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A Study of a Fractional-Order Cholera Model

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Abstract: In this work, we investigate the dynamical behavior of a fractional order cholera model. All the feasible equilibria for the system are obtained and the conditions for the existence of interior equilibrium are determined. Local stability analysis of the cholera model is studied by using the fractional Routh-Hurwitz stability conditions. Our results indicate the potential of fractional-order cholera models to cope with modern epidemics.

Keywords: Cholera model, stability, numerical simulation, fractional order, dynamical system

1 Introduction

Cholera is an acute intestinal infection caused by ingestion of food or water contaminated with the bacterium Vibrio cholerae. It has a short incubation period, from less than one day to five days, and produces an enterotoxin that causes a copious, painless, watery diarrhoea that can quickly lead to severe dehydration and death if treatment is not promptly given. Vomiting also occurs in most patients. Cholera is an ancient disease that continues to cause epidemic and pandemic infection despite ongoing efforts to limit its spread ([1]–[7]). Historically, six out of the seven cholera pandemics have swept the globe since 1816 [8,9]. Most recently, the seventh pandemic started from Indonesia in 1961, spread into Europe, South Pacific and Japan in the late 1970s, reached South America in 1990s, and has continued (though much diminished) to the present. The last few years have witnessed many cholera outbreaks in developing countries, including Liberia (2002), Mali (2003), Senegal and Chad (2004), West Africa (2005), Angola and Sudan (2006), India (2007), Iraq and Congo (2008), Zimbabwe (2008–2009), Vietnam (2009), Nigeria, Central Africa, Pakistan and Haiti (2010), Sierra Leone (2012). Every year there are an estimated 3 to 5 million cholera cases and 100 000 to 120 000 deaths due to cholera [9,10]. Particularly, cholera represents a significant public health burden to developing countries and cholera continues receiving worldwide attention. Cholera is an extremely virulent disease. It affects both children and adults and can kill within hours. About 75% of people infected with Vibrio cholerae do not develop any symptoms, although the bacteria are present in their faeces for 7-14 days after infection and are shed back into the environment, potentially infecting other people.

Many mathematical models have been proposed to investigate the complex epidemic and endemic behavior of cholera. The earliest mathematical model was proposed by Capasso and Paveri-Fontana [11] to study a cholera epidemic occurred in the Mediterranean in 1973. Codeco [12] in 2001 extended the work in [11] and explicitly accounted for the role of the aquatic reservoir in cholera dynamics. Using similar non-linear incidence in Codeos model, Hartley et al. [13] incorporated a hyper-infective stage of V. cholerae in 2006. This model emphasizes the stage of explosive infectivity of V. cholerae, based on the laboratory measurements that freshly shed V. cholerae from human intestines outcompeted other V. cholerae by as much as 700-fold for the first few hours in the environment [1,5]. Recently, Mukandavire et al. [14] proposed a model to estimate the reproduction number for the 2008-2009 cholera outbreak in Zimbabwe. Their model includes both environment-to-human and human-to-human transmission pathways. In 2010, Tien and Earn [15] published a water-borne disease model which also included the dual transmission pathways, with bilinear incidence rates employed for both the environment-to-human and human-to-human infection routes. Jensen et al. [16] proposed a mathematical model

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to study how lytic bacteriophage specific for V. cholerae affects cholera outbreaks. In [17], the authors proposed a new and unified deterministic model that incorporates a general incidence rate and a general formulation of the pathogen concentration to analyse the dynamics of cholera. This work unifies many existing cholera models proposed by different authors. In [18], the authors proposed global stability analysis for several deterministic cholera epidemic models. These models, incorporating both human population and pathogen V. cholerae concentration, constitute four-dimensional non-linear autonomous systems where the classical Poincar-Bendixson theory is not applicable.

Fractional-order differentiation is regarded as the generalization of classical integer-order differentiation to real or complex orders. There has been much interest in developing the theoretical analysis and numerical methods for fractional differential equations as fractional calculus is found to be a valuable tool in various fields of science and engineering. Indeed, we can find numerous applications in polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, etc. ([19]–[21]). For some recent work on fractional differential equations and inclusions, see ([22]–[34]) and the references therein.

Recently, several investigators have studied the qualitative properties and numerical solutions of fractional-order biological models, for instance, see [35]. It has been mainly due to the reason that fractional-order equations are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. Yan and Kou [36] investigated stability properties of fractional-order differential equations and applied their results to analyze the stability of the equilibria for the model of HIV-1 infection.

In [37], the existence and uniqueness of solutions, stability of equilibria and numerical solutions for the fractional-order predator-prey model and rabies model were investigated. In [38], stability properties for a fractional-order model of nonlocal epidemics were studied and the results were found to be relevant to foot-and-mouth disease, SARS and avian flu. In addition, Ding and Ye have also introduced some kinds of models for HIV infection and discussed the stability of equilibria for the corresponding systems [39, 40].

The objective of this paper is to investigate a fractional-order cholera model by means of an efficient numerical method, based on an idea of transforming the given model to a system of ordinary differential equations of integer order. All the feasible equilibria for the system are discussed. The conditions ensuring the existence of interior equilibrium are also given. Local stability analysis of the cholera model is carried out by applying the fractional Routh–Hurwitz criterion. Our results show the worth of fractional-order cholera models to represent modern epidemics.

The paper is organized as follows. First of all, we describe our model. Section 3 contains some preliminary concepts. The stability for equilibria of the system is discussed in Section 4. In Section 5, the given model is studied numerically and the graphical results are presented in Section 6.

