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Exponential Type p–Convex Function with Some Related Inequalities and their Applications

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Abstract: In this paper, the idea of exponential type p–convex function and its algebraic properties have been investigated.The authors proved new trapezium type inequality for this new class of functions and derived many refinements of the trapezium type inequality for functions whose first derivative in absolute value at certain power is exponential type p–convex. Finally, some new bounds for special means of different positive real numbers are provided as well. The findings show some generalizations of the known results.

Keywords: Hermite–Hadamard inequality, Convexity, Exponential type convexity

1 Introduction

Theory of convexity played significant role in the development of theory of inequalities.

Definition 1. *[1] A function $\psi : I \to \Re$ is said to be convex, if*

$$\psi(\chi\theta_1 + (1-\chi)\theta_2) \le \chi\psi(\theta_1) + (1-\chi)\psi(\theta_2) \quad (1)$$

holds for all $\theta_1, \theta_2 \in I$ and $\chi \in [0,1]$.

Many known results in inequalities theory can be obtained using the convexity property of the functions, see [2,3,4] and the references therein.
Hermite–Hadamard's inequality (H–H inequality) is one of the well known investigated results involving convex functions and it asserts that, if a function $\psi : I \subset \Re \to \Re$ is convex in I for $\theta_1, \theta_2 \in I$ and $\theta_1 < \theta_2$, then

$$\psi\left(\frac{\theta_1+\theta_2}{2}\right) \le \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\psi(\chi)d\chi \le \frac{\psi(\theta_1)+\psi(\theta_2)}{2}. \quad (2)$$

Interested readers can see [5]–[23].

Definition 2. *[24] A function $\psi : I \subset (0,+\infty) \to \Re$ is called $h-$convex, if*

$$\psi(\chi\theta_1 + (1-\chi)\theta_2) \le h(\chi)\psi(\theta_1) + h(1-\chi)\psi(\theta_2) \quad (3)$$

holds for all $\theta_1, \theta_2 \in I$ and $\chi \in [0,1]$.

If the above-mentioned definition is reversed, then ψ is said to be h–concave. Clearly, if we substitute $h(\chi) = \chi$, then the h–convex functions give the classical convex functions, see [25,26].

Definition 3.*[27,28] A $p-$convex function $\psi : I \subset (0,+\infty) \to \Re$ on $p-$convex set I is defined by*

$$\psi\left(\left[\chi\theta_1^p + (1-\chi)\theta_2^p\right]^{\frac{1}{p}}\right) \le \chi\psi(\theta_1) + (1-\chi)\psi(\theta_2)$$

for all $\theta_1, \theta_2 \in I$ and $\chi \in [0,1]$. If above inequality is reversed, then ψ is said to be $p-$concave.

Definition 4.*[29] A nonnegative function $\psi : I \to \Re$ is said to be exponential type convex, if*

$$\psi(\chi\theta_1 + (1-\chi)\theta_2) \le (e^{\chi}-1)\psi(\theta_1) + \left(e^{1-\chi}-1\right)\psi(\theta_2) \quad (4)$$

holds for all $\theta_1, \theta_2 \in I$ and $\chi \in [0,1]$.

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The family of all exponential type convex functions on I is represented by $EXPC(I)$.
We recall the following hypergeometric function:

$$
\begin{aligned}
&{}_2F_1(\theta_1,\theta_2;\theta_3;\theta)\\
&=\frac{1}{\beta(\theta_2,\theta_3-\theta_2)}\int_0^1 \chi^{\theta_2-1}(1-\chi)^{\theta_3-\theta_2-1}(1-\theta\chi)^{-\theta_1}\,d\chi,
\end{aligned}
$$

where $\theta_3>\theta_2>0$, $|\theta|<1$ and $\beta(\cdot,\cdot)$ is Euler beta function.
Motivated by above results and literature, we present in Section 2, the idea of exponential type p-convex function and its algebraic properties. In Section 3, we prove new trapezium type inequality for the exponential type p–convex function ψ. In Section 4, we obtain some refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power are exponential type p–convex. In Section 5, some new bounds for special means are presented. Section 6 is devoted to conclusion.

