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Fixed point approaches for cyclic contraction mappings in C^*-algebra-valued b-metric spaces

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Abstract: In this manuscript, we concentrated on extracting some fixed point results for cyclic contraction mappings under mild contractive conditions in the context of $\mathbf{C}^*$-algebra. These results are considered a generalization and extension of some results in the scientific literature such as the results of Banach, Kannan and Chatterjea in the setting of $\mathbf{C}^*$-algebra valued b-metric spaces. We present some definitions, supportive examples and an application related to nonlinear integral equation to support our main results.

Keywords: Cyclic contraction mapping; fixed point technique; $\mathbf{C}^*$-algebra valued b-metric spaces, nonlinear integral equation.

1 Scientific Introduction

Fixed-point (**FP**) theory is an active, exciting, and important component in the field of functional analysis. Owing to the smoothness and flexibility of the **FP** method, it has not only found many important applications in mathematics, but also has applications in various sciences such as modern optimization, Computer science, mechanical engineering, economics, astronomy, biology, and chemistry. For instance, **FP** theory has numerous pure and applied mathematics applications such as root finding and iterative methods (Numerical Analysis), studying dynamic systems described by ordinary differential equations (**ODEs**) in addition analyzing the stability and behavior of equilibrium states (Differential Equations), determining the equilibrium points of control systems (Control Theory), studying isometries and symmetries of geometric shapes and figures (Geometry), studying geodesics and minimal surfaces (Differential Geometry), and showing the existence and uniqueness of solutions to integral, differential, integro-deifferential, and fractional differenial equations in function spaces (Functional Analysis).

The **FP** method was able to solve many problems of mathematical modeling and nonlinear analysis using Banach's contraction principle (**BCP**) [1]. This principle is considered to be the bridge that researchers cross to obtain reliable results and super-applications in this direction. As known, **BCP** is a very straightforward, practical, and traditional method utilized in modern analysis. In general, **BCP** has been broadly generalized in two directions. On one hand, Weakly contractive (expansive) condition is used instead of usual contractive (expansive) condition. On the opposite hand, metric spaces with an ordered or partially ordered structure take the role of action spaces. **BCP** is more realistic when the operator is continuous but in the non-continuous state, there is an obvious shortcoming. Hence, many researchers tend to overcome this drawback by using new constraints (e.g. [2,3,4,5,6,7,8]) or by generalizing the space (e.g. [9,10,11,12,13,14,15,16,19,17,18]).

BCP is a critically significant theorem and a useful tool in the study of metric spaces, b-metric spaces and etc. Numerous generalizations of the concept of metric space have been constructed, and several fixed-point theorems have been established in these spaces (see [20,21,22,23]. In particular, Bakhtin [24] and Czerwik [25] proposed the b-metric spaces, so that the triangle was replaced by the b-triangle inequality. A b-metric space is not always a metric space, but

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any metric space is also a b-metric space. Numerous fixed-point results have been found on these spaces (see [26,27,28]).

$\mathbf{C}^*$-algebra valued b-metric spaces ($\mathbf{C}^*$-algebra $Vb-MSs$) are a fascinating and strong topic of research at the point of intersection between functional analysis and metric spaces. The concept of metric space is extended to the context in which distances are established using operators from $\mathbf{C}^*$-algebras in $\mathbf{C}^*$-algebra valued metric spaces ($\mathbf{C}^*$-algebra $VMSs$) and after that in $\mathbf{C}^*$-algebra $Vb-MSs$. In 2014, Ma et al.[29] replaced the set of real numbers with the set of all positive elements of unital $\mathbf{C}^*$-algebra to obtain a $\mathbf{C}^*$-algebra $VMSs$. They studied the topological properties of this space and obtained some nice fixed point results under suitable contractive conditions, for more details, see [30,31]. Recently, a lot of researchers introduced their work depending on the extension of **BCP** for $\mathbf{C}^*$-algebra $VMSs$. Later, Batul and Kamran [32] presented the concept of $\mathbf{C}^*$-valued contractive type mapping and investigated a **FP** result in this context. Inspired by the concepts and results presented in[29,32,33,34], Kamran et al.[35] established a new concept of $\mathbf{C}^*$-algebra $Vb-MSs$ and introduced a **FP** result in such spaces. The significance and essential characteristics of the idea of $\mathbf{C}^*$-algebra $Vb-MSs$ will be discussed in this article.

Integral equations (**IEs**) are mathematical equations that involve integral expressions with unknown functions. Due to their capacity to model complicated processes, these equations have been extensively investigated and used in a wide range of scientific and technical fields. **IEs** provide power solutions to many problems that would be difficult to resolve with traditional differential equations. There are two fundamental categories of **IEs**: Fredholm and Volterra equations. In Fredholm integral equations (**FIEs**), the unknown function only appears within the integral, in contrast to Volterra integral equations (**VIEs**), it appears both within and outside the integral. These equations have a wide range of practical applications in many disciplines, including physics, engineering, biology, economics, and others. For instance, they are employed in the study of a variety of topics such as fluid dynamics, signal processing, heat conduction, population dynamics, and electrical circuits, among others. The relationship between **FP** theorems and **IEs** is quite interesting, as **FP** theorems represent an effective mathematical technique for showing and illustrating the existence and uniqueness of solutions to many types of **IEs**, in order to acquire more information (see[36,37,38,39,40,41,42,43,44,45,46,47]).

The main purpose of this article is to discuss **FP** results within the context of cyclic contraction mappings in $\mathbf{C}^*$-algebra $Vb-MSs$. Our work aims to advance and acquire the knowledge of **FP** theory in this general, influential, and powerful context. We can develop a strong theoretical basis that can be used to solve a variety of mathematical and practical problems by investigating **FP** results in the context of cyclic contraction mappings and $\mathbf{C}^*$-algebra $Vb-MSs$. To support and illustrate our theoretical results, we present a variety examples and applictation that display the applicability and effectiveness of our methodologies.

