Appl. Math. Inf. Sci. **18**, No. 5, 1075-1080 (2024)

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180513

Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations of infinite order

*S. A. El-Zahaby and Samira El-Tamimy**

Department of Mathematics, Faculty of Science, Al-Azhar University [For Girls], Nasr City, Cairo, Egypt

Received: 12 May 2024, Revised: 12 Jun. 2024, Accepted: 15 Jul. 2024
Published online: 1 Sep. 2024

Abstract: In this paper, a semilinear elliptic optimal control problem of infinite order with pointwise state constraints is studied. The existence of regular Lagrange multipliers is obtained, we derive first-order necessary optimality conditions for elliptic control problem with pointwise state constraint generated by elliptic operator of an infinite order with finite dimension . Second order sufficient optimality conditions are established.

Keywords: Optimal control, semilinear elliptic equation, infinite order operator, state constraints, necessary optimality conditions, second order optimality conditions

1 Introduction

It is known that in the case of nonlinear equations the first order conditions are not in general sufficient for optimality so that we are going to derive a second order conditions. In this paper, we study an optimal control problem for a semilinear elliptic distributed control problem governed by elliptic operator of infinite order with pointwise constraints on the state. The aim is to derive the first-order necessary and second order sufficient optimality conditions by using [1,2,3,4,5,6]

The Cauchy Dirichlet problem studied by Dubinskii [7,8]

$$L(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma} u) = h(x), \quad x \in \Omega$$

$$D^{|\omega|} u(x)|_{\partial\Omega} = 0, \quad |\omega| = 0, 1, 2, \cdots$$

The Sobolev space of infinite order which define by

$$W^{\infty}\{a_{\alpha}, p_{\alpha}\}(\Omega) = \{u(x) \in C_0^{\infty}(\Omega) : p(u) \equiv \sum_{|\alpha|=0}^{\infty} \|D^{\alpha} u\|_{p_{\alpha}}^{p_{\alpha}} < \infty\}$$

where $a_{\alpha} \geq 0$ and $p_{\alpha} \geq 1$ are numerical sequences and $\|.\|_p$ is the canonical norm in the space $L_p(G)$.

Gali et al. [9] presented a set of inequalities defining an optimal control of a system governed by self-adjoint elliptic operators with an infinite number of variables.

Subsequently Lions suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimensions.

Gali has solved this problem, the result has been published in [10].

Moreover, I. M. Gali et. al. [10,9,11,12] presented some control problems generated by both elliptic and hyperbolic linear operator of an infinite order with finite number of variables.

El-Zahaby et al [13] obtained the optimal control of problems governed by variational inequalities of an infinite order with finite domain.

We refers for instance, to Cases [6] for the first-order necessary optimality conditions, Casas, Tröltzsch and Unger [3] for the second-order sufficient condition.

For the elliptic case with quadratic objective and linear equation of infinite order, this obtained by El-Zahaby et. al. [14] while the state constrained problem was investigated in [15], and a semilinear problem of infinite order with finite dimension, this obtained by El-Zahaby [16], and [17,18], where elliptic problem with pointwise control constraints.

* Corresponding author e-mail: samiraahmed@azhar.edu.eg

The existence of regular Lagrange multipliers was discussed by Röch and Tröltzsch [19] for elliptic case.

Necessary optimality conditions in function spaces can be directly deduced form the Karush-Kuhn-Tuker theory for optimization problems in Banach spaces. This method, which called the Lagrange method, will be investigated in [20].

In [3] that was, the first paper on second order sufficient conditions for problems with pointwise state constraints, the constriction of the critical cone was quite complicated yet. Several improvements were made that culminated so far in [21] for state-constrained problems with semilinear equations.

For the papers which a close connection to our work, we refer to [16,2,1,3,21,22,23] and reference given there in.

2 Some Function Spaces

The embedding problems for non-trivial Sobolev spaces of infinite order are investigated in [8,7].

An embedding criterion established in terms of the characteristic functions of these space.