2 some algebraic properties of exponential type p–convex functions

In this section, we to add a new definition i.e. exponential type p–convex function and its basic algebraic properties.

Definition 5. *A nonnegative function $\psi : I \to \Re$ is said to be exponential type p–convex, if*

$$
\begin{aligned}
&\psi\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&\qquad\le (e^\chi-1)\psi(\theta_1)+\left(e^{1-\chi}-1\right)\psi(\theta_2) \qquad (5)
\end{aligned}
$$

holds for all $\theta_1,\theta_2\in I$ and $\chi\in[0,1]$.

Remark. If we put $p=1$, we get exponential type convexity given by İşcan in [29].

Remark. The new class of functions defined in Definition 5 has range $[0,+\infty)$.

Proof. Let $\theta\in I$ be arbitrary. Using Definition 5 for $\chi=1$, we have

$$\psi(\theta)\le(e-1)\psi(\theta)\implies(e-2)\psi(\theta)\ge0\implies\psi(\theta)\ge0.$$

Lemma 1. *The following inequalities $(e^\chi-1)\ge\chi$ and $(e^{1-\chi}-1)\ge(1-\chi)$ hold for all $\chi\in[0,1]$.*

Proof. The proof is completed.

Proposition 1. *Let $I\subset(0,+\infty)$ be a p–convex set. Every p–convex function on a p–convex set is exponential type p–convex function.*

Proof. Using Definition of p-convex function and Lemma 1, since $\chi\le(e^\chi-1)$ and $(1-\chi)\le(e^{1-\chi}-1)$ for all $\chi\in[0,1]$, we have

$$
\begin{aligned}
\psi\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)&\le\chi\psi(\theta_1)+(1-\chi)\psi(\theta_2)\\
&\le(e^\chi-1)\psi(\theta_1)+\left(e^{1-\chi}-1\right)\psi(\theta_2).
\end{aligned}
$$

Remark. Taking $p=1$ in Proposition 1, then we get Proposition 2.1 in [29].

Proposition 2. *Every exponential type p–convex function is an h–convex function with $h(\chi)=(e^\chi-1)$.*

Proof. If we put $(e^\chi-1)=h(\chi)$ and $\left(e^{1-\chi}-1\right)=h(1-\chi)$ in the Definition 5, then Definition 2 is easily obtained.

Theorem 1. *Let $\psi,\phi:[\theta_1,\theta_2]\to\Re$. If ψ and ϕ are two exponential type p–convex functions, then*

1. *$\psi+\phi$ is exponential type p–convex function;*
2. *For nonnegative real number c, $c\psi$ is exponential type p–convex function.*

Proof. (1) Let ψ and ϕ be two exponential type p–convex functions, then

$$
\begin{aligned}
&(\psi+\phi)\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&=\psi\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)+\phi\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&\le(e^\chi-1)\psi(\theta_1)+\left(e^{1-\chi}-1\right)\psi(\theta_2)+(e^\chi-1)\phi(\theta_1)\\
&\qquad+\left(e^{1-\chi}-1\right)\phi(\theta_2)\\
&=(e^\chi-1)\left[\psi(\theta_1)+\phi(\theta_1)\right]+\left(e^{1-\chi}-1\right)\left[\psi(\theta_2)+\phi(\theta_2)\right]\\
&=(e^\chi-1)(\psi+\phi)(\theta_1)+\left(e^{1-\chi}-1\right)(\psi+\phi)(\theta_2).
\end{aligned}
$$

(2) Let ψ be exponential type p–convex, then

$$
\begin{aligned}
&(c\psi)\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&\le c\left[(e^\chi-1)\psi(\theta_1)+\left(e^{1-\chi}-1\right)\psi(\theta_2)\right]\\
&=(e^\chi-1)c\psi(\theta_1)+\left(e^{1-\chi}-1\right)c\psi(\theta_2)\\
&=(e^\chi-1)(c\psi)(\theta_1)+\left(e^{1-\chi}-1\right)(c\psi)(\theta_2).
\end{aligned}
$$

Remark. Choosing $p=1$ in Theorem 1, then we get Theorem 2.1 in [29].