2 Basic facts

This part is devoted to introducing some fundamental properties of $\mathbf{C}^*$-algebra $Vb-MSs$ and b-metric space, which will be used in the research.

Definition 1. *[25] Assume that $\mathscr{X}$ is a non-empty set. A mapping $d_b : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R}^+$ is called b-valued metric on $\mathscr{X}$ if there exists $b \geq 1$ verifies the following assumptions: For $\nu, \varpi, \vartheta \in \mathscr{X}$,*
(b_1) $d_b(\nu,\varpi) \succeq 0_b$ and $d_b(\nu,\varpi) = 0_b$ iff $\nu = \varpi$;
(b_2) $d_b(\nu,\varpi) = d_b(\varpi,\nu)$;
(b_3) $d_b(\nu,\varpi) \preceq b\,[d_b(\nu,\vartheta) + d_b(\vartheta,\varpi)]$,
for all ν,ϖ,ϑ. The pair $(\mathscr{X}, d_b)$is called a b-metric space.

Example 1. [48] Suppose that $\mathscr{X} = [1,\infty)$. Define $d_b : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R}^+$ by $d_b(\nu,\varpi) = |\nu - \varpi|^2$. Then $(\mathscr{X}, d_b)$ is a b-metric space with coefficient $b = 2$.

Definition 2. *[49] Assume that $\mathbb{A}$ is unital algebra with unit $\mathscr{I}$. Then, for all $\hbar, \ell \in \mathbb{A}$*
(i) A conjugate linear mapping $\hbar \mapsto \hbar^*$ on $\mathbb{A}$ such that $\hbar = \hbar^{**}$ and $(\hbar\ell)^* = \ell^*\hbar^*$ is considered an involution on $\mathbb{A}$, then the pair $(\mathbb{A},*)$ is called $*$-algebra;
(ii) A $*$-algebra $\mathbb{A}$ with complete submultiplicative norm such that $\|\hbar^*\| = \|\hbar\|$ is called Banach $*$-algebra;
(iii) A $\mathbf{C}^*$-algebra is a Banach $*$-algebra such that $\|\hbar^*\hbar\| = \|\hbar\|^2$.

Remark. [29] A $\mathbf{C}^*$-algebra has many examples such as the set of complex numbers, the set of $n \times n$-matrices, $\mathscr{M}_n(\mathbb{C})$, and the set of all bounded linear operators on a Hilbert space $\mathscr{H}, \mathscr{L}(\mathscr{H})$.

Definition 3. *[50] Let $\mathscr{X}$ be a non-empty set. A mapping $d : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{A}$ is called $\mathbf{C}^*$-algebra valued b-metric on $\mathscr{X}$ if there exists $b \in \mathbb{A}'_+$ such that $b \succeq \mathscr{I}$ satisfies the following assertions: For $\nu, \varpi, \vartheta \in \mathscr{X}$,*

($\mathbf{C}^*b_1$) $d(\nu,\varpi)\succeq 0_{\mathbb{A}}$ and $d(\nu,\varpi)=0_{\mathbb{A}}$ iff $\nu=\varpi$;
($\mathbf{C}^*b_2$) $d(\nu,\varpi)=d(\varpi,\nu)$;
($\mathbf{C}^*b_3$) $d(\nu,\varpi)\preceq b\,[d(\nu,\vartheta)+d(\vartheta,\varpi)]$.
Then, $(\mathscr{X},\mathbb{A},d)$ is called $\mathbf{C}^*$-algebra $Vb-MSs$.

Below is a new example on the $\mathbf{C}^*$-algebra $Vb-MSs$.

Example 2. Let $\mathscr{X}=\mathbb{C}$ and $\mathbb{A}=\mathscr{M}_n(\mathbb{C})$ be a $\mathbf{C}^*$-algebra. Define a mapping $d:\mathscr{X}\times\mathscr{X}\longrightarrow M_n(\mathbb{C})$ by

$$d(\vartheta_1,\vartheta_2)=\begin{pmatrix} c_1\,|\vartheta_1-\vartheta_2|^p & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & c_n\,|\vartheta_n-\vartheta_m|^p\end{pmatrix},$$

where $c_1,c_2,\ldots,c_n$ are constants and $\vartheta_1,\vartheta_2,\ldots,\vartheta_m\in\mathscr{X}$. Hypotheses ($\mathbf{C}^*b_1$) and ($\mathbf{C}^*b_2$) are easy to achieve. To prove the assertion ($\mathbf{C}^*b_3$), we find that the below inequality holds for all $p>1$.

$$|\vartheta_1-\vartheta_3|^p\preceq 2^{p-1}\Big(|\vartheta_1-\vartheta_2|^p+|\vartheta_2-\vartheta_3|^p\Big)$$

Hence, we have

$$d(\vartheta_1,\vartheta_3)\preceq 2^{p-1}\Big(d(\vartheta_1,\vartheta_2)+d(\vartheta_2,\vartheta_3)\Big).$$

Therefore, $(\mathscr{X},\mathbb{A},d)$ is a $\mathbf{C}^*$-algebra $Vb-MSs$.

Remark. [49] When $\mathbb{A}$ is a unital $\mathbf{C}^*$-algebra, then for every $\nu\in\mathbb{A}_+$ we have $\nu\preceq\mathscr{I}\Leftrightarrow\|\nu\|\leq 1$.