2 The Model

We consider the fractional-order Codeco model involving Caputo derivative given by [12]:

$$\begin{cases} \frac{d^\alpha S}{dt^\alpha} = n(H-S) - a\frac{SB}{K+B}, \\ \frac{d^\alpha I}{dt^\alpha} = a\frac{SB}{K+B} - rI, \\ \frac{d^\alpha B}{dt^\alpha} = B(nb-mb) + eI, \quad nb < mb, \end{cases} \tag{1}$$

$$S(\delta) = S_0, \quad I(\delta) = I_0 > 0, \quad B(\delta) = B_0 > 0,$$

where the symbols appearing in this model are listed in Table 1. The first equation of (1) describes the dynamics of susceptibles in a community of constant size H. Susceptible individuals are renewed at a rate n. Renewal may occur as result of birth, immigration and/or loss of acquired immunity (cholera apparently does not confer life-long immunity). Susceptible people becomes infected at a rate $a\frac{B}{K+B}$, where a is the rate of contact with untreated water and $\frac{B}{K+B}$ is the probability of a person to catch cholera. Probability of catching cholera depends on the concentration of *V.* cholerae in the consumed water.

Table 1. Symbols used in the model

Symbol	*Description*
State Variables	
S	number of susceptibles
I	number of infected
B	concentration of toxigenic V. cholerae in water (cells/ml)
Parameters	
H	total human population
n	Human birth and death rates (day-1)
a	rate of exposure to contaminated water (day-1)
K	concentration of V. cholerae in water that yields 50% chance of catching cholera (cells/ml)
r	rate at which people recover from cholera (day-1)
nb	growth rate of V. cholerae in the aquatic environment (day-1)
mb	loss rate of V. cholerae in the aquatic environment (day-1)
e	contribution of each infected person to the population of V. cholerae in the aquatic environment (cell/ml day-1 person-1)

3 Preliminaries

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

The Riemann-Liouville and Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ for a given continuous function f are defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds,$$

$$D_{t_0}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds.$$

In case of Caputo derivative, the function $f \in AC^{n-1}$. The initial value problem related to the above defnition is

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), \\ x(t)|_{t=0^+} = x_0, \end{cases} \tag{2}$$

where $0 < \alpha < 1$ and $D^\alpha = D_0^\alpha$.

Now, we recall some stability theorems on fractional-order systems.

Theorem 1 ([41]). The following autonomous system:

$$\frac{d^\alpha x}{dt^\alpha} = Ax, \quad x(0) = x_0, \tag{3}$$

with $0 < \alpha \leq 1$, $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues of matrix A. Also, this system is stable if and only if $|\arg(\lambda)| \geq \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues of matrix A with those critical eigenvalues satisfying $|\arg(\lambda)| = \frac{\alpha\pi}{2}$ and having geometric multiplicity of one. The geometric multiplicity of an eigenvalue λ of the matrix A is the dimension of the subspace of vectors v for which $Av = \lambda v$.

Theorem 2 ([42]). Consider the following commensurate fractional-order system:

$$\frac{d^\alpha x}{dt^\alpha} = f(x), \quad x(0) = x_0, \tag{4}$$

with $0 < \alpha \leq 1$ and $x \in \mathbb{R}^n$. The equilibrium points of system (4) are calculated by solving the equation: $f(x) = 0$. These points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy: $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

4 Stability of equilibrium

In this section, we analyze model (1) by finding its equilibria and studying their stability. Steady states of the model satisfy the following equations:

$$\frac{d^\alpha S}{dt^\alpha} = 0, \frac{d^\alpha I}{dt^\alpha} = 0, \frac{d^\alpha B}{dt^\alpha} = 0. \tag{5}$$

(5) has a trivial equilibrium $E_0 = (H, 0, 0)$. Let $R_0 = \frac{aeH}{rK(mb-nb)}$ be the basic reproduction number. If $R_0 > 1$ then (5) has a nontrivial equilibrium $E_1 = (\overline{S}, \overline{I}, \overline{B})$, where

$$\overline{S} = \frac{rK(mb-nb)+neH}{e(n+a)} = \frac{H(a+R_0)}{R_0(a+n)},$$

$$\overline{B} = \frac{n(aeH - rK(mb-nb)}{er(n+a)} = \frac{nK(mb-nb)(R_0-1)}{e(n+a)},$$

$$\overline{I} = \frac{(mb-nb)\overline{B}}{e} = \frac{nK(mb-nb)^2(R_0-1)}{e^2(n+a)}.$$

To investigate the local behavior of system (1) about each of the equilibrium points, the Jacobian matrix J of the equilibrium point $E = (S, I, B)$ is computed as

$$J(E) = \begin{pmatrix} -n - \frac{aB}{K+B} & 0 & -\frac{aSK}{(K+B)^2} \\ \frac{aB}{K+B} & -r & \frac{aSK}{(K+B)^2} \\ 0 & e & -(\text{nb-mb}) \end{pmatrix}$$

Now we consider the asymptotically stability of system (1) at the equilibrium point E_0. The equilibrium point E_0 is asymptotically stable if $R_0 < 1$.

The Jacobian matrix of (1) at equilibrium point E_0 is

$$J(E_0) = \begin{pmatrix} -n & 0 & -\frac{aH}{K} \\ 0 & -r & \frac{aH}{K} \\ 0 & e & (\text{nb-mb}) \end{pmatrix}$$

with the characteristic equation

$$P(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{6}$$

where

$$b_1 = mb - nb + n + r,$$

$$b_2 = (mb-nb)(n+r) + nr - \frac{eaH}{K}$$

$$= n(mb-nb+r) + r(mb-nb)(1-R_0),$$

$$b_3 = -\frac{eaH}{K} + nr(mb-nb)$$

$$= rn(mb-nb)(1-R_0),$$

$$b_1b_2 - b_3 = n(mb-nb+n+r)(mb-nb+r)$$

$$+ (mb-nb+r)r(mb-nb)(1-R_0).$$

Therefore, the eigenvalues corresponding to the equilibrium E_0 are

$$\lambda_1 = -n,$$

$$\lambda_{2,3} = \frac{-(mb-nb+r) \pm \sqrt{(mb-nb+r)^2 + 4\frac{eaH}{K}}}{2}. \tag{7}$$