Theorem 2. *Let $\psi:I\to J$ be p–convex function and $\phi:J\to\Re$ are non-decreasing and exponential type convex function. Then the function $\phi\circ\psi:I\to\Re$ is exponential type p–convex.*

Proof. For all $\theta_1,\theta_2\in I$, and $\chi\in[0,1]$, we get

$$
\begin{aligned}
&(\phi\circ\psi)\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&=\phi\left(\psi\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right)\\
&\le\phi\left(\chi\psi(\theta_1)+(1-\chi)\psi(\theta_2)\right)\\
&\le(e^\chi-1)\phi(\psi(\theta_1))+\left(e^{1-\chi}-1\right)\phi(\psi(\theta_2))\\
&=(e^\chi-1)(\phi\circ\psi)(\theta_1)+\left(e^{1-\chi}-1\right)(\phi\circ\psi)(\theta_2).
\end{aligned}
$$

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Remark. If we put $p = 1$ in Theorem 2, then we obtain Theorem 2.2 in [29].

Theorem 3. *Let $\psi_i : [\theta_1, \theta_2] \to \Re$ be an arbitrary family of exponential type p–convex functions and let $\psi(\theta) = \sup_i \psi_i(\theta)$. If $O = \{\theta \in [\theta_1, \theta_2] : \psi(\theta) < +\infty\} \neq \emptyset$, then O is an interval and ψ is exponential type p–convex function on O.*

Proof. For all $\theta_1, \theta_2 \in O$ and $\chi \in [0,1]$, we obtain

$$\begin{aligned}
&\psi\left(\left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&= \sup_i \psi_i\left(\left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&\leq \sup_i \left[(e^\chi - 1)\psi_i(\theta_1) + \left(e^{1-\chi} - 1\right)\psi_i(\theta_2)\right]\\
&\leq (e^\chi - 1)\sup_i \psi_i(\theta_1) + \left(e^{1-\chi} - 1\right)\sup_i \psi_i(\theta_2)\\
&= (e^\chi - 1)\psi(\theta_1) + \left(e^{1-\chi} - 1\right)\psi(\theta_2) < +\infty.
\end{aligned}$$

Remark. Taking $p = 1$ in Theorem 3, then we have Theorem 2.3 in [29].

Theorem 4. *If the function $\psi : [\theta_1, \theta_2] \to \Re$ is exponential type p–convex then ψ is bounded on $[\theta_1, \theta_2]$.*

Proof. Let $L = \max\left\{\psi(\theta_1), \psi(\theta_2)\right\}$ and $x \in [\theta_1, \theta_2]$ be an arbitrary point. Then there exists $\chi \in [0,1]$ such that $x = \left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}$. Thus, since $e^\chi \leq e$ and $e^{1-\chi} \leq e$, we have

$$\begin{aligned}
\psi(x) &= \psi\left(\left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&\leq (e^\chi - 1)\psi(\theta_1) + \left(e^{1-\chi} - 1\right)\psi(\theta_2)\\
&\leq \left(e^\chi + e^{1-\chi} - 2\right)\cdot L\\
&\leq 2(e-1)\cdot L = M.
\end{aligned}$$

We have shown that ψ is bounded above from real number M. Interested reader can also prove the fact that ψ is bounded below using the same idea as in Theorem 2.4 in [29].

Remark. Choosing $p = 1$ in Theorem 4, then we get Theorem 2.4 in [29].

3 Hermite–Hadamard inequality for exponential type p–convex functions

This section aims to derive a new inequality of Hermite–Hadamard type for the exponential type p–convex function ψ.