Definition 4. *[50] Let $(\mathscr{X},\mathbb{A},d)$ be a $\mathbf{C}^*$-algebra $Vb-MSs$. Then*
(i) $\{\nu_n\}$ converges to $\nu\in\mathscr{X}$ with respect to (shortly w.r.t.) $\mathbb{A}$, if for any $\varepsilon>0$ there is N such that for all $n>N$, $d(\nu_n,\nu)\leq\varepsilon$, and we say that ν is the limit of $\{\nu_n\}$, i.e., $\lim_{n\to\infty}\nu_n=\nu$;
(ii) $\{\nu_n\}$ is called Cauchy sequence w.r.t. $\mathbb{A}$ iff for any $\varepsilon>0$ there is N such that for all $n,m>N$, $\|d(\nu_n,\nu_m)\|\leq\varepsilon$;
(iii) $(\mathscr{X},\mathbb{A},d)$ is called a complete if every Cauchy sequence w.r.t. $\mathbb{A}$ is convergent.

The following results introduced in the work of [51]:

Theorem 1. *Suppose that $\mathbb{A}$ be a $\mathbf{C}^*$-algebra, then:*
(1) The set $\mathbb{A}_+$ is closed cone in $\mathbb{A}$ (a cone $\mathbf{C}$ in a complex or real vector space is a subset closed under addition and scalar multiplication by $\mathbb{R}_+$).
(2) The set $\{\hbar^*\hbar:\hbar\in\mathbb{A}\}$ is equal to the set $\mathbb{A}_+$.
(3) If $0_{\mathbb{A}}\preceq\hbar\preceq\ell$, then $\|\hbar\|\preceq\|\ell\|$.
(4) If $\hbar$ and ℓ are positive invertible elements and $\mathbb{A}$ is unital, then $\hbar\preceq\ell\Rightarrow 0_{\mathbb{A}}\preceq\ell^{-1}\preceq\hbar^{-1}$.

Lemma 1. *Assume that $\mathbb{A}$ be a unital $\mathbf{C}^*$-algebra with a unit $\mathscr{I}$, then:*
(1) If $\hbar\in\mathbb{A}_+$ with $\|\hbar\|<1/2$, then $\mathscr{I}-\hbar$ is invertible and $\left\|\hbar(\mathscr{I}-\hbar)^{-1}\right\|<1$.
(2) Let $\hbar,\ell\in\mathbb{A}$ with $\hbar,\ell\succeq 0_{\mathbb{A}}$ and $\hbar\ell=\ell\hbar$; then $\hbar\ell\succeq 0_{\mathbb{A}}$.
(3) Define $\mathbb{A}'=\{\hbar\in\mathbb{A}:\hbar\ell=\ell\hbar,\ \forall\,\ell\in\mathbb{A}\}$. Suppose that $\hbar\in\mathbb{A}_+$; if $\ell,u\in\mathbb{A}$ with $\ell\succeq u\succeq 0_{\mathbb{A}}$ and $\mathscr{I}-\hbar\in\mathbb{A}'_+$ is invertible operator, then

$$(\mathscr{I}-\hbar)^{-1}\ell\ \succeq\ (\mathscr{I}-\hbar)^{-1}u.$$

In 2003, Kirk et al. [52] investigated mappings that satisfy the cyclic contraction condition in a fixed-point setting.

Definition 5. *Suppose that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty sets of a metric space $(\mathscr{X},d)$. Then*
(i) A mapping $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\longrightarrow\mathscr{P}\cup\mathscr{Q}$ is called a cyclic if $\mathscr{T}(\mathscr{Q})\subseteq\mathscr{P}$ and $\mathscr{T}(\mathscr{P})\subseteq\mathscr{Q}$;
(ii) $\mathscr{T}$ is called cyclic contraction if there exists $h\in(0,1)$ such that $d(\mathscr{T}\nu,\mathscr{T}\varpi)\leq h\,d(\nu,\varpi)$ for all $\nu\in\mathscr{P}$ and $\varpi\in\mathscr{Q}$.

In 2011, Karapinar et al. [37] proposed Kannan-type cyclic contractions and Chatterjea-type cyclic contractions. Furthermore, they summarize some fixed point theorems for cyclic contractions in the complete metric space as follows:

Theorem 2. *(Kannan type). Let $\mathscr{P}$ and $\mathscr{Q}$ be non-empty subsets of a metric spaces $(\mathscr{X},d)$ and $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\longrightarrow\mathscr{P}\cup\mathscr{Q}$ be a cyclic mapping (shortly* **CM***) such that*

$$d(\mathscr{T}\nu,\mathscr{T}\varpi)\leq k\Big[d(\nu,\mathscr{T}\nu)+d(\varpi,\mathscr{T}\varpi)\Big],\qquad \textit{for all}\quad \nu\in\mathscr{P},\ \varpi\in\mathscr{Q},$$

Then $\mathscr{T}$ owns a unique fixed point (shortly **UFP***) in $\mathscr{P}\cap\mathscr{Q}$ provided that $k\in[0,\frac{1}{2})$.*

Theorem 3. *(Chatterjea type). Let $\mathscr{P}$ and $\mathscr{Q}$ be non-empty subsets of a metric spaces $(\mathscr{X},d)$ and $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\longrightarrow\mathscr{P}\cup\mathscr{Q}$ be a* **CM** *such that*

$$d(\mathscr{T}\nu,\mathscr{T}\varpi)\leq k\Big[d(\varpi,\mathscr{T}\nu)+d(\nu,\mathscr{T}\varpi)\Big],\qquad \textit{for all}\quad \nu\in\mathscr{P},\ \varpi\in\mathscr{Q},$$

Then $\mathscr{T}$ owns a **UFP** *in $\mathscr{P}\cap\mathscr{Q}$ provided that $k\in[0,\frac{1}{2})$.*