Clearly, if $R_0 < 1$, then $\lambda_1, \lambda_2, \lambda_3 < 0$.
Let $D(P)$ denote the discriminant of a polynomial $P(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3$. Then

$$D(P) = 18b_1b_2b_3 + (b_1b_2)^2 - 4b_3b_1^3 - 4b_2^3 - 27b_3^3.$$

The equilibrium point E_1 is asymptotically stable if one of the following conditions holds for polynomial P and $D(P)$:
$(i) D(P) > 0, b_1 > 0, b_3 > 0$ and $b_1b_2 > b_3$.
$(ii) D(P) < 0, b_1 \geq 0, b_2 \geq 0, b_3 > 0$ and $\alpha < \frac{2}{3}$.
The Jacobian matrix of (1) at equilibrium point E_1 is

$$J(E_1) = \begin{pmatrix} -n - \frac{a\overline{B}}{K+\overline{B}} & 0 & -\frac{r(\text{mb-nb})}{e}\frac{K}{(K+\overline{B})} \\ \frac{a\overline{B}}{K+\overline{B}} & -r & \frac{r(\text{mb-nb})}{e}\frac{K}{(K+\overline{B})} \\ 0 & e & -(\text{nb-mb}) \end{pmatrix}$$

with the characteristic equation

$$P(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{8}$$

where

$$b_1 = mb - nb + n + r + \frac{a\overline{B}}{K+\overline{B}},$$

$$b_2 = r(n + \frac{a\overline{B}}{K+\overline{B}}) + (mb-nb)(r+n+\frac{a\overline{B}}{K+\overline{B}}) - \frac{rK(mb-nb)}{K+\overline{B}},$$

$$b_3 = (r(mb-nb) - \frac{r(mb-nb)K}{K+\overline{B}})(n + \frac{a\overline{B}}{K+\overline{B}}) + \frac{r(mb-nb)Ka\overline{B}}{K+\overline{B}}.$$

Observe that $b_1 > 0$. Manipulating b_2 and b_3, we have

$$b_2 = n(mb-nb) + \frac{rK(mb-nb) + [r(n+mb-nb) + a(mb-nb+r)]\overline{B}}{K+\overline{B}},$$

$$b_3 = \frac{C_0 + C_1\overline{B} + C_2\overline{B}^2}{(K+\overline{B})^2}.$$

where

$$C_0 = rn^2K^2 + r^2nK^2 + K(mb-nb)r^2n,$$

$$C_1 = rnKa + nKr^2 + rn^2K + Kr(n+r+mb-nb)(n+mb-nb)$$

$$+ Ka(mb-nb+r)(n+r+mb-nb) + r^2n(mb-nb),$$

$$C_2 = (r+mb-nb)[r(n+mb-nb) + a(mb-nb+r)] + (a+n)[nr + a(mb-nb+r)].$$

It is clear that $C_0, C_1, C_2 > 0$. Therefore $b_2, b_3 > 0$.

5 Numerical method

Here, we shall use a numerical method introduced by Atanackovic and Stankovic [43,44] to solve the fractional-order nonlinear system (1). In [43], it was shown that the fractional derivative of order α $(0 < \alpha \leq 1)$ for a function $f(t)$ may be expressed as

$$D^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)}\{\frac{f^{(1)}(t)}{t^{\alpha-1}}[1 + \sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}]$$

$$-[\frac{\alpha-1}{t^\alpha}f(t) + \sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}(\frac{f(t)}{t^\alpha} + \frac{V_p(f)(t)}{t^{p-1+\alpha}})]\}, \tag{9}$$

where

$$V_p(f)(t) = -(p-1)\int_0^t \tau^{p-2}f(\tau)d\tau,\ p = 2,3,\cdots, \tag{10}$$

$$\frac{d}{dt}V_p(f) = -(p-1)t^{p-2}f(t),\ p = 2,3,\cdots. \tag{11}$$

We approximate $D^\alpha f(t)$ by using M terms in sums appearing in (9) as follows

$$D^\alpha f(t) \simeq \frac{1}{\Gamma(2-\alpha)}\{\frac{f^{(1)}(t)}{t^{\alpha-1}}[1 + \sum_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}]$$

$$-[\frac{\alpha-1}{t^\alpha}f(t) + \sum_{p=2}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}(\frac{f(t)}{t^\alpha} + \frac{V_p(f)(t)}{t^{p-1+\alpha}})]\}. \tag{12}$$

We can rewrite (12) as

$$D^\alpha f(t) \simeq \Omega(\alpha,t,M)f^{(1)}(t) + \Phi(\alpha,t,M)f(t) + \sum_{p=2}^{M}A(\alpha,t,p)\frac{V_p(f)(t)}{t^{p-1+\alpha}}, \tag{13}$$

where

$$\Omega(\alpha,t,M) = \frac{1 + \sum_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}}{\Gamma(2-\alpha)t^{\alpha-1}},$$

$$R(\alpha,t) = \frac{1-\alpha}{t^\alpha\Gamma(2-\alpha)}, A(\alpha,t,p) = -\frac{\Gamma(p-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)p!},$$

$$\Phi(\alpha,t,M) = R(\alpha,t) + \sum_{p=2}^{M}\frac{A(\alpha,t,p)}{t^\alpha}.$$