Theorem 5. *Let $\psi : [\theta_1, \theta_2] \to \Re$ be exponential type p–convex function. If $\psi \in L_1([\theta_1, \theta_2])$, then*

$$\frac{1}{2(\sqrt{e}-1)}\psi\left(\left[\frac{\theta_1^p + \theta_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\theta_2^p - \theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx \tag{6}$$
$$\leq (e-2)[\psi(\theta_1) + \psi(\theta_2)].$$

Proof. Using exponential type p–convexity of ψ, we have

$$\begin{aligned}
&\psi\left(\left[\frac{\theta_1^p + \theta_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\leq \psi\left(\frac{1}{2}[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p]^{\frac{1}{p}} + \frac{1}{2}[(1-\chi)\theta_1{}^p + \chi\theta_2{}^p]^{\frac{1}{p}}\right)\\
&\leq (\sqrt{e}-1)\psi\left(\left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\\
&\qquad + (\sqrt{e}-1)\psi\left(\left[(1-\chi)\theta_1{}^p + \chi\theta_2{}^p\right]^{\frac{1}{p}}\right).
\end{aligned}$$

Now integrating the above inequality with respect to $\chi \in [0,1]$, we obtain

$$\begin{aligned}
&\psi\left(\left[\frac{\theta_1^p + \theta_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\leq \left[(\sqrt{e}-1)\int_0^1 \psi\left(\left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)d\chi\right.\\
&\left. + (\sqrt{e}-1)\int_0^1 \psi\left(\left[(1-\chi)\theta_1{}^p + \chi\theta_2{}^p\right]^{\frac{1}{p}}\right)d\chi\right]\\
&= \frac{2p(\sqrt{e}-1)}{\theta_2^p - \theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx,
\end{aligned}$$

which completes the left side inequality. For the right side inequality, changing the variable of integration as $x = \left(\left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)$, and using the definition of the exponential type p–convex function ψ, we obtain

$$\begin{aligned}
&\frac{p}{\theta_2^p - \theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\\
&= \int_0^1 \psi\left(\left[\chi\theta_1{}^p + (1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)d\chi\\
&\leq \int_0^1 \left((e^\chi - 1)\psi(\theta_1) + \left(e^{1-\chi} - 1\right)\psi(\theta_2)\right)d\chi\\
&= (e-2)[\psi(\theta_1) + \psi(\theta_2)],
\end{aligned}$$

which gives the right side inequality.

Remark. If we put $p = 1$ in Theorem 5, then we get Theorem 3.1 in [29].

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4 Refinements of Hermite–Hadamard (or trapezium type inequality) type inequality

Let us recall the following crucial Lemma that we will use in the sequel.

Lemma 2. *[30] Let $\psi : I \to \Re$ be differentiable function on I° with $\theta_1, \theta_2 \in I$ and $\theta_1 < \theta_2$. If $\psi' \in L_1[\theta_1, \theta_2]$, then*

$$\frac{\psi(\theta_1)+\psi(\theta_2)}{2} - \frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx \tag{7}$$
$$= \left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\times$$
$$\int_0^1 \frac{1-2\chi}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)d\chi.$$

Theorem 6. *Let $\psi : I \to \Re$ be differentiable function on I° with $\theta_1, \theta_2 \in I$ and $\theta_1 < \theta_2$. If $\psi' \in L_1[\theta_1, \theta_2]$ and $|\psi'|^q$ is exponentially type p–convex on $[\theta_1, \theta_2]$ for $q \geq 1$, then*

$$\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2} - \frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right| \tag{8}$$
$$\leq \left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left[B_1(p,\theta_1,\theta_2)\right]^{1-\frac{1}{q}}\times$$
$$\left[B_2(p,\theta_1,\theta_2)|\psi'(\theta_1)|^q + B_3(p,\theta_1,\theta_2)|\psi'(\theta_2)|^q\right]^{\frac{1}{q}},$$

where

$$B_1(p,\theta_1,\theta_2) = \int_0^1 \frac{|1-2\chi|}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi,$$

$$B_2(p,\theta_1,\theta_2) = \int_0^1 \frac{|1-2\chi|(e^\chi-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi$$

and

$$B_3(p,\theta_1,\theta_2) = \int_0^1 \frac{|1-2\chi|(e^{1-\chi}-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi.$$