Definition 6. *[53] Assume that Δ refer to the set of all functions $\Delta:\mathbb{R}_+^4\longrightarrow\mathbb{R}_+$ that are satisfying the following conditions:*
$\Delta_1:$ *$\mho$ is continuous;*
$\Delta_2:$ *$\mho(t_1,t_2,t_3,t_4)=0$ iff $t_1t_2t_3t_4=0$.*

Example 3. [53] The following functions are included of Δ :
(1) $\mho(t_1,t_2,t_3,t_4)=L\min\{t_1,t_2,t_3,t_4\}$, where $L\neq 0$ is a constant;
(2) $\mho(t_1,t_2,t_3,t_4)=t_1t_2t_3t_4$;
(3) $\mho(t_1,t_2,t_3,t_4)=\ln(1+t_1t_2t_3t_4)$;
(4) $\mho(t_1,t_2,t_3,t_4)=e^{t_1t_2t_3t_4}-1$.

Example 4. The following functions are part of Δ :
(1) $\mho(t_1,t_2,t_3,t_4)=L\sin\big(t_1t_2t_3t_4\big)$;
(2) $\mho(t_1,t_2,t_3,t_4)=\sinh\big(t_1t_2t_3t_4\big)$;
(3) $\mho(t_1,t_2,t_3,t_4)=\cos\big(t_1t_2t_3t_4\big)-1$;
(4) $\mho(t_1,t_2,t_3,t_4)=\exp\big(1-\cosh(t_1t_2t_3t_4)\big)-1$;
(5) $\mho(t_1,t_2,t_3,t_4)=\ln\big(1+\tan(t_1t_2t_3t_4)\big)$.

3 Main results

In the first part, we combine the results of Banach and Kannan with continuous function $\mho$ to obtain the following definition in $\mathbf{C}^*$-algebra $Vb-MSs$.

Definition 7. *Let $(\mathscr{X},\mathbb{A},d)$ be a $\mathbf{C}^*$-algebra $Vb-MSs$. We say that a mapping $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\to\mathscr{P}\cup\mathscr{Q}$ is a Δ-cyclic Banach-Kannan type (Δ-***CBKT***) mapping if the following inequality holds:*

$$\begin{aligned}d(\mathscr{T}\nu,\mathscr{T}\varpi)&\preceq\lambda\Big[d(\nu,\varpi)+d(\nu,\mathscr{T}\nu)+d(\varpi,\mathscr{T}\varpi)\Big]\\&\quad+\beta\,\mho\Big(d(\nu,\mathscr{T}\nu),\ d(\varpi,\mathscr{T}\varpi),\ d(\nu,\mathscr{T}\varpi),\ \frac{c\,d(\varpi,\mathscr{T}\nu)}{1+c^2}\Big),\end{aligned}\tag{1}$$

for all $\nu\in\mathscr{P}$, $\varpi\in\mathscr{Q}$, $c\geq 1$ and $\lambda,\beta\in\mathbb{A}_+$ such that $\|\lambda\|<\dfrac{1}{3\,\|b\|}$.

In order to discuss the existence and uniqueness of the fixed point for the mapping $\mathscr{T}$, we give the following theorem:

Theorem 4. *Let $\mathscr{P}$ and $\mathscr{Q}$ be non-empty closed subset of a complete $\mathbf{C}^*$-algebra $Vb-MS$ $(\mathscr{X},\mathbb{A},d)$ and $\mathscr{T}$ be a Δ-***CBKT*** mapping. Then $\mathscr{T}$ possesses a* **UFP** *in $\mathscr{P}\cap\mathscr{Q}$.*

Proof. Let ν_0 be arbitrary element in $\mathscr{P}$. Since $\mathscr{T}$ is **CM**, we get $\mathscr{T}\nu_0 \in \mathscr{Q}$ and $\mathscr{T}^2\nu_0 \in \mathscr{P}$. Set $\nu_{n+1} = \mathscr{T}\nu_n = \mathscr{T}^{n+1}\nu_0,\ n \in \mathbb{N}$, and applying the condition (1), we have

$$\begin{aligned} d(\nu_n,\nu_{n+1}) &= d(\mathscr{T}\nu_{n-1},\mathscr{T}\nu_n) \\ &\preceq \lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_{n-1},\mathscr{T}\nu_{n-1})+d(\nu_n,\mathscr{T}\nu_n)\right] \\ &\quad + \beta\,\mho\left(d(\nu_{n-1},\mathscr{T}\nu_{n-1}),\ d(\nu_n,\mathscr{T}\nu_n),\ d(\nu_{n-1},\mathscr{T}\nu_n),\ \frac{c\,d(\nu_n,\mathscr{T}\nu_{n-1})}{1+c^2}\right) \\ &\preceq \lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_{n-1},\nu_n)+d(\nu_n,\nu_{n+1})\right] \\ &\quad + \beta\,\mho\left(d(\nu_{n-1},\nu_n),\ d(\nu_n,\nu_{n+1}),\ d(\nu_{n-1},\nu_{n+1}),\ 0\right) \\ &\preceq d(\nu_{n-1},\nu_{n+1})+2\lambda\,d(\nu_{n-1},\nu_n)+\lambda\,d(\nu_n,\nu_{n+1}), \end{aligned}$$

that is

$$(\mathscr{I}-\lambda)\,d(\nu_n,\nu_{n+1}) \ \preceq\ 2\lambda\,d(\nu_{n-1},\nu_n),$$

where $\lambda \in \mathbb{A}_+$ and $\|\lambda\| < \dfrac{1}{3\|b\|} < \dfrac{1}{2}$.