In this case

$$W^{\infty}\{a_\alpha,2\} \subseteq L_2(R^n) \subseteq W^{-\infty}\{a_\alpha,2\}$$

where,

$$W^{\infty}\{a_\alpha,2\} = \{\phi \in C^{\infty}(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \phi\|_2^2 < \infty\}$$

be Sobolev space of infinite order of periodic function defined on all of R^n and $W^{-\infty}\{a_\alpha,2\}$ denotes their topological dual with respect to $L_2(R^n)$, we recall that $\alpha = (\alpha_1,\cdots,\alpha_n)$ is a multi-index for differentiation and $|\alpha| = \alpha_1 + \cdots + \alpha_n$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots (\partial x_n^{\alpha_n})}, \qquad a_\alpha > 0,$$

is a numerical sequence, and $\|.\|_2$ is the canonical norm in the space $L_2(R^n)$, (all functions are assumed to be real value).

3 Problem Statement

In the study of semilinear elliptic control problem of infinite order and pointwise state and control constraints

$$(P)\begin{cases} \min J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{k}{2}\|u\|^2_{L^2(\Omega)}, & (1)\\ \text{subject to} & \\ Ay + d(x,y) = u \quad \text{in} \quad \Omega, & (2)\\ y^{|w|}|_{\Gamma} = 0 \quad |w| = 0,1,2,\cdots, & \\ \text{and the state constraint} & \\ u_a(x) \le u(x) \le u_b(x), \quad y_u(x) \le \gamma & (3)\\ \text{for} \quad \text{almost all} \quad x \in \Omega & \end{cases}$$

where A is the elliptic operator of infinite order $A \in L(W^{\infty}\{a_\alpha,2\}, W^{-\infty}\{a_\alpha,2\})$ for which has self-adjoint closure

$$Ay(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y, \qquad a_\alpha > 0 \qquad (4)$$

we introduce the following continuous bilinear form

$$a(u,v) = \sum_{|\alpha|=0}^{\infty} \left((-1)^{|\alpha|} a_\alpha D^{2\alpha} u(x), v(x)\right)_{L^2(R^n)} \text{on } W^{\infty}\{a_\alpha,2\}$$

It is well known that the ellipticity of A is sufficient for the coerciveness of bilinear form $a(u,v)$ i.e.,

$$a(y,y) \ge \nu \|y\|^2_{W_0^{\infty}\{a_\alpha,2\}}$$

We adapt the following assumption

Assumption 1:

- The function $y_d \in L^2(\Omega)$ and $k > 0, N \neq 0, \gamma \neq 0$ are real numbers and the bound u_a and u_b are fixed function in $L^2(\Omega)$ with $u_a(x) \le u_b(x)$.
- $\Omega \subset R^N$ be bounded Lipschitz domain with boundary Γ.
- A denotes elliptic operator of infinite order with finite dimension take the form (4).
- This operator is bounded self-adjoint mapping $W_0^{\infty}\{a_\alpha,2\}$ onto $W_0^{\infty}\{a_\alpha,2\}$ and satisfy the condition of ellipticity [17]

$$\begin{aligned} a(y(x),y(x)) &= \sum_{|\alpha|=0}^{\infty} (a_\alpha D^\alpha y(x), D^\alpha y(x))_{L^2(R^n)} \\ &\ge \nu \|y\|^2_{W_0^{\infty}\{a_\alpha,2\}}, \quad 1 \ge \nu > 0. \end{aligned}$$

- The function $d = d(x,y) : \Omega \times R \to R$ is measurable with respect to $x \in \Omega$ for all fixed $y \in R$, and twice continuously differentiable with respect to y, for almost all $x \in \Omega$.
- Moreover, for d is satisfy the boundedness condition of order two with respect to x, there exists $C > 0$ such that

$$\|d(.,0)\|_\infty + \|d_y(.,0)\|_\infty + \|d_{yy}(.,0)\|_\infty \le C$$

- Further, for $a.a.x \in \Omega$ and $y \in R$, it holds that $d_y(x,y) \ge 0$.
- Also, the derivative of d w.r.t. y up to order two are local Lipschitz condition, i.e. for all $M > 0$ there exists $L_M > 0$ such that d satisfies

$$\|d_{yy}(.,y_1) - d_{yy}(.,y_2)\|_\infty \le L_M |y_1 - y_2|$$

for all $y_i \in R$ with $|y_i| \le M$, $i = 1,2$.
- There is a subset $E_\Omega \subset \Omega$ of positive measure with $d_y(x,y) > 0$ in $E_\Omega \times R$.