Proof. From Lemma 2, power mean inequality, exponentially type p–convexity of $|\psi'|^q$ and properties of modulus; we have

$$\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2} - \frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\times$$
$$\int_0^1 \left|\frac{1-2\chi}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}\right|\times$$
$$\left|\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right|d\chi$$
$$\leq \left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\chi|}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \frac{|1-2\chi|}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}\times\right.$$
$$\left.\left|\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right|^q d\chi\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\chi|}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \frac{|1-2\chi|}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}\times\right.$$
$$\left.\left\{(e^\chi-1)|\psi'(\theta_1)|^q+\left(e^{1-\chi}-1\right)|\psi'(\theta_2)|^q\right\}d\chi\right)^{\frac{1}{q}}$$
$$= \left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left[B_1(p,\theta_1,\theta_2)\right]^{1-\frac{1}{q}}\times$$
$$\left[B_2(p,\theta_1,\theta_2)|\psi'(\theta_1)|^q + B_3(p,\theta_1,\theta_2)|\psi'(\theta_2)|^q\right]^{\frac{1}{q}},$$

which completes the proof.

Theorem 7. *Let $\psi : I \to \Re$ be differentiable function on I° with $\theta_1, \theta_2 \in I$ and $\theta_1 < \theta_2$. If $\psi' \in L_1[\theta_1, \theta_2]$ and $|\psi'|^q$ is exponentially type p–convex on $[\theta_1, \theta_2]$ for $q > 1$ and $\frac{1}{l}+\frac{1}{q}=1$, then*

$$\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2} - \frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right| \tag{9}$$
$$\leq \left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}\times$$
$$\left[B_4(p,q,\theta_1,\theta_2)|\psi'(\theta_1)|^q + B_5(p,q,\theta_1,\theta_2)|\psi'(\theta_2)|^q\right]^{\frac{1}{q}},$$

where

$$B_4(p,q,\theta_1,\theta_2) = \int_0^1 \frac{(e^\chi-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}}d\chi$$

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and

$$B_5(p,q,\theta_1,\theta_2)=\int_0^1 \frac{(e^{1-\chi}-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}}d\chi.$$

Proof. From Lemma 2, Hölder's inequality, exponentially type p–convexity of $|\psi'|^q$ and properties of modulus; we have

$$\begin{aligned}&\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2}-\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right|\\
&\le\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\int_0^1|1-2\chi|^l d\chi\right)^{\frac{1}{l}}\times\\
&\left(\int_0^1\frac{1}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{q(1-\frac{1}{p})}}\times\right.\\
&\qquad\left.\left|\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right|^q d\chi\right)^{\frac{1}{q}}\\
&\le\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}\times\\
&\quad\left(\int_0^1\frac{(e^{\chi}-1)|\psi'(\theta_1)|^q+(e^{1-\chi}-1)|\psi'(\theta_2)|^q}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}}d\chi\right)^{\frac{1}{q}}\\
&=\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}}\times\\
&\quad\left[B_4(p,q,\theta_1,\theta_2)|\psi'(\theta_1)|^q+B_5(p,q,\theta_1,\theta_2)|\psi'(\theta_2)|^q\right]^{\frac{1}{q}},\end{aligned}$$

which completes the proof.