Using the condition (1) of Lemma 1, we get $\mathscr{I}-\lambda$ is invertible and $\left\|2\lambda(\mathscr{I}-\lambda)^{-1}\right\| < 1$. Thus

$$d(\nu_n,\nu_{n+1}) \preceq 2\lambda\,(\mathscr{I}-\lambda)^{-1}\,d(\nu_{n-1},\nu_n),$$

this implies that

$$\begin{aligned} d(\nu_n,\nu_{n+1}) &\preceq \mu\,d(\nu_{n-1},\nu_n) \\ &\preceq \mu^2\,d(\nu_{n-2},\nu_{n-1}) \\ &\ \vdots \\ &\preceq \mu^n\,d(\nu_0,\nu_1) = \mu^n\,\xi. \end{aligned}$$

where $\mu = 2\lambda\,(\mathscr{I}-\lambda)^{-1}$ and $\xi = d(\nu_0,\nu_1)$.

Next, we claim that $\{v_m\}$ is Cauchy sequence with respect to $\mathbb{A}$. Assume that $m \geq n$, for every $m, n \in \mathbb{N}$, then we get

$$
\begin{aligned}
d(v_n, v_m) &\preceq b\, d(v_n, v_{n+1}) + b^2\, d(v_{n+1}, v_{n+2}) + \ldots + b^{m-n}\, d(v_{m-1}, v_m) \\
&\preceq b\, \mu^n\, \xi + b^2\, \mu^{n+1}\, \xi + \ldots + b^{m-n}\, \mu^{m-1}\, \xi \\
&= \sum_{k=n}^{m-1} b^{k-n+1}\, \mu^k\, \xi. \\
&= \sum_{k=n}^{m-1} \left| b^{\frac{k-n+1}{2}}\, \mu^{\frac{k}{2}}\, \xi^{\frac{1}{2}} \right|^2 \\
&\preceq \left\| \sum_{k=n}^{m-1} \left| b^{\frac{k-n+1}{2}}\, \mu^{\frac{k}{2}}\, \xi^{\frac{1}{2}} \right|^2 \right\| \mathscr{I} \\
&\preceq \sum_{k=n}^{m-1} \left\| b^{\frac{k-n+1}{2}} \right\|^2 \left\| \mu^{\frac{k}{2}} \right\|^2 \left\| \xi^{\frac{1}{2}} \right\|^2 \mathscr{I} \\
&= \|\xi\| \sum_{k=n}^{m-1} \|b\|^{k-n+1}\, \|\mu\|^k\, \mathscr{I} \\
&\preceq \|\xi\| \sum_{k=n}^{m-1} \|b\|^k\, \|\mu\|^k\, \mathscr{I} \\
&= \|\xi\| \sum_{k=n}^{m-1} (\|b\|\, \|\mu\|)^k\, \mathscr{I} \\
&\preceq \|\xi\| \sum_{k=n}^{\infty} (\|b\|\, \|\mu\|)^k\, \mathscr{I} \\
&\preceq \|\xi\| \frac{(\|b\|\, \|\mu\|)^n}{1 - (\|b\|\, \|\mu\|)} \mathscr{I}.
\end{aligned}
$$

Consider that

$$
\begin{aligned}
\|b\|\, \|\mu\| &= \|b\| \left\| 2\lambda\, (\mathscr{I} - \lambda)^{-1} \right\| \leq 2\, \|b\|\, \|\lambda\|\, \left\| (\mathscr{I} - \lambda)^{-1} \right\| \\
&= 2\, \|b\|\, \|\lambda\| \left\| \sum_{j=0}^{\infty} (\lambda)^j \right\| \leq 2\, \|b\|\, \|\lambda\| \sum_{j=0}^{\infty} \|(\lambda)\|^j \\
&< 2\, \|b\| \left(\frac{1}{3\, \|b\|} \right) \frac{1}{1 - \|\lambda\|} < \frac{2}{3} \frac{1}{1 - \frac{1}{3}} = 1.
\end{aligned}
$$

Thus

$$
\|\xi\| \frac{(\|b\|\, \|\mu\|)^n}{1 - (\|b\|\, \|\mu\|)} \mathscr{I} \longrightarrow 0_{\mathbb{A}} \quad \text{as} \quad n \longrightarrow \infty.
$$

Consequently, $\{v_m\}$ is Cauchy sequence w.r.t. $\mathbb{A}$. From the completeness of $(\mathscr{X}, \mathbb{A}, d)$, there exists an element $v \in \mathscr{X}$ such that $\lim_{m \to \infty} v_m = v$.

Since $\{v_{2m}\}$ is a sequence in $\mathscr{P}$ and $\{v_{2m-1}\}$ in $\mathscr{Q}$, we find that both sequences converge to the same limit v. Furthermore, since $\mathscr{P}$ and $\mathscr{Q}$ are closed sets, this leads to $v \in \mathscr{P} \cap \mathscr{Q}$.