Theorem 1 *Under Assumption 1 the semilinear elliptic control problem (2) admits for every $u \in L^2(\Omega)$ a unique solution $y \in W^{\infty}\{a_\alpha,2\} \cap C(\bar{\Omega})$ with*

$$\|y\|_{W^{\infty}\{a_\alpha,2\}} + \|y\|_{C(\bar{\Omega})} \le C\|u\|_{L^2(\Omega)}.$$

Based on this theorem, we introduced the control to state operator $G : L^2(\Omega) \to W^\infty\{a_\alpha,2\} \cap C(\bar{\Omega})$, $u \to y$. Let us reformulate the problem (P) with the help of the solution operator G to obtain the reduced formulation

$$(PV)\begin{cases} \min f(u) = J(G(u),u) \\ \qquad := \frac{1}{2}\|G(u)-y_d\|^2_{L^2(\Omega)} + \frac{k}{2}\|u\|^2_{L^2(\Omega)}, \\ \text{and the state constraint} \\ u_a(x) \le u(x) \le u_b(x), \quad G(u)(x) \le \gamma \\ \text{for} \quad \text{a.a} \quad x \in \Omega \end{cases} \tag{5}$$

Let us define the set of admissible handling the box constraints on the control

$$U_{ad} = \{u \in L^2(\Omega) : \ u_a(x) \le u(x) \le u_b(x) \ a.e \ in \ \Omega\}$$

For the proof we refer to [16,20].

4 First-Order Optimality Conditions

We consider the state y_u associated with a given control u in the space $W^\infty\{a_\alpha,2\} \cap C(\bar{\Omega})$. It is known that the mapping $u \to y_u$ is twice continuously Frechet differentiable.

Definition 1*A control $\bar{u} \in U_{ad}$ satisfy $y_{\bar{u}(x)} \le \gamma$ in Ω is called a local solution of problem (PV) if there exists a $\rho > 0$ such that $f(\bar{u}) \le f(u)$ for all $u \in U_{ad}$ and $\|u-\bar{u}\| \le \rho$.*

The next theorem states the existence of an optimal solution for(PV).

Theorem 2*Let the Assumption 1 be satisfied. If the admissible set is not empty, then Problem (PV) admits at least one a local solution in the sense of Definition 1.*

The proof follows from a more general result see [20]

The first and second order optimality conditions can now be easily transferred from the elliptic case. We obtain for the first and second order derivatives of f in direction $v \in L^\infty(\Omega)$

Lemma 1*Let Assumption 1 be fulfilled. Then the mapping $G : L^2(\Omega) \to W^\infty\{a_\alpha,2\} \cap C(\bar{\Omega})$ is twice continuously Frechet differentiable. Moreover, for all $u \in L^2(\Omega)$, $w = G'(u)h$ is defined as the solution of*

$$\begin{aligned} &Aw + d_y(x,y)w = h && in \quad \Omega \\ &w^{|\alpha|}|_\Gamma = 0 && |\alpha| = 0,1,2,\cdots \end{aligned} \tag{6}$$

Furthermore, for every $z \in L^2(\Omega)$, $z = G''(u)[u_1,u_2]$ is the solution of

$$\begin{aligned} &Az + d_y(x,y)z = -d_{yy}(x,y)y_1y_2 && in \quad \Omega \\ &z^{|w|}|_\Gamma = 0 && |w| = 0,1,2,\cdots \end{aligned} \tag{7}$$

where $y_i = G'(u)u_i$.

The proof can be obtained by using the implicit function theorem.

Since f is twice continuously Frechet differentiable and with differentiability of G, this yields the following Lemma.

Lemma 2*Under Assumption 1, f is twice continuously Frechet differentiable from $L^2(\Omega)$ to R. Its first derivative is given by*

$$f'(u)h = \int_\Omega (ku + p(x))h(x)dx \tag{8}$$

where p solves the adjoint equation

$$\begin{aligned} &Ap + d_y(x,y)p = y - y_d && in \quad \Omega \\ &p^{|w|}|_\Gamma = 0 && |w| = 0,1,2,\cdots \end{aligned} \tag{9}$$

where p is the adjoint solution and A is the adjoint operator which take the same form in (4).