Theorem 8. *Let $\psi: I\to\Re$ be differentiable function on I° with $\theta_1,\theta_2\in I$ and $\theta_1<\theta_2$. If $\psi'\in L_1[\theta_1,\theta_2]$ and $|\psi'|^q$ is exponentially type p–convex on $[\theta_1,\theta_2]$ for $q>1$ and $\frac{1}{l}+\frac{1}{q}=1$, then*

$$\begin{aligned}&\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2}-\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right| \\
&\le\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left[B_6(p,l,\theta_1,\theta_2)\right]^{\frac{1}{l}}\times\\
&\qquad\left(B_7(q)|\psi'(\theta_1)|^q+B_8(q)|\psi'(\theta_2)|^q\right)^{\frac{1}{q}},\end{aligned}\tag{10}$$

where

$$B_6(p,l,\theta_1,\theta_2)=\int_0^1\frac{1}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{l\left(1-\frac{1}{p}\right)}}d\chi$$

$$=\begin{cases}\frac{1}{2\theta_1^{l(p-1)}}\times{}_2F_1\left(l-\frac{l}{p},1;2;1-\left(\frac{\theta_2}{\theta_1}\right)^p\right), & p<0;\\ \frac{1}{2\theta_2^{l(p-1)}}\times{}_2F_1\left(l-\frac{l}{p},1;2;1-\left(\frac{\theta_1}{\theta_2}\right)^p\right), & p>0,\end{cases}$$

and

$$B_7(q)=\int_0^1(e^{\chi}-1)|1-2\chi|^q d\chi,$$
$$B_8(q)=\int_0^1(e^{1-\chi}-1)|1-2\chi|^q d\chi.$$

Proof. From Lemma 2, Hölder's inequality, exponentially type p–convexity of $|\psi'|^q$ and properties of modulus; we have

$$\begin{aligned}&\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2}-\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right|\\
&\le\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\int_0^1\frac{1}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{l\left(1-\frac{1}{p}\right)}}d\chi\right)^{\frac{1}{l}}\\
&\times\left(\int_0^1|1-2\chi|^q\left|\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right|^q d\chi\right)^{\frac{1}{q}}\\
&=\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left[B_6(p,l,\theta_1,\theta_2)\right]^{\frac{1}{l}}\times\\
&\qquad\left(B_7(q)|\psi'(\theta_1)|^q+B_8(q)|\psi'(\theta_2)|^q\right)^{\frac{1}{q}},\end{aligned}$$

which completes the proof.

Theorem 9. *Let $\psi: I\to\Re$ be differentiable function on I° with $\theta_1,\theta_2\in I$ and $\theta_1<\theta_2$. If $\psi'\in L_1[\theta_1,\theta_2]$ and $|\psi'|$ is exponentially type p–convex on $[\theta_1,\theta_2]$, then*

$$\begin{aligned}&\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2}-\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right|\\
&\le\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\times\\
&\left(B_9(p,\theta_1,\theta_2)|\psi'(\theta_1)|+B_{10}(p,\theta_1,\theta_2)|\psi'(\theta_2)|\right),\end{aligned}\tag{11}$$

where

$$B_9(p,\theta_1,\theta_2)=\int_0^1\frac{|1-2\chi|(e^{\chi}-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi,$$
$$B_{10}(p,\theta_1,\theta_2)=\int_0^1\frac{|1-2\chi|(e^{1-\chi}-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi.$$

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Proof. From Lemma 2, exponentially type p–convexity of $|\psi'|^q$ and properties of modulus; we get

$$\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2}-\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right|$$
$$\leq\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\int_0^1\left|\frac{1-2\chi}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}\right|\left|\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right|d\chi$$
$$\leq\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\times\int_0^1\left|\frac{1-2\chi}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}\right|\times\left[(e^{\chi}-1)|\psi'(\theta_1)|+(e^{1-\chi}-1)|\psi'(\theta_2)|\right]d\chi$$
$$=\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\times\left[|\psi'(\theta_1)|\int_0^1\frac{|1-2\chi|(e^{\chi}-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi+|\psi'(\theta_2)|\int_0^1\frac{|1-2\chi|(e^{1-\chi}-1)}{\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{1-\frac{1}{p}}}d\chi\right]$$
$$=\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(B_9|\psi'(\theta_1)|+B_{10}|\psi'(\theta_2)|\right),$$

which completes the proof.