Now, consider

$$
\begin{aligned}
d(\mathscr{T}\nu,\nu) &\preceq b\left[d(\mathscr{T}\nu,\mathscr{T}\nu_{2m})+d(\mathscr{T}\nu_{2m},\nu)\right]\\
&= b\,d(\mathscr{T}\nu,\mathscr{T}(\mathscr{T}\nu_{2m-1}))+b\,d(\mathscr{T}\nu_{2m},\nu)\\
&\preceq b\lambda\left[d(\nu,\mathscr{T}\nu_{2m-1})+d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu)+d(\nu,\mathscr{T}\nu_{2m})\right]\\
&\quad +b\,\beta\,\mho\Big(d(\nu,\mathscr{T}\nu),\ d(\mathscr{T}\nu_{2m-1},\mathscr{T}(\mathscr{T}\nu_{2m-1})),\ d(\nu,\mathscr{T}(\mathscr{T}\nu_{2m-1})),\ \frac{c\,d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu)}{1+c^2}\Big)\\
&\quad +b\,d(\mathscr{T}\nu_{2m},\nu)\\
&= b\lambda\left[d(\nu,\nu_{2m})+d(\nu_{2m},\mathscr{T}\nu)+d(\nu,\nu_{2m+1})\right]\\
&\quad +b\,\beta\,\mho\Big(d(\nu,\mathscr{T}\nu),\ d(\nu_{2m},\nu_{2m+1}),\ d(\nu,\nu_{2m+1}),\ \frac{c\,d(\nu_{2m},\mathscr{T}\nu)}{1+c^2}\Big)\ +\ b\,d(\nu_{2m+1},\nu).
\end{aligned}
$$

Taking $m\longrightarrow\infty$, we get

$$
\begin{aligned}
d(\mathscr{T}\nu,\nu) &\preceq \|b\|\,\|\lambda\|\,d(\mathscr{T}\nu,\nu)\\
&\prec \|b\|\left(\frac{1}{3\,\|b\|}\right)d(\mathscr{T}\nu,\nu)=\frac{1}{3}\,d(\mathscr{T}\nu,\nu),
\end{aligned}
$$

which is a contradiction, then $d(\mathscr{T}\nu,\nu)=0_{\mathbb{A}}$, that is, $\nu=\mathscr{T}\nu$. This means ν is a fixed point (**FP**) of $\mathscr{T}$. For uniqueness, let ϖ is another fixed point (**FP**) of $\mathscr{T}$ such that $\varpi\neq\nu$, then

$$
\begin{aligned}
0_{\mathbb{A}}\preceq d(\nu,\varpi)=d(\mathscr{T}\nu,\mathscr{T}\varpi) &\preceq \lambda\left[d(\nu,\varpi)+d(\nu,\mathscr{T}\nu)+d(\varpi,\mathscr{T}\varpi)\right]\\
&\quad +\beta\,\mho\Big(d(\nu,\mathscr{T}\nu),\ d(\varpi,\mathscr{T}\varpi),\ d(\nu,\mathscr{T}\varpi),\ \frac{c\,d(\varpi,\mathscr{T}\nu)}{1+c^2}\Big)\\
&= \lambda\,d(\nu,\varpi)\\
&\preceq \|\lambda\|\,d(\nu,\varpi)\\
&\prec \left(\frac{1}{3\,\|b\|}\right)d(\nu,\varpi)\ \prec\ d(\nu,\varpi),
\end{aligned}
$$

a contradiction again where $\|b\|>1$. Hence $\nu=\varpi$ is a **UFP** in $\mathscr{P}\cap\mathscr{Q}$ and this completes the proof.

If we combine the results of Banach and Kannan only, we will get the following definition and corollary in $\mathbf{C}^*$-algebra $Vb-MSs$.

Definition 8. *Let $(\mathscr{X},\mathbb{A},d)$ be a $\mathbf{C}^*$-algebra $Vb-MSs$. We say that a mapping $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\to\mathscr{P}\cup\mathscr{Q}$ is a cyclic Banach-Kannan type (**CBKT**) mapping if the following inequality holds:*

$$
d(\mathscr{T}\nu,\mathscr{T}\varpi)\preceq\lambda\left[d(\nu,\varpi)+d(\nu,\mathscr{T}\nu)+d(\varpi,\mathscr{T}\varpi)\right], \tag{2}
$$

for all $\nu\in\mathscr{P}$, $\varpi\in\mathscr{Q}$ and $\lambda\in\mathbb{A}_+$ such that $\|\lambda\|<\dfrac{1}{3\,\|b\|}$.

Corollary 1. *Let $\mathscr{P}$ and $\mathscr{Q}$ be non-empty closed subset of a complete $\mathbf{C}^*$-algebra $Vb-MS$ $(\mathscr{X},\mathbb{A},d)$ and $\mathscr{T}$ be a **CBKT** mapping. Then $\mathscr{T}$ possesses a **UFP** in $\mathscr{P}\cap\mathscr{Q}$.*

In the second part, we merge a continuous function $\mho$ with the results of Banach and Chatterjea to get the following definition in $\mathbf{C}^*$-algebra $Vb-MSs$:

Definition 9. *Assume that $(\mathscr{X},\mathbb{A},d)$ be a $\mathbf{C}^*$-algebra $Vb-MSs$. A mapping $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\to\mathscr{P}\cup\mathscr{Q}$ is called a a Δ-cyclic Banach-Chatterjea type (Δ-**CBCT**) mapping if the following inequality verifies:*

$$
\begin{aligned}
d(\mathscr{T}\nu,\mathscr{T}\varpi) &\preceq \lambda\left[d(\nu,\varpi)+d(\varpi,\mathscr{T}\nu)+d(\nu,\mathscr{T}\varpi)\right],\\
&\quad +\beta\,\mho\Big(d(\nu,\mathscr{T}\nu),\ d(\varpi,\mathscr{T}\varpi),\ d(\nu,\mathscr{T}\varpi),\ \frac{c\,d(\varpi,\mathscr{T}\nu)}{1+c^2}\Big),
\end{aligned} \tag{3}
$$

for all $\nu\in\mathscr{P}$, $\varpi\in\mathscr{Q}$, $c\geq 1$ and $\lambda,\beta\in\mathbb{A}_+$ such that $\|\lambda\|<\dfrac{1}{3\,\|b\|\,(1+2\,\|b\|)}$.