We set

$$\Theta_1(t) = S(t), \Theta_{M+1}(t) = I(t),$$

$$\Theta_{2M+1}(t) = B(t), \Theta_p(t) = V_p(S)(t),$$

$$\Theta_{p+M}(t) = V_p(I)(t), \Theta_{2M+p}(t) = V_p(B)(t),$$

for $p = 2,3,\cdots$.
We can rewrite system (1) in the following form

$$\begin{aligned}
&\Omega(\alpha,t,M)\Theta_1'(t) + \Phi(\alpha,t,M)\Theta_1(t) + \sum_{p=2}^{M}A(\alpha,t,p)\frac{\Theta_p(t)}{t^{p-1+\alpha}} \\
&= n(H - \Theta_1(t)) - a\frac{\Theta_1(t)\Theta_{2M+1}(t)}{K+\Theta_{2M+1}(t)}, \\
&\Omega(\alpha,t,M)\Theta_{M+1}'(t) + \Phi(\alpha,t,M)\Theta_{M+1}(t) + \sum_{p=2}^{M}A(\alpha,t,p)\frac{\Theta_{M+p}(t)}{t^{p-1+\alpha}} \\
&= a\frac{\Theta_1(t)\Theta_{2M+1}(t)}{K+\Theta_{2M+1}(t)} - r\Theta_{M+1}(t), \\
&\Omega(\alpha,t,M)\Theta_{2M+1}'(t) + \Phi(\alpha,t,M)\Theta_{2M+1}(t) + \sum_{p=2}^{M}A(\alpha,t,p)\frac{\Theta_{2M+p}(t)}{t^{p-1+\alpha}} \\
&= (nb-mb)\Theta_{2M+1} + e\Theta_{M+1}(t),
\end{aligned} \tag{14}$$

where

$$\Theta_p(t) = -(p-1)\int_0^t \tau^{p-2}\Theta_1(\tau)d\tau,$$

$$\Theta_{M+p}(t) = -(p-1)\int_0^t \tau^{p-2}\Theta_{M+1}(\tau)d\tau, \quad (15)$$

$$\Theta_{2M+p}(t) = -(p-1)\int_0^t \tau^{p-2}\Theta_{2M+1}(\tau)d\tau,$$

$p = 2,3,\cdots,M.$

Finally (14) and (15) can be rewritten as

$$\Theta_1'(t) = \frac{1}{\Omega(\alpha,t,M)}(n(H-\Theta_1(t)) - a\frac{\Theta_1(t)\Theta_{2M+1}(t)}{K+\Theta_{2M+1}(t)}) - \Phi(\alpha,t,M)\Theta_1(t) - \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_p(t)}{t^{p-1+\alpha}}),$$

$$\Theta_p'(t) = -(p-1)t^{p-2}\Theta_1(t),\ p = 2,3,\cdots,M,$$

$$\Theta_{M+1}'(t) = \frac{1}{\Omega(\alpha,t,M)}(a\frac{\Theta_1(t)\Theta_{2M+1}(t)}{K+\Theta_{2M+1}(t)} - r\Theta_{M+1}(t)) - \Phi(\alpha,t,M)\Theta_{M+1}(t) - \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_{M+p}(t)}{t^{p-1+\alpha}}), \quad (16)$$

$$\Theta_{M+p}'(t) = -(p-1)t^{p-2}\Theta_{M+1}(t),\ p = 2,3,\cdots,M,$$

$$\Theta_{2M+1}'(t) = \frac{1}{\Omega(\alpha,t,M)}((nb-mb)\Theta_{2M+1} + e\Theta_{M+1}(t)) - \Phi(\alpha,t,M)\Theta_{2M+1}(t) - \sum_{p=2}^{M} A(\alpha,t,p)\frac{\Theta_{2M+p}(t)}{t^{p-1+\alpha}}),$$

$$\Theta_{2M+p}'(t) = -(p-1)t^{p-2}\Theta_{2M+1}(t),\ p = 2,3,\cdots,M,$$

with the following initial conditions

$$\Theta_1(\delta) = S_0,$$

$$\Theta_p(\delta) = -\frac{p-1}{2}\Delta t^{p-1}S_0,\ p = 2,3,\cdots,M,$$

$$\Theta_{M+1}(\delta) = I_0,$$

$$\Theta_{M+p}(\delta) = -\frac{p-1}{2}\Delta t^{p-1}I_0,\ p = 2,3,\cdots,M, \quad (17)$$

$$\Theta_{2M+1}(\delta) = B_0,$$

$$\Theta_{2M+p}(\delta) = \frac{p-1}{2}\Delta t^{p-1}B_0,\ p = 2,3,\cdots,M.$$

In the next section, we solve the system (16) with the initial conditions (17) by using the well known Runge–Kutta method of order fourth.

6 Numerical Simulation and discussion

To facilitate the interpretation of our mathematical results developed for the model (1) so far, we proceed to investigate it by numerical simulations. We solve the system (1) numerically by using the method proposed in the previous section. In all numerical runs, the solution has been approximated at $\delta = \Delta t = 0.01,\ M = 5,\ \eta = mb - nb$. We illustrate our numerical results by considering a variety of examples.

Example 1.
We consider the following set of parameters:
$H = 1000, N = 0.001, a = 0.1, K = 1000, r = 0.4, \eta = 0.4, e = 1.$
The stability of equilibria E_0 can be seen in Figs. 1-2, with the initial conditions $[S_0, I_0, B_0] = [986,10,4],[974,20,6],[965,30,5],[932,60,8]$ and simulation time $T = 1200$ for $\alpha = 0.95$ and $\alpha = 0.99$ respectively. It is easy to compute that $R_0 = 0.6250 < 1$.