For the second derivative, we obtain

$$\begin{aligned} f''(u)[u_1,u_2] =& (y_1,y_2)_{L^2(\Omega)} + k(u_1,u_2)_{L^2(\Omega)} \\ & - \int_\Omega d_{yy}(x,y)y_1y_2pdx \end{aligned} \tag{10}$$

*Proof.*From definition of the reduced cost functional

$$J(G(u),u) = \frac{1}{2}\int_\Omega (G(u)-y_d(x))^2 + \frac{k}{2}\int_\Omega u^2(x)dx$$

We get

$$f'(u)h = (y-y_d,w)_{L^2(\Omega)} + k(u,h)_{L^2(\Omega)},$$

where $y = G(u)$ and $w = G'(u)h$ denotes the weak solution of the linearized equation (6) with the right hand side h.

Now, choosing p as a test function in the weak formulation of (6) and inserting y in the weak formulation of equation (9), we obtain

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}\int_\Omega a_\alpha D^{2\alpha}wpdx + \int_\Omega d_y(x,y)wpdx = \int_\Omega hpdx$$

$$\sum_{|\alpha|=0}^{\infty}\int_\Omega a_\alpha(D^\alpha w)(x)(D^\alpha p)(x)dx + \int_\Omega d_y(x,y)wpdx$$
$$= \int_\Omega hpdx$$

and

$$\sum_{|\alpha|=0}^{\infty}\int_\Omega a_\alpha(D^\alpha p)(x)(D^\alpha w)(x)dx + \int_\Omega d_y(x,y)pwdx$$
$$= \int_\Omega (y-y_d)wdx$$

Substracting one equation from the other finally yields

$$(y-y_d,w)_{L^2(\Omega)} = (h,p)_{L^2(\Omega)}$$

As a simple conclusion, the following expression for the directional derivative of the objective functional f at $\bar{u}$ in the direction $h \in L^2(\Omega)$ yields

$$f'(\bar{u})h = \int_\Omega (p(x) + k\bar{u}(x))h(x)dx$$

We obtain the desired necessary optimality condition.

Applying again the Chain rule, we can calculate the second derivative as follow. First, we obtain

$$f'(u)u_1 = D_yJ(G(u),u)G'(u)u_1 + D_uJ(G(u),u)u_1$$

Next, we calculate the direction derivative of $f'(u)u_1$ in direction u_2, we find that

$$f''(u)[u_1,u_2] \\ = J''(y,u)[(y_1,u_1),(y_2,u_2)] + D_yJ(y,u)G''(u)[u_1,u_2].$$

A similar discussion as above, where the abbreviation $z := G''(u)[u_1,u_2]$ denotes the weak solution of (7) with this, we obtain the expression

$$D_yJ(y,u)z = \int_\Omega (y - y_d)z(x)dx$$

which can be transformed by using the adjoint state p, which is the weak solution to (9) hence, we have $(y-y_d,z)_{L^2(\Omega)} = -(d_{yy}(x,y)y_1y_2,p)_{L^2(\Omega)}$.

5 Lagrange Multiplier Rule

To formulate first necessary optimality conditions, the pointwise state constraints are included in a Lagrangian function by associated Lagrange multipliers. We define Lagrangian $\mathscr{L}$ by adding the inequality constraints in the following way

$$\mathscr{L}(y,u,\mu) := J(y,u) + \int_\Omega (y_u(x) - \gamma)d\mu(x) \tag{11}$$

where $\mu \in M(\Omega)$ is a regular Boral measure . Such multipliers exist under so called regularity condition that is here taken as complementary slackness conditions.

Then there exist an associated Lagrange multiplier $\bar{\mu} \in M(\Omega)$ such that

$$\bar{\mu} \geq 0 \quad and \quad \int_\Omega (y_u(x) - \gamma)d\bar{\mu}(x) = 0$$

Then the equation

$$\int_\Omega (k\bar{u} + q)(u - \bar{u}) \geq 0 \tag{12}$$

can be expressed in the form $\dfrac{\partial \mathscr{L}}{\partial u}(\bar{y},\bar{u},q,\bar{\mu}) \geq 0$ for all $u \in L^2(\Omega)$.