Theorem 10. *Let $\psi: I\to\Re$ be differentiable function on I° with $\theta_1,\theta_2\in I$ and $\theta_1<\theta_2$. If $\psi'\in L_1[\theta_1,\theta_2]$ and $|\psi'|^q$ is exponentially type p–convex on $[\theta_1,\theta_2]$ for $q>1$ and $\frac{1}{l}+\frac{1}{q}=1$, then*

$$\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2}-\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right| \tag{12}$$
$$\leq\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)[2(e-2)]^{\frac{1}{q}}\times[B_{11}(p,l,\theta_1,\theta_2)]^{\frac{1}{l}}A^{\frac{1}{q}}\left(|\psi'(\theta_1)|^q,|\psi'(\theta_2)|^q\right),$$

where $A(\cdot,\cdot)$ is the arithmetic mean and

$$B_{11}(p,l,\theta_1,\theta_2)=\int_0^1\frac{|1-2\chi|^l}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}d\chi.$$

Proof. From Lemma 2, Hölder's inequality, exponentially type p–convexity of $|\psi'|^q$ and properties of modulus; we have

$$\left|\frac{\psi(\theta_1)+\psi(\theta_2)}{2}-\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\frac{\psi(x)}{x^{1-p}}dx\right|$$
$$\leq\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\times\int_0^1\left|\frac{1-2\chi}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}\right|\times\left|\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right|d\chi$$
$$\leq\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\int_0^1\frac{|1-2\chi|^l}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}d\chi\right)^{\frac{1}{l}}\times\left(\int_0^1\left|\psi'\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]^{\frac{1}{p}}\right)\right|^q\right)^{\frac{1}{q}}d\chi$$
$$\leq\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\int_0^1\frac{|1-2\chi|^l}{\left(\left[\chi\theta_1{}^p+(1-\chi)\theta_2{}^p\right]\right)^{1-\frac{1}{p}}}d\chi\right)^{\frac{1}{l}}\times\left(\int_0^1\left[(e^{\chi}-1)|\psi'(\theta_1)|^q+(e^{1-\chi}-1)|\psi'(\theta_2)|^q\right]d\chi\right)^{\frac{1}{q}}$$
$$=\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)[2(e-2)]^{\frac{1}{q}}\times[B_{11}(p,l,\theta_1,\theta_2)]^{\frac{1}{l}}A^{\frac{1}{q}}\left(|\psi'(\theta_1)|^q,|\psi'(\theta_2)|^q\right),$$

which completes the proof.

5 Applications

Consider the following special means of two positive real numbers θ_1,θ_2 $(\theta_2>\theta_1)$:

1.The arithmetic mean

$$A(\theta_1,\theta_2)=\frac{\theta_1+\theta_2}{2}.$$

2.The harmonic mean

$$H(\theta_1,\theta_2)=\frac{2\theta_1\theta_2}{\theta_1+\theta_2}.$$

3.The power mean

$$M_q(\theta_1,\theta_2)=\left(\frac{\theta_1^q+\theta_2^q}{2}\right)^{\frac{1}{q}},\quad q\neq0.$$

4.The logarithmic mean

$$L(\theta_1,\theta_2)=\frac{\theta_2-\theta_1}{\ln\theta_2-\ln\theta_1}.$$

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5.The p–logarithmic mean

$$L_p(\theta_1,\theta_2)=\left(\frac{\theta_2^{p+1}-\theta_1^{p+1}}{(p+1)(\theta_2-\theta_1)}\right)^{\frac{1}{p}},\quad p\in\Re\backslash\{-1,0\}.$$

Proposition 3. *If* $0<\theta_1<\theta_2$ *and* $p\in\Re\backslash\{-1,0,1\}$, *then*

$$\frac{L_{p-1}^{p-1}(\theta_1,\theta_2)}{2(\sqrt{e}-1)}M_q(\theta_1,\theta_2)\le L_p^p(\theta_1,\theta_2) \tag{13}$$
$$\le 2(e-2)A(\theta_1,\theta_2)L_{p-1}^{p-1}(\theta_1,\theta_2).$$

*Proof.*Taking $\psi(x)=x$ for $x>0$ in Theorem 5, then inequality (13) is easily obtained.