Example 2.
Let us choose a set of parameters:
$H = 100, N = 0.001, a = 0.5, K = 100, r = 0.4, \eta = 0.02, e = 1,$
and the initial conditions $[S_0, I_0, B_0] == [81,15,4],[69,25,6],[60,35,5],[32,60,8]$ with simulation time $T = 1800$ for $\alpha = 0.6 < \frac{2}{3}$ and $\alpha = 0.5 < \frac{2}{3}$. The stability of equilibria $E_1 = (1.7964, 0.2455, 12.2754)$ can be observed in Figs. 3–4. One can easily find that $R_0 = 62.5 > 1, b_1 = 0.4757 > 0, b_2 = 0.0243 > 0, b_3 = 4.3821e-004 > 0, b_1b_2 - b_3 = 0.0111 > 0, D(P) = -2.1616e-005.$

Example 3.
In this case, we consider the following set of parameters:

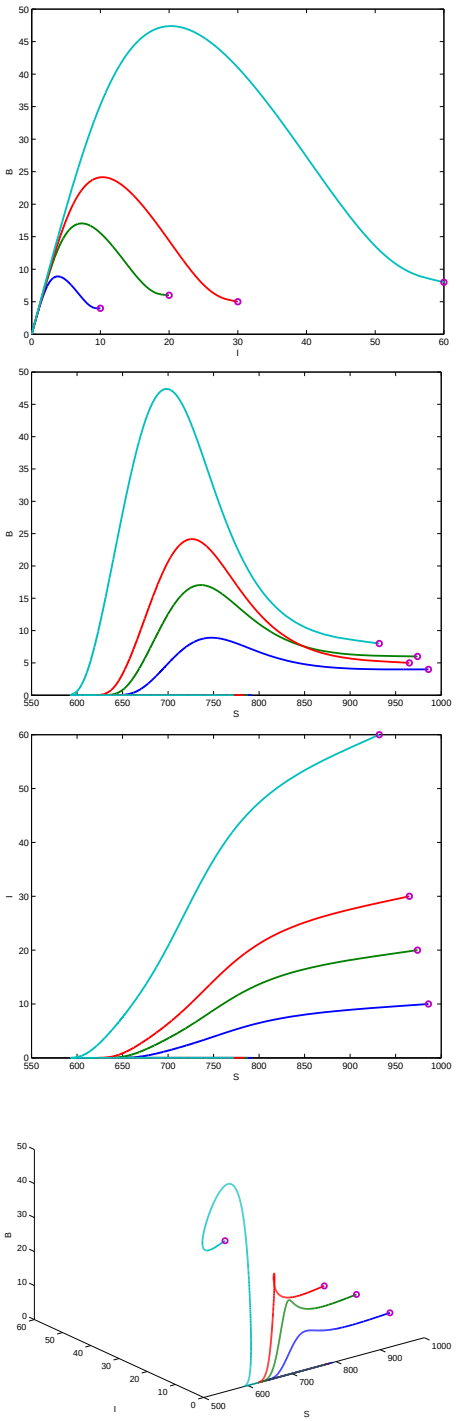

Fig. 1: Stability of the equilibria E_0. Consider the following choice of parametric values: $H = 1000, N = 0.001, a = 0.1, K = 1000, r = 0.4, \eta = 0.4, e = 1$. For $\alpha = 0.95$ and $R_0 = 0.6250 < 1$.

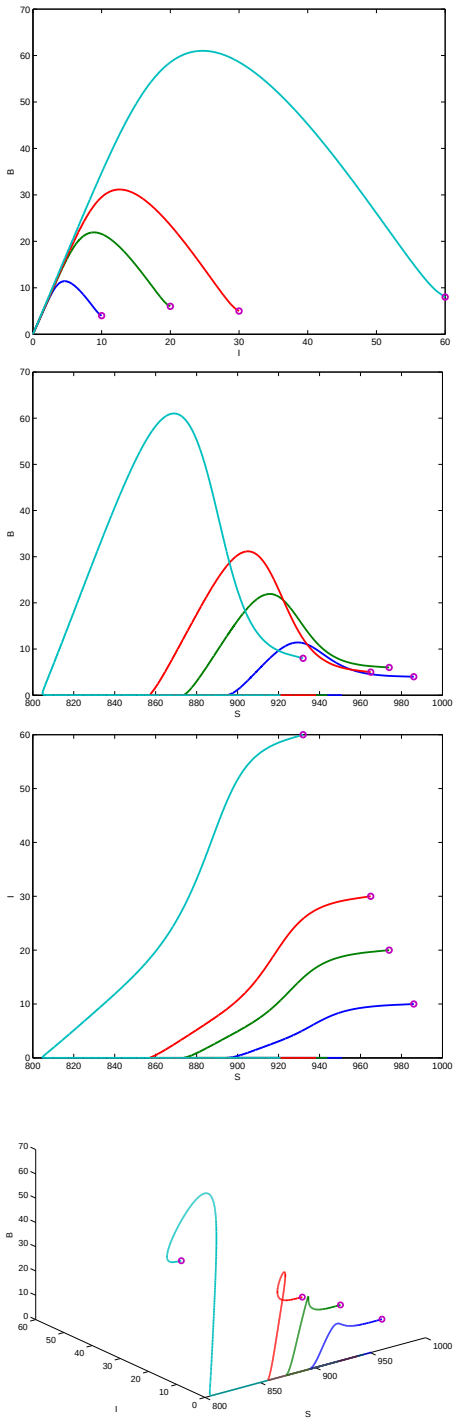

Fig. 2: Stability of the equilibria E_0. Consider the following choice of parametric values: $H = 1000, N = 0.001, a = 0.1, K = 1000, r = 0.4, \eta = 0.4, e = 1$. For $\alpha = 0.99$ and $R_0 = 0.6250 < 1$.