For discussing the existence and uniqueness of the **FP** for the mapping $\mathscr{T}$, we introduce the following theorem:

Theorem 5. *Assume that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty closed subset of a complete $\mathbf{C}^*$-algebra $Vb-MSs$ $(\mathscr{X},\mathbb{A},d)$ and $\mathscr{T}$ be a Δ-**CBCT** mapping. Then $\mathscr{T}$ possesses a **UFP** in $\mathscr{P}\cap\mathscr{Q}$.*

Proof. Let ν_0 be arbitrary element in $\mathscr{P}$. Since $\mathscr{T}$ is **CM**, we get $\mathscr{T}\nu_0\in\mathscr{Q}$ and $\mathscr{T}^2\nu_0\in\mathscr{P}$. Set $\nu_{n+1}=\mathscr{T}\nu_n=\mathscr{T}^{n+1}\nu_0,\ n\in\mathbb{N}$, and applying the condition (3), we have

$$\begin{aligned}
d(\nu_n,\nu_{n+1}) &= d(\mathscr{T}\nu_{n-1},\mathscr{T}\nu_n)\\
&\preceq \lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_n,\mathscr{T}\nu_{n-1})+d(\nu_{n-1},\mathscr{T}\nu_n)\right]\\
&\quad+\beta\,\mho\left(d(\nu_{n-1},\mathscr{T}\nu_{n-1}),\ d(\nu_n,\mathscr{T}\nu_n),\ d(\nu_{n-1},\mathscr{T}\nu_n),\ \frac{c\,d(\nu_n,\mathscr{T}\nu_{n-1})}{1+c^2}\right)\\
&\preceq \lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_n,\nu_n)+d(\nu_{n-1},\nu_{n+1})\right]\\
&\quad+\beta\,\mho\left(d(\nu_{n-1},\nu_n),\ d(\nu_n,\nu_{n+1}),\ d(\nu_{n-1},\nu_{n+1}),\ 0\right)\\
&\preceq \lambda\,d(\nu_{n-1},\nu_n)+b\lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_n,\nu_{n+1})\right],
\end{aligned}$$

that is

$$(\mathscr{I}-b\lambda)\,d(\nu_n,\nu_{n+1})\preceq(\mathscr{I}+b)\,\lambda\,d(\nu_{n-1},\nu_n),$$

where $b,\lambda\in\mathbb{A}_+$ with $\lambda<\dfrac{1}{3\|b\|(1+2\|b\|)}<\dfrac{1}{2}$.

Taking the condition (2) of Lemma 1 in consideration, we find

$$(\mathscr{I}+b)\,\lambda\in\mathbb{A}_+\quad\text{and}\quad\|b\lambda\|<\|b\|\frac{1}{3\|b\|(1+2\|b\|)}<\frac{1}{2}.$$

By the condition (1) of Lemma 1, we obtain

$$(\mathscr{I}-\lambda)^{-1}\in\mathbb{A}_+\quad\text{and}\quad(\mathscr{I}+b)\,\lambda\,(\mathscr{I}-b\lambda)^{-1}\in\mathbb{A}_+,$$

with

$$\left\|(\mathscr{I}+b)\,\lambda\,(\mathscr{I}-\lambda)^{-1}\right\|<1.$$

Then, from the condition (3) of Lemma 1, we have

$$d(\nu_n,\nu_{n+1})\preceq(\mathscr{I}+b)\,\lambda\,(\mathscr{I}-b\lambda)^{-1}\,d(\nu_{n-1},\nu_n).$$

that is

$$\begin{aligned}
d(\nu_n,\nu_{n+1}) &\preceq \eta\,d(\nu_{n-1},\nu_n)\\
&\preceq \eta^2\,d(\nu_{n-2},\nu_{n-1})\\
&\ \ \vdots\\
&\preceq \eta^n\,d(\nu_0,\nu_1)=\eta^n\,\xi.
\end{aligned}$$

where $\eta=(\mathscr{I}+b)\,\lambda\,(\mathscr{I}-b\lambda)^{-1}$ and $\xi=d(\nu_0,\nu_1)$.

Next, we claim that $\{\nu_m\}$ is Cauchy sequence with respect to $\mathbb{A}$. Assume that $m \geq n$, for every $m,n \in \mathbb{N}$, then we obtain

$$\begin{aligned}
d(\nu_n,\nu_m) &\preceq b\,d(\nu_n,\nu_{n+1}) + b^2\,d(\nu_{n+1},\nu_{n+2}) + \ldots + b^{m-n}\,d(\nu_{m-1},\nu_m) \\
&\preceq b\,\eta^n\,\xi + b^2\,\eta^{n+1}\,\xi + \ldots + b^{m-n}\,\eta^{m-1}\,\xi \\
&= \sum_{k=n}^{m-1} b^{m-n}\eta^k\,\xi. \\
&= \sum_{k=n}^{m-1} \left| b^{\frac{k-n+1}{2}}\,\eta^{\frac{k}{2}}\,\xi^{\frac{1}{2}} \right|^2 \\
&\preceq \|\xi\| \sum_{k=n}^{m-1} \|b\|^{k-n+1}\,\|\eta\|^k\,\mathscr{I} \\
&\preceq \|\xi\| \sum_{k=n}^{m-1} \|b\|^k\,\|\eta\|^k\,\mathscr{I} \\
&= \|\xi\| \sum_{k=n}^{m-1} (\|b\|\,\|\eta\|)^k\,\mathscr{I} \\
&\preceq \|\xi\| \sum_{k=n}^{\infty} (\|b\|\,\|\eta\|)^k\,\mathscr{I} \\
&\preceq \|\xi\|\,\frac{(\|b\|\,\|\eta\|)^n}{1-(\|b\|\,\|\eta\|)}\,\mathscr{I}.
\end{aligned}$$