Proposition 4.*If* $0<\theta_1<\theta_2$ *and* $p>1$, *then*

$$\frac{1}{2(\sqrt{e}-1)}H\left(\theta_1^p,\theta_2^p\right)L_{p-1}^{p-1}(\theta_1,\theta_2)\le L^{-1}(\theta_1,\theta_2) \tag{14}$$
$$\le 2(e-2)\,A\left(\theta_1^p,\theta_2^p\right)L_{p-1}^{p-1}(\theta_1,\theta_2).$$

*Proof.*Choosing $\psi(x)=x^p$ for $x>0$ in Theorem 5, then inequality (14) is easily derived.

Proposition 5.*If* $0<\theta_1<\theta_2$, $q\neq 0$ *and* $p\in\Re\backslash\{0,1,2\}$, *then*

$$\frac{1}{2(\sqrt{e}-1)}L_{p-1}^{p-1}(\theta_1,\theta_2)M_q^{-1}(\theta_1,\theta_2)\le L_{p-2}^{p-2}(\theta_1,\theta_2) \tag{15}$$
$$\le 2(e-2)A(\theta_1,\theta_2)L_{p-1}^{p-1}(\theta_1,\theta_2).$$

*Proof.*Taking $\psi(x)=\frac{1}{x}$ for $x>0$ in Theorem 5, then inequality (15) is easily captured.

Proposition 6.*If* $0<\theta_1<\theta_2$, $q\ge 1$ *and* $p>1$, *then*

$$\left|H^{-1}\left(\theta_1^p,\theta_2^p\right)-\frac{L^{-1}(\theta_1,\theta_2)}{L_{p-1}^{p-1}(\theta_1,\theta_2)}\right|$$
$$\le\left(\frac{\theta_2^p-\theta_1^p}{2}\right)\left[B_1(p,\theta_1,\theta_2)\right]^{1-\frac{1}{q}} \tag{16}$$
$$\times\left[B_2(p,\theta_1,\theta_2)\,\theta_1^{q(p-1)}+B_3(p,\theta_1,\theta_2)\,\theta_2^{q(p-1)}\right]^{\frac{1}{q}}.$$

*Proof.*Choosing $\psi(x)=x^p$ for $x>0$ in Theorem 6, then inequality (16) is easily derived.

Proposition 7.*If* $0<\theta_1<\theta_2$, $q>1$, $\frac{1}{l}+\frac{1}{q}=1$ *and* $p\in\Re\backslash\{0,1,2\}$, *then*

$$\left|H^{-1}(\theta_1,\theta_2)-\frac{L_{p-2}^{p-2}(\theta_1,\theta_2)}{L_{p-1}^{p-1}(\theta_1,\theta_2)}\right|\le\left(\frac{\theta_2^p-\theta_1^p}{2p}\right)\left(\frac{1}{l+1}\right)^{\frac{1}{l}} \tag{17}$$
$$\times\left[B_4(p,q,\theta_1,\theta_2)\,\theta_1^{2q}+B_5(p,q,\theta_1,\theta_2)\,\theta_2^{2q}\right]^{\frac{1}{q}}.$$

*Proof.*Taking $\psi(x)=\frac{1}{x}$ for $x>0$ in Theorem 7, then inequality (17) is easily captured.

6 Conclusion

The present paper showed new Hermite–Hadamard (or trapezium type inequality) type inequality for the new class of functions, the so–called exponential type p–convex function ψ and obtained some interesting refinements. The interested reader can find other new results using other suitable functions ψ and new bounds for special means and error estimates for the trapezoidal and midpoint formula. To the best of our knowledge these results are new in the literature. Since convex functions has large applications in many mathematical areas, we hope that our new results can be applied in convex analysis, special functions, quantum analysis, quantum mechanics, post quantum analysis, etc.

Conflict of Interest

The authors declare that they have no conflict of interest.

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