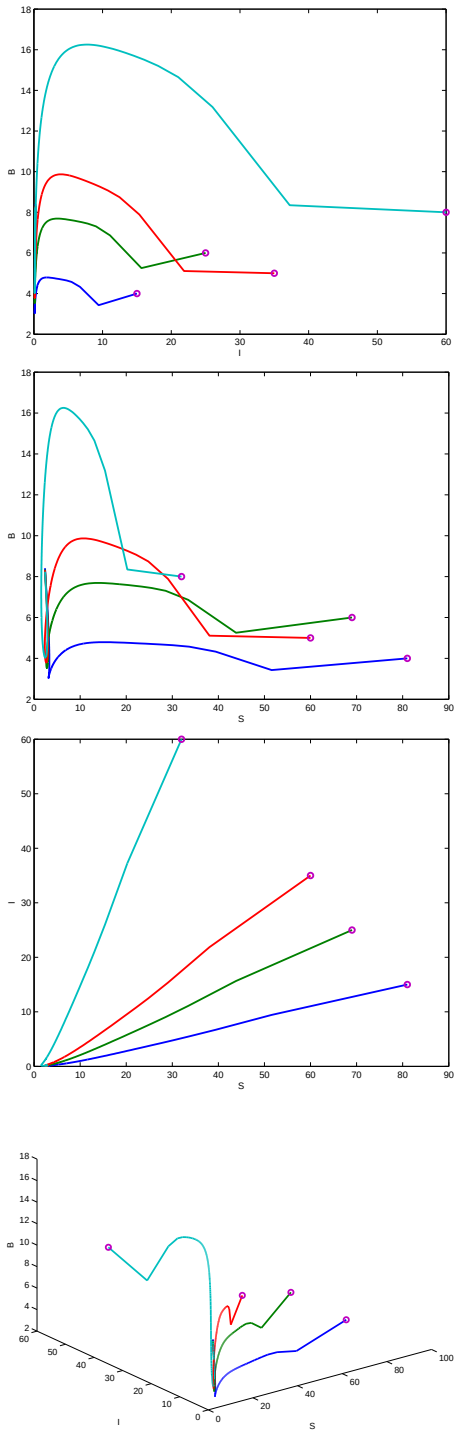

Fig. 3: Stability of the equilibria E_1. Consider the following choice of parametric values: $H = 100, N = 0.001, a = 0.5, K = 100, r = 0.4, \eta = 0.02, e = 1$. For $\alpha = 0.6$ and $R_0 = 62.5$.

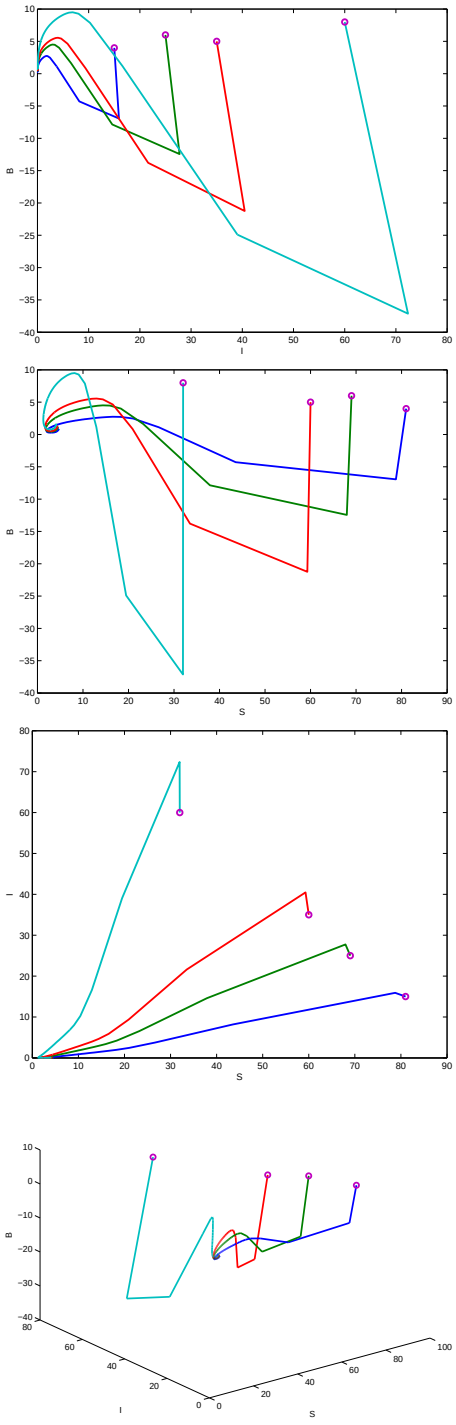

Fig. 4: Stability of the equilibria E_1. Consider the following choice of parametric values: $H = 100, N = 0.001, a = 0.5, K = 100, r = 0.4, \eta = 0.02, e = 1$. For $\alpha = 0.5$ and $R_0 = 62.5$.

$H = 1000, N = 0.001, a = 0.7, K = 1000, r = 0.4, \eta = 0.02, e = 1$.
with the initial conditions $[S_0, I_0, B_0] = [986, 155, 25], [705, 235, 60], [475, 435, 50], [860, 60, 80]$ and simulation time $T = 1000$ for $\alpha = 0.95, 0.9, 0.8$ and $\alpha = 0.7$. The stability of equilibria $E_1 = (12.8388, 2.4679, 123.3951)$ is shown in Figs. 5-8. With the given data, we find that $R_0 = 87.5 > 1, b_1 = 0.4979 > 0, b_2 = 0.0336 > 0, b_3 = 6.1599e-004 > 0, b_1 b_2 - b_3 = 0.0161 > 0, D(P) = 9.4334e-006$.

Example 4.
Consider the following values of parameters: $N = 0.002, a = 0.2, K = H, r = 0.05, \eta = 0.13, e = 20$, and initial conditions: $[S_0, I_0, B_0] = [H - 180, 155, 25]$ with simulation time: $T = 250$ for $\alpha = 0.9$ and $H = 1000, 3000, 5000, 7000$. It is easy to compute that $R_0 = 615.3846 > 1$. The numerical solution of (1) is shown by Fig. 9.

