}^{-}(a).
\end{aligned}$$

Thus, $\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}\leq\widetilde{\lambda}^{-}$. Hence $\widetilde{\lambda}^{-}=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$.

(2)⇒(3)⇒(4): are obvious.

(4)⇒(1): Let A be a quasi-ideal of S. Then by Lemma 4.13, $\widetilde{C_A}$ is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S. Thus by hypothesis

$$\begin{aligned}
\widetilde{C}_A^{-}&=\left(\widetilde{C}_A\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_A\circ\widetilde{\mathscr{S}}\circ\widetilde{C}_A\right)^{-}\\
&=\left(\widetilde{C}_A\circ\widetilde{C}_S\circ\widetilde{C}_A\circ\widetilde{C}_S\circ\widetilde{C}_A\right)^{-}\\
&=\widetilde{C}_{ASASA}^{-}.
\end{aligned}$$

This implies that $A=ASASA$. Hence it follows by Theorem 2.5, S is regular.

By using similar arguments as in the proof of the Theorem 5.11, we can prove the following theorem.

5.12 Theorem

For a ternary semigroup S, the following conditions are equivalent:

(1) S is regular;

(2) $\widetilde{\lambda}^{-}=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal $\widetilde{\lambda}$ of S;

(3) $\widetilde{\lambda}^{-}=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal $\widetilde{\lambda}$ of S.

5.13 Theorem

For a ternary semigroup S, the following statements are equivalent:

(1) S is regular;

(2) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in,\in\vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S;

(3) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in,\in\vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S;

(4) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in,\in\vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.

Proof.(1)⇒(4): Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal and $\widetilde{\mu}$ an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S. Since S is regular, so for every $a\in S$ there exists $x\in S$ such that $a=axa$. Thus,

$$\begin{aligned}
\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}(a)&=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)(a)\wedge[0.5,0.5]\\
&=\left(\vee_{a=pqr}\left\{\widetilde{\lambda}(p)\wedge\widetilde{\mathscr{S}}(q)\wedge\widetilde{\mu}(r)\right\}\right)\\
&\quad\wedge[0.5,0.5]\\
&\geq\left(\widetilde{\lambda}(a)\wedge\widetilde{\mathscr{S}}(x)\wedge\widetilde{\mu}(a)\right)\wedge[0.5,0.5]\\
&=\left(\widetilde{\lambda}(a)\wedge[1,1]\wedge\widetilde{\mu}(a)\right)\wedge[0.5,0.5]\\
&=\left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\right)\wedge[0.5,0.5]\\
&=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)(a)\wedge[0.5,0.5]\\
&=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}(a).
\end{aligned}$$

So, $\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}\geq\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}$.

(4)⇒(3)⇒(2): are obvious.

(2)⇒(1): Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy right ideal and $\widetilde{\mu}$ an interval-valued $(\in,\in\vee q)$-fuzzy left ideal of S. Since every interval-valued $(\in,\in\vee q)$-fuzzy right ideal is an interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal of S. So, $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}$.

Let $a\in S$ and consider,

$$\begin{aligned}
\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}(a)&=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)(a)\wedge[0.5,0.5]\\
&=\left(\vee_{a=xyz}\left(\widetilde{\lambda}(x)\wedge\widetilde{\mathscr{S}}(y)\wedge\widetilde{\mu}(z)\right)\right)\\
&\quad\wedge[0.5,0.5]\\
&=\left(\vee_{a=xyz}\left(\widetilde{\lambda}(x)\wedge\widetilde{\mu}(z)\right)\right)\wedge[0.5,0.5]\\
&\leq\left(\vee_{a=xyz}\left(\widetilde{\lambda}(xyz)\wedge\widetilde{\mu}(xyz)\right)\right)\wedge[0.5,0.5]\\
&=\left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\right)\wedge[0.5,0.5]\\
&=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)(a)\wedge[0.5,0.5]=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}(a).
\end{aligned}$$

So $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\geq\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}$. Thus, $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}=\left(\widetilde{\lambda}\circ\widetilde{\mathscr{S}}\circ\widetilde{\mu}\right)^{-}$. Therefore, by Theorem 5.10, S is regular.

5.14 Theorem

For a ternary semigroup S, the following statements are equivalent:

(1) S is regular;

(2) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy quasi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in,\in\vee q)$-fuzzy right ideal $\widetilde{\mu}$ of S;

(3) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}$ for every interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in,\in\vee q)$-fuzzy right ideal $\widetilde{\mu}$ of S.

Proof.(1)⇒(3): Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal and $\widetilde{\mu}$ an interval-valued $(\in,\in\vee q)$-fuzzy right ideal of S. Since S is regular, so for any element $a\in S$ there exists $x\in S$ such that $a=axa$. Now,

$$\begin{aligned}\left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}(a) &= \left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge[0.5,0.5]\\ &= \left(\vee_{a=xpqr}\left(\widetilde{\mu}(p)\wedge\widetilde{\mathscr{S}}(q)\wedge\widetilde{\lambda}(r)\right)\right)\\ &\quad\wedge[0.5,0.5]\\ &\geq \left(\widetilde{\mu}(a)\wedge\widetilde{\mathscr{S}}(x)\wedge\widetilde{\lambda}(a)\right)\wedge[0.5,0.5]\\ &= \left(\widetilde{\mu}(a)\wedge\widetilde{\lambda}(a)\right)\wedge[0.5,0.5]\\ &= \left(\widetilde{\mu}\wedge\widetilde{\lambda}\right)(a)\wedge[0.5,0.5]=\left(\widetilde{\mu}\wedge\widetilde{\lambda}\right)^{-}(a).\end{aligned}$$

So

$$\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}\leq\left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}.$$

(3)⇒(2): Obvious.

(2)⇒(1): Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy left ideal and $\widetilde{\mu}$ an interval-valued $(\in,\in\vee q)$-fuzzy right ideal of S and $a\in S$. Now

$$\begin{aligned}\left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}(a) &= \left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)(a)\wedge[0.5,0.5]\\ &= \left(\vee_{a=xyz}\left(\widetilde{\mu}(x)\wedge\widetilde{\mathscr{S}}(y)\wedge\widetilde{\lambda}(z)\right)\right)\\ &\quad\wedge[0.5,0.5]\\ &= \left(\vee_{a=xyz}\left(\widetilde{\mu}(x)\wedge\widetilde{\lambda}(z)\right)\right)\wedge[0.5,0.5]\\ &\leq \left(\vee_{a=xyz}\left(\widetilde{\mu}(xyz)\wedge\widetilde{\lambda}(xyz)\right)\right)\wedge[0.5,0.5]\\ &= \left(\widetilde{\mu}(a)\wedge\widetilde{\lambda}(a)\right)\wedge[0.5,0.5]=\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}(a).\end{aligned}$$

Thus,

$$\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)^{-}=\left(\widetilde{\mu}\circ\widetilde{\mathscr{S}}\circ\widetilde{\lambda}\right)^{-}.$$

Hence by Theorem 5.10, S is regular.

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