Moreover, the adjoint equation

$$\begin{aligned} &Aq + d_y(x,\bar{y})q = \bar{y}_{\bar{u}} - y_d + \bar{\mu} \quad in \quad \Omega \\ &q^{|w|}|_\Gamma = 0 \qquad\qquad |w| = 0,1,2,\cdots \end{aligned} \tag{13}$$

is equivalent to the equation

$$\frac{\partial \mathscr{L}}{\partial y}(\bar{y},\bar{u},q,\bar{\mu}) = 0$$

for all $y \in W^\infty\{a_\alpha,2\}$.

The optimality system can therefore be rewritten in the following form

$$\frac{\partial \mathscr{L}}{\partial y} \geq 0, \qquad \frac{\partial \mathscr{L}}{\partial u} \geq 0, \qquad \bar{\mu} \geq 0$$
$$\int_\Omega (y_u(x) - \gamma)d\bar{\mu}(x) = 0$$

where the last two conditions are called complementary slackness conditions.

The next theorem show that (12), (13) and complementary slackness conditions follow from the variational inequality .

Theorem 3*Suppose that $\bar{u}$ is a local solution of an optimal control problem, then there exist an element $u_0 \in U_{ad}$ such that*

$$y_{\bar{u}}(x) + w_{u_0-\bar{u}} \leq \gamma \quad f.a \quad x \in \Omega$$

Then there exist an associated Lagrange multipliers $\bar{\mu} \in M(\Omega)$ and the adjoint state $q \in W^\infty\{a_\alpha,2\} \cap C(\bar{\Omega})$ such that the condition (12), the adjoint equation (13) and complementing slackness conditions are satisfied.

6 Second-Order Sufficient Conditions

We discuss a sufficient second-order optimality condition (SSC) . For this purpose, we define the following critical cone as follows:

Definition 2*The critical cone was introduced in the context for state-constrained problems with semilinear elliptic equations. The critical cone is defined by*

$$C_{\bar{u},\bar{\mu}} = \left\{ v \in L^2(\Omega) : \int_\Omega (k\bar{u} + p(x))vdx = 0, \right. \\ \left. v \quad satisfies\ the\ conditions (14),(15),(16) \right\}$$

The further conditions defining $C_{\bar{u},\bar{\mu}}$ are the sign conditions

$$v(x) := \left\{ v \in L^2(\Omega) \middle| \begin{array}{l} v(x) = 0\ (k\bar{u} + p(x)) \neq 0 \\ v(x) \geq 0\ where\ \bar{u}(x) = u_a(x) \\ v(x) \leq 0\ where\ \bar{u}(x) = u_b(x) \end{array} \right. \tag{14}$$

and

$$w_v(x) \leq 0 \quad if \quad \bar{y}(x) = \gamma \tag{15}$$

$$\int_{\Omega} w_v(x) \quad d\bar{\mu}(x) = 0 \tag{16}$$

The sufficient second-order optimality conditions are given by the equation (17) in the next theorem covering the local optimality of $\bar{v}$ [20] .

Theorem 4*Let $\bar{v}$ be a feasible control for problem (P) and satisfies the variational inequality (12). Assume that the coercivity condition*

$$f''(\bar{v})v^2 \geq \hat{\delta}\|v\|^2_{L^2(\Omega)} \qquad \forall\, v \in C_{\bar{u},\bar{\mu}} \tag{17}$$

is satisfied with $\hat{\delta} > 0$. Then there exist $\varepsilon > 0$ s.t. we have the quadratic growth condition holds:

$$\frac{\delta}{4}\|v-\bar{v}\|^2_{L^2(\Omega)} + f(\bar{v}) \leq f(v) \; if \; \|v-\bar{v}\|_{L^\infty(\Omega)} < \varepsilon \; \forall v \in V_{ad} \tag{18}$$

So that $\bar{v}$ is a local solution of (P) with respect to the norm $\|.\|_\infty$

*Proof.*We select $v \in L^2(\Omega)$ and perform a Taylor expansion at $\bar{v}$. We get

$$f(v) = f(\bar{v}) + f'(\bar{v})(v-\bar{v}) + \frac{1}{2}f''(\bar{v}+\theta(v-\bar{v}))(v-\bar{v})^2$$

with $\theta \in (0,1)$.