7 Conclusions

In this paper, we have studied several features of a fractional-order cholera model. These features can be summarized as follows. *(i)* We present criteria for the existence of infected-free equilibria and concentration of toxigenic V. cholerae in water equilibria. *(ii)* Stability of the equilibria for the system (1) has been discussed in terms of the reproduction number $R_0 = \frac{aeH}{rK(mb-nb)}$. Precisely, we have established the following facts: if $R_0 < 1$, then the equilibrium E_0 of system (1) is locally asymptotically stable for all $0 < \alpha < 1$; the equilibrium E_1 of system (1) is locally asymptotically stable if $R_0 > 0$. Also the stability analysis for the system (1) is carried out by applying the fractional Routh-Hurwitz method. *(iii)* The fractional-order model (1) is converted to a system of ordinary differential equations of integer-order and is then solved numerically by using the fourth order Runge-Kutta method. The graphical solutions are presented for several choices of the parameters and conditions involved in the model.

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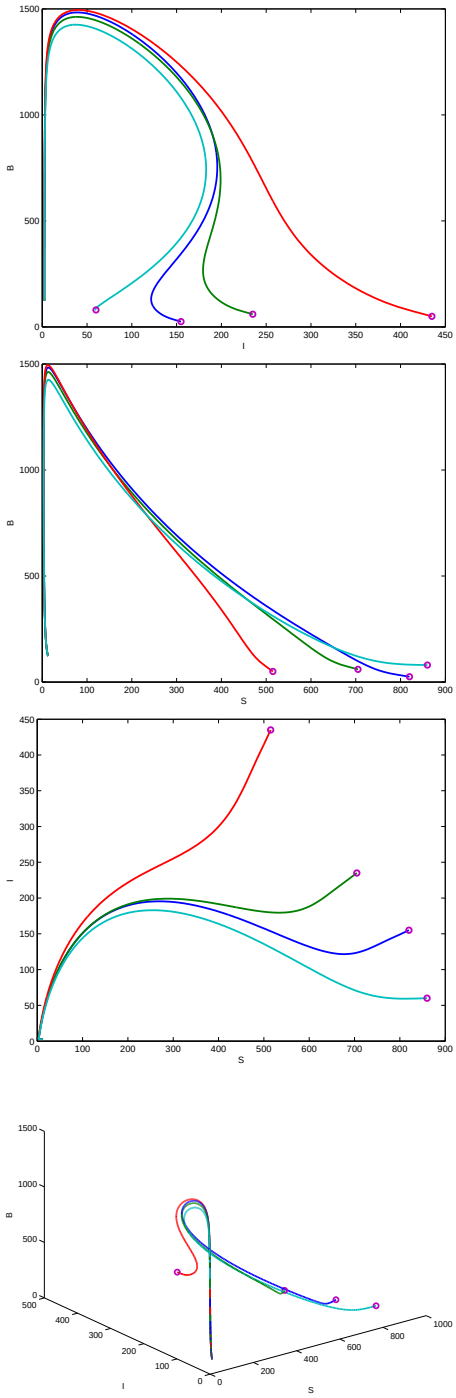

Fig. 5: Stability of the equilibria E_1. Consider the following choice of parametric values: $H = 1000, N = 0.001, a = 0.7, K = 1000, r = 0.4, \eta = 0.02, e = 1$. For $\alpha = 0.95$ and $R_0 = 87.5$.

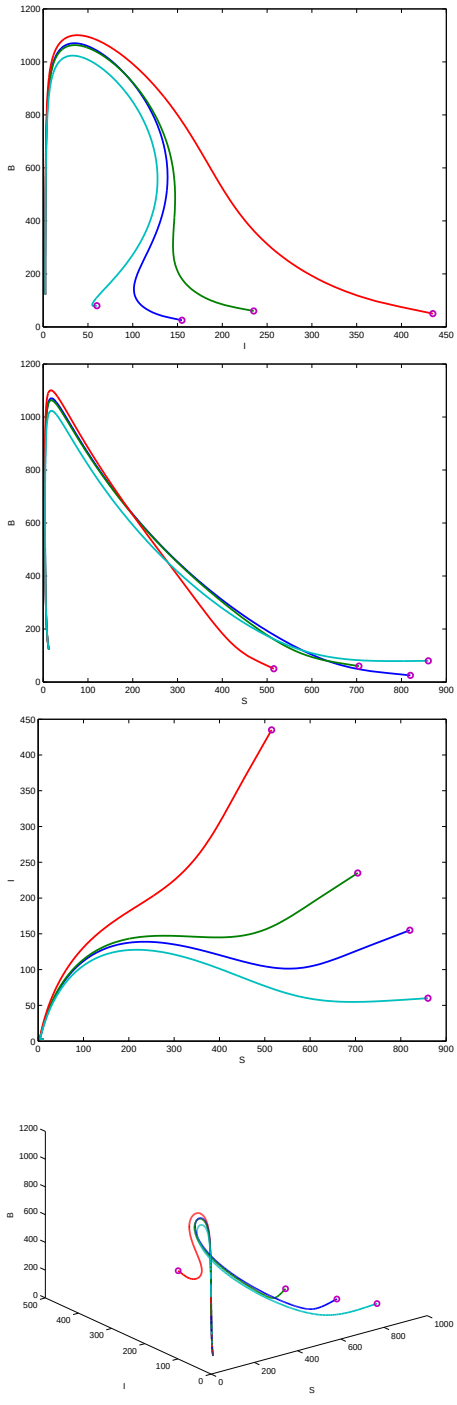

Fig. 6: Stability of the equilibria E_1. Consider the following choice of parametric values: $H = 1000, N = 0.001, a = 0.7, K = 1000, r = 0.4, \eta = 0.02, e = 1$. For $\alpha = 0.9$ and $R_0 = 87.5$.

