A Matrix Inequality Concerning Weakly Connected and Balanced Digraphs

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Abstract: Based on spectral properties of Laplacian matrix, we present a new matrix inequality concerning weakly connected and balanced digraphs.

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1 Introduction

Let $G = (V(G), E(G), A(G))$ denote a weighted digraph (directed graph) of order $n$ with the set of vertices $V(G) = \{1, 2, \cdots, n\}$, edges $E(G) \subseteq V(G) \times V(G)$, and the $n \times n$ weighted adjacency matrix $A(G) = (a_{ij})$. A directed edge from $j$ to $i$ exists if and only if $a_{ij} > 0$. We assume that $a_{ii} = 0$ for all $i \in V(G)$. The graph Laplacian (or Laplacian matrix) $L(G) = (l_{ij})$ induced by the digraph $G$ is defined by (see e.g. [1])

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{k=1}^{n} a_{ik}, & i = j. \end{cases}$$

(1)

A digraph $G$ is called balanced [2] if $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ for all $i \in V(G)$. In other words, a digraph is balanced if and only if the total weight of edges entering a vertex and leaving the same vertex are equal for all vertices. By definition, any undirected graph is balanced. An important property of balanced digraphs is that $I = (1, \cdots, 1)^{T} \in \mathbb{R}^{n}$ is a left eigenvector of the Laplacian, i.e., $1^{T} L(G) = 0$.

Recall that a digraph is strongly connected if, between every pair of distinct vertices, there is a directed path. On the other hand, a digraph is called weakly connected if it is connected when viewed as a graph (replacing directed edges by undirected ones). An interesting result is that a balanced digraph is weakly connected if and only if it is strongly connected [3]. Moreover, weakly connected and balanced digraphs play an important role in the consensus coordination of multi-agent systems. It is shown that ([2] or [4, Theorem 3.17,]) the agreement protocol over a digraph reaches the average consensus for every initial condition if and only if it is weakly connected and balanced.

The goal of this paper is to present a matrix inequality concerning weakly connected and balanced digraphs by using spectral properties of Laplacian matrix. It is hoped that the result may find potential applications in multi-agent coordination (see the concluding remarks in Section 2).

2 The matrix inequality

We begin this section with some notations and definitions. A nonnegative matrix $A = (a_{ij})$ with all entries on the main diagonal equal to zero can be associated naturally with a digraph $G = (V, E, A)$ in such a way that $(j, i) \in E$ if and only if $a_{ij} > 0$. Consider two symmetric matrices $X$ and $Y$ of the same dimension, we say $X \succ Y$ if and only if $X - Y$ is positive definite for $X \in \mathbb{R}^{m \times m}$, $X$ can be viewed as a linear map $X : \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ with kernel defined by $\text{Ker}X = \{x \in \mathbb{R}^{m} : Xx = 0\}$.

For an undirected graph $G$, $L(G)$ is a symmetric matrix with real eigenvalues and, hence, the set of eigenvalues of $L(G)$ can be ordered sequentially in an ascending order as

$$0 = \lambda_1(L(G)) \leq \lambda_2(L(G)) \leq \cdots \leq \lambda_n(L(G)).$$

(2)

$G$ is connected if and only if $\lambda_2(L(G)) > 0$ [1]. For a digraph $G$, the following lemma is shown in [2].
Lemma 2.1. ([2]) Assume that $G$ is a strongly connected digraph. Then all eigenvalues but one simple eigenvalue at zero of $L(G)$ have positive real-parts.

Theorem 2.1. Assume that $G_1$ and $G_2$ are two digraphs of order $n$. If the digraph associated with $A(G_1) - A(G_2)$ is weakly connected and balanced, for any matrix $F \in \mathbb{R}^{n \times m}$ satisfying $\text{Ker}F = 0$ and $1^T F = 0$,

$$F^T (L(G_1) + L(G_1)^T)F > F^T (L(G_2) + L(G_2)^T)F. \quad (3)$$

Proof. Let $G$ be the digraph associated with $A(G_1) - A(G_2)$. Thus, $G$ is weakly connected and balanced, and $L(G) = L(G_1) - L(G_2)$. It suffices to show that

$$F^T (L(G) + L(G)^T)F > 0. \quad (4)$$

According to the aforementioned comment, we obtain $1^T L(G) = 0$. Since $(L(G)1 = 0$, it follows that $1^T (L(G) + L(G)^T) = (L(G) + L(G)^T)1 = 0$. Hence, the digraph $\hat{G}$ with the Laplacian matrix $L(G) + L(G)^T$ is also balanced. On the other hand, it is clear that $\hat{G}$ is weakly connected (and automatically strongly connected, by our above comment).

Lemma 2.1 then implies that $\lambda_2(L(G) + L(G)^T) > 0$, where

$$0 = \lambda_1(L(G) + L(G)^T) < \lambda_2(L(G) + L(G)^T) \leq \cdots \leq \lambda_n(L(G) + L(G)^T) \quad (5)$$

are the eigenvalues of $L(G) + L(G)^T$. By the Courant-Fischer theorem [1], we obtain

$$x^T (L(G) + L(G)^T)x \geq \lambda_2(L(G) + L(G)^T)x^T x, \quad (6)$$

for $x \in \mathbb{R}^n$ satisfying $1^T x = 0$. For any $y \in \mathbb{R}^m$ and $y \neq 0$, we know that $1^T (Fy) = 0$ by the assumption $1^T F = 0$. Therefore, we obtain

$$y^T F^T (L(G) + L(G)^T)Fy$$

$$= (Fy)^T (L(G) + L(G)^T)(Fy)$$

$$\geq \lambda_2(L(G) + L(G)^T)(Fy)^T (Fy) > 0, \quad (7)$$

where the second inequality follows from (6), and the last one follows from (5) and the assumption $\text{Ker}F = 0$. This implies (4), and the proof of Theorem 2.1 is complete. □

We give some remarks here.

Remark 2.1. If we take $G_2$ as an empty graph, i.e., $A(G_2) = 0$, we have the following corollary: Assume that $G_1$ of order $n$ is weakly connected and balanced, then we have

$$F^T (L(G_1) + L(G_1)^T)F > 0 \quad (8)$$

for any matrix $F \in \mathbb{R}^{n \times m}$ satisfying $\text{Ker}F = 0$ and $1^T F = 0$.

Remark 2.2. The digraph $\hat{G}$ with the Laplacian $L(G) + L(G)^T$ is essentially undirected with the new weights given by $\hat{a}_{ij} = \hat{a}_{ji} = a_{ij} + a_{ji}$. $\hat{G}$ is also known as disoriented digraph [4], which often appears in multi-agent coordination (see e.g. [5, 6, 7]).

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References


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