Under Assumption 1 and since f is twice continuously Frechet differentiable, there exists some $\varepsilon > 0$ such that

$$\begin{aligned} &|f''(\bar{v})v_1^2 - f''(v)v_1^2| \leq \frac{\delta}{2}\|v_1\|^2_{L^2(\Omega)} \; \forall\, v_1 \in L^2(\Omega) \\ &if \; \|v-\bar{v}\|_{L^\infty(\Omega)} \leq \varepsilon \end{aligned} \tag{19}$$

It follow from the variational inequality (12), we get with some intermediate point v_θ between v and $\bar{v}$

$$\begin{aligned} f(v) - f(\bar{v}) &= \frac{1}{2}f''(v_\theta)(v-\bar{v})^2 \\ &= \frac{1}{2}\left(f''(\bar{v})(v-\bar{v})^2 + \left[f''(v_\theta) - f''(\bar{v})\right](v-\bar{v})^2\right) \\ &\geq \frac{1}{2}\left(\delta\|v-\bar{v}\|^2_{L^2(\Omega)} - \frac{\delta}{2}\|v-\bar{v}\|^2_{L^2(\Omega)}\right) \\ &= \frac{\delta}{4}\|v-\bar{v}\|^2_{L^2(\Omega)} \end{aligned}$$

in view of (17) and (19), provided that

$$\|v-\bar{v}\|_{L^\infty(\Omega)} \leq \varepsilon$$

7 Conclusion

In contrast to the optimal control of linear systems with a convex objective, where first order necessary optimality conditions are already sufficient for optimality, higher order conditions such as second order sufficient optimality conditions (SSC) should be employed to verify optimality for nonlinear systems. In this paper, we have investigated how second order conditions can be formulated for state constrained control problems governed by semilinear elliptic equations of infinite order with respect to second order sufficient optimality conditions (SSC). We obtained that a first order necessary optimality conditions of pointwise state by associated Lagrange multipliers. Also, we derived a sufficient second order optimality conditions by introduced the critical cone which covering the local optimality of control.

References

[1] Eduardo Casas and Fredi Tröltzsch. Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory. *SIAM Journal on Optimization*, 13(2):406–431, 2002.

[2] Jacqueline Mossino. An application of duality to distributed optimal control problems with constraints on the control and the state. *Journal of Mathematical Analysis and Applications*, 50(2):223–242, 1975.

[3] Eduardo Casas, Fredi Tröltzsch, and Unger. Second order sufficient optimality conditions for some state-constrained control problems for semilinear equations. *SAIMJ.Control. Optim.*, 38(5):1369–1391, 2000.

[4] Eduardo Casas. Control of an elliptic problem with pointwise state constraints. *SIAM Journal on Control and Optimization*, 24(6):1309–1318, 1986.

[5] Eduardo Casas and Mariano Mateos. Second order optimality conditions for semilinear elliptic control problems with finitely many state constraints. *SIAM journal on control and optimization*, 40(5):1431–1454, 2002.

[6] JF Bonnans and E Casas. On the choice of the function spaces for some state-constrained control problems. *Numerical functional analysis and optimization*, 7(4):333–348, 1985.

[7] Yu A Dubinskii. Sobolev spaces of infinite order. *Soviet Math. Dokl.*, 19:1271–1274, 1979.

[8] Yulii Andreevich Dubinskii. Sobolev spaces of infinite order and the behavior of solutions of some boundary value problems with unbounded increase of the order of the equation. *Matematicheskii Sbornik*, 140(2):163–184, 1975.

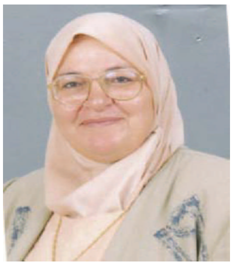
[9] IM Gali and HA El-Saify. Distributed control of a system governed by dirichlet and neumann problems for a self-adjoint elliptic operator with an infinite number of variables. *Journal of Optimization theory and Applications*, 39:293–298, 1983.

[10] IM Gali. Optimal control of systems governed by elliptic operator of infinite order. In *Ordinary and Partial Differential Equations: Proceedings of the Seventh Conference Held at Dundee, Scotland, March 29–April 2, 1982*, pages 263–271. Springer, 1982.