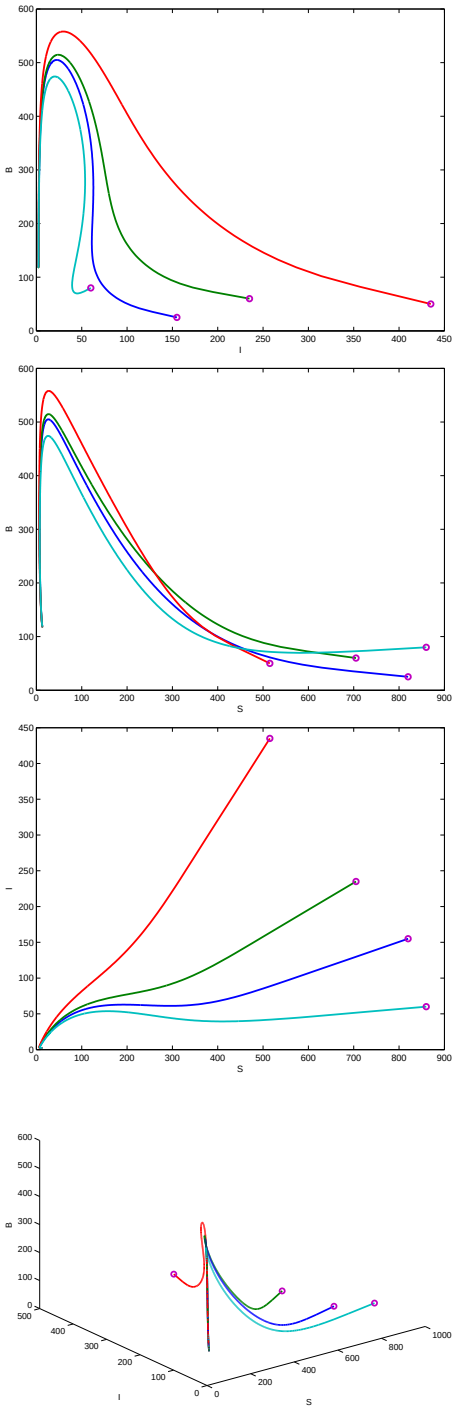

Fig. 7: Stability of the equilibria E_1. Consider the following choice of parametric values: $H = 1000, N = 0.001, a = 0.7, K = 1000, r = 0.4, \eta = 0.02, e = 1$. For $\alpha = 0.8$ and $R_0 = 87.5$.

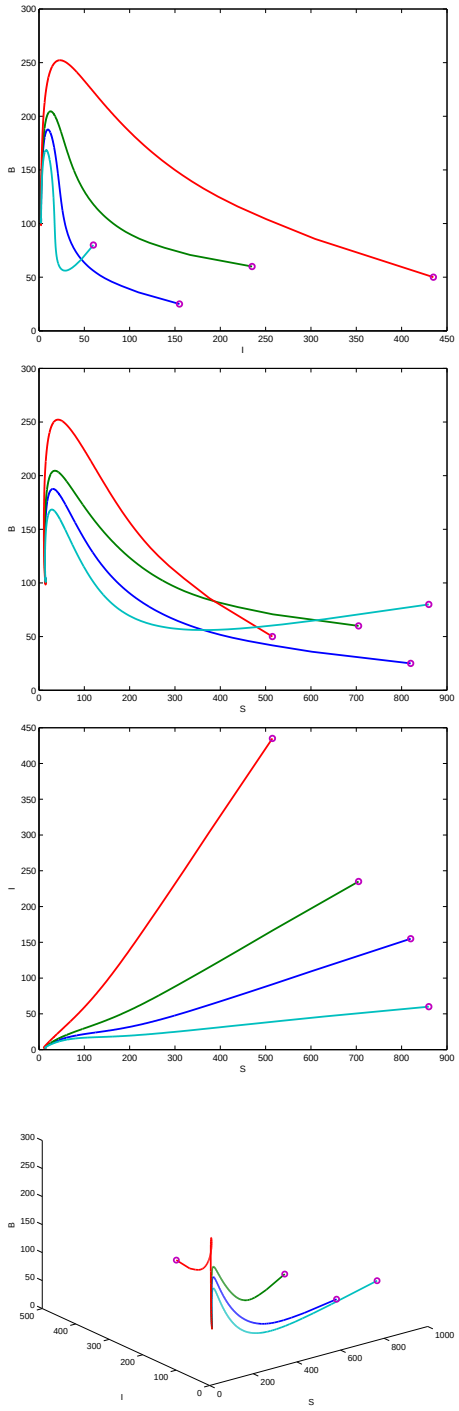

Fig. 8: Stability of the equilibria E_1. Consider the following choice of parametric values: $H = 1000, N = 0.001, a = 0.7, K = 1000, r = 0.4, \eta = 0.02, e = 1$. For $\alpha = 0.7$ and $R_0 = 87.5$.

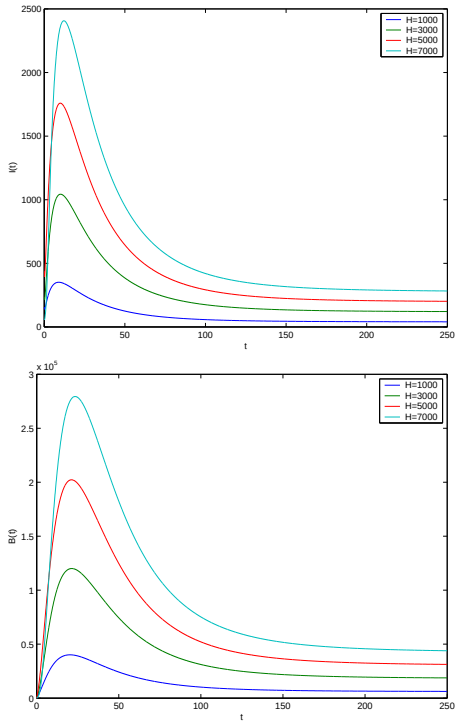

Fig. 9: Numerical solution of (1). Consider the following choice of parametric values: $N = 0.002, a = 0.2, K = H, r = 0.05, \eta = 0.13, e = 20$ For $\alpha = 0.9, R_0 = 615.3846$ and $H = 1000, 3000, 5000, 7000$.

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