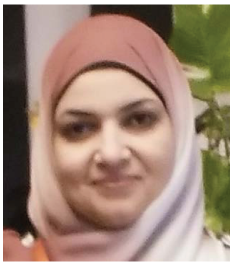
[11] IM Gali, HA El-Saify, and SA El-Zahabi. Optimal control of a system governed by hyperbolic operator. In *Ordinary and Partial Differential Equations: Proceedings of the Eighth Conference held at Dundee, Scotland, June 25–29, 1984*, pages 157–163. Springer, 1985.

[12] IM Gali, HA El-Saify, and SA El-Zahaby. Distributed control of a system governed by dirichlet and neumann

problems for elliptic equations of infinite order. In *Proceeding of the International Conference on "Functional-Differential Systems and Related Topics*, volume 3, page 83e87, 1983.
[13] IM Gali and SA El-Zahaby. Optimal control of a system governed by variational inequalities for infinite order elliptic operator. *Kyungpook Mathematical Journal*, 31(2):193–200, 1991.
[14] SA El-Zahaby and Samira El-Tamimy. Optimal control problems of systems governed by elliptic operator of infinite order with pointwise control constraint. *Journal of Mathematics and Statistical Science (ISSN2411-2518, USA)*, pages 199–212, 2015.
[15] Samira El-Tamimy. Linear elliptic control problems of infinite order with pointwise state constraints. *Appl. Math*, 16(6):1005–1008, 2022.
[16] SA El-Zahaby. Second order sufficient optimality conditions for semilinear elliptic control problem of infinite order. *International Journal of Pure and Applied Mathematics*, 100(2):169–192, 2015.
[17] SA El-Zahaby and Samira El-Tamimy. Second order optimality conditions for a semilinear elliptic control problem of infinite order with pointwise control constraints. *Journal of Mathematics and Statistical Science (ISSN2411-2518, USA)*, 5:263–273, 2019.
[18] SA El-Zahaby and Samira El-Tamimy. Optimal control of systems governed by semilinear elliptic equation for infinite order operator with finite dimension. *Journal of Al-Azhar Bulletin of Science*, 9:115–121, 2017.
[19] A. Roch and F. Tröltzsch. xistence of regular lagrange multipliers for elliptic optimal control with pointwise control-state constraints. *Preprint*, 2005.
[20] Fredi Tröltzsch. *Optimal control of partial differential equations: theory, methods, and applications*, volume 112. American Mathematical Soc., 2010.
[21] Eduardo Casas, Juan Carlos De Los Reyes, and Fredi Tröltzsch. Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints. *SIAM Journal on Optimization*, 19(2):616–643, 2008.
[22] Eduardo Casas and Fredi Tröltzsch. Recent advances in the analysis of pointwise state-constrained elliptic optimal control problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 16(3):581–600, 2010.
[23] Eduardo Casas and Fredi Tröltzsch. Second-order and stability analysis for state-constrained elliptic optimal control problems with sparse controls. *SIAM Journal on Control and Optimization*, 52(2):1010–1033, 2014.



Somaya Aly El-Zahaby Prof.Dr. in Department of Mathematics, Faculty of Science (Girls Branch), Al-Azhar Universit, Cairo, Egypt. Received the PHD for pure Math. From Al-Azhar Unv. She research interests are in Functional Analysis in Optimal control. Previous visits abroad: 1) Technische University Berlin , Institute of Mathematics Prof. Dr. Fredi Tröltzsch, grants from DFG, Jun –march 2012, 2) Potsdam University, Institute of Mathematics, Pr. Dr. H Junek, grants from DAAD, Oct-Dec 2003. During the visit I participated in a workshop of partial differential equations (Berlin 11-13 Dec. 2003). Institute of Mathematics, University of Potsdam Pr. and by grants from DFG for two Months. 2006-2013: Head of Mathematics Department of Faculty of Science, El-Azhar University, Cairo, Egypt. Member of the Standing Committee for the Promotion of Professors, Member of the Egyptian Society for Mathematics.



Samira El-Tamimy is Teaching Assistance of Pure Mathematics in Mathematics Department at Al-Azhar University (Girl Branch), Cairo, Egypt. She was secretary of the Mathematics Department. She received the Ph. D. degree in "Pure Mathematics", "Functional Analysis", at Al-Azhar University (Girl Branch), Cairo, Egypt in 2019.