Consider that

$$\begin{aligned}
\|b\|\,\|\eta\| = \|b\|\left\|(\mathscr{I}+b)\,\lambda\,(\mathscr{I}-b\lambda)^{-1}\right\| &\leq \|b\|\,\|(\mathscr{I}+b)\|\,\|\lambda\|\left\|(\mathscr{I}-b\lambda)^{-1}\right\| \\
&\leq \|b\|\,(1+\|b\|)\,\|\lambda\|\left\|\sum_{j=0}^{\infty}(b\lambda)^j\right\| < \|b\|\,(1+\|b\|)\,\frac{1}{3\,\|b\|\,(1+2\,\|b\|)}\sum_{j=0}^{\infty}\|(b\lambda)\|^j \\
&< \|b\|\,(1+\|b\|)\,\frac{1}{3\,\|b\|\,(1+\|b\|)}\,\frac{1}{1-\|b\lambda\|} < \frac{1}{2}\,\frac{1}{1-\frac{1}{2}} = 1.
\end{aligned}$$

Thus

$$\|\xi\|\,\frac{(\|b\|\,\|\eta\|)^n}{1-(\|b\|\,\|\eta\|)}\,\mathscr{I} \longrightarrow 0_{\mathbb{A}} \quad \text{as} \quad n \longrightarrow \infty.$$

Consequently, $\{\nu_m\}$ is Cauchy sequence w.r.t. $\mathbb{A}$. From the completeness of $(\mathscr{X},\mathbb{A},d)$, there exists an element $\nu \in \mathscr{X}$ such that $\lim_{m\to\infty} \nu_m = \nu$.

Since $\{\nu_{2m}\}$ is a sequence in $\mathscr{P}$ and $\{\nu_{2m-1}\}$ in $\mathscr{Q}$, we find that both sequences converge to the same limit ν. Furthermore, since $\mathscr{P}$ and $\mathscr{Q}$ are closed sets, this leads to $\nu \in \mathscr{P} \cap \mathscr{Q}$.

Now, consider

$$
\begin{aligned}
d(\mathscr{T}\nu,\nu) &\preceq b\left[d(\mathscr{T}\nu,\mathscr{T}\nu_{2m})+d(\mathscr{T}\nu_{2m},\nu)\right]\\
&= b\,d(\mathscr{T}\nu,\mathscr{T}(\mathscr{T}\nu_{2m-1}))+b\,d(\mathscr{T}\nu_{2m},\nu)\\
&\preceq b\lambda\left[d(\nu,\mathscr{T}\nu_{2m-1})+d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu)+d(\nu,\mathscr{T}\nu_{2m})\right]\\
&\quad + b\,\beta\,\mho\left(d(\nu,\mathscr{T}\nu),\ d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu_{2m})),\ d(\nu,\mathscr{T}\nu_{2m}),\ \frac{c\,d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu)}{1+c^2}\right)\\
&\quad + b\,d(\mathscr{T}\nu_{2m},\nu)\\
&\preceq b\lambda\left[d(\nu,\nu_{2m})+d(\nu_{2m},\mathscr{T}\nu)+d(\nu,\nu_{2m+1})\right]\\
&\quad + b\,\beta\,\mho\left(d(\nu,\mathscr{T}\nu),\ d(\nu_{2m},\nu_{2m+1}),\ d(\nu,\nu_{2m+1}),\ \frac{c\,d(\nu_{2m},\mathscr{T}\nu)}{1+c^2}\right)\\
&\quad + b\,d(\nu_{2m+1},\nu).
\end{aligned}
$$

Putting $m\longrightarrow\infty$, we get

$$
\begin{aligned}
d(\mathscr{T}\nu,\nu) &\preceq b\lambda\,d(\mathscr{T}\nu,\nu)\\
&\preceq \|b\|\,\|\lambda\|\,d(\mathscr{T}\nu,\nu)\\
&\prec \|b\|\left(\frac{1}{3\,\|b\|\,(1+2\,\|b\|)}\right)d(\mathscr{T}\nu,\nu)\ \prec\ \frac{1}{3}\,d(\mathscr{T}\nu,\nu),
\end{aligned}
$$

which is a contradiction, then $d(\mathscr{T}\nu,\nu)=0_{\mathbb{A}}$, that is, $\nu=\mathscr{T}\nu$. This means ν is a **FP** of $\mathscr{T}$. For uniqueness, let ϖ is another **FP** of $\mathscr{T}$ such that $\varpi\neq\nu$, then

$$
\begin{aligned}
0_{\mathbb{A}}\ \preceq\ d(\nu,\varpi)\ =\ d(\mathscr{T}\nu,\mathscr{T}\varpi) &\preceq \lambda\left[d(\nu,\varpi)+d(\varpi,\mathscr{T}\nu)+d(\nu,\mathscr{T}\varpi)\right]\\
&\quad + \beta\,\mho\left(d(\nu,\mathscr{T}\nu),\ d(\varpi,\mathscr{T}\varpi),\ d(\nu,\mathscr{T}\varpi),\ \frac{c\,d(\varpi,\mathscr{T}\nu)}{1+c^2}\right)\\
&= 3\lambda\,d(\nu,\varpi)\\
&\preceq 3\,\|\lambda\|\,d(\nu,\varpi)\\
&\prec 3\left(\frac{1}{3\,\|b\|\,(1+2\,\|b\|)}\right)d(\nu,\varpi)\ \prec\ d(\nu,\varpi),
\end{aligned}
$$

a contradiction again. Thus, $\nu=\varpi$ and the proof is complete.

If we merge a continuous function $\mho$ with the results of Banach and Chatterjea, we will get a new definition and corollary in $\mathbf{C}^*$-algebra $Vb-MSs$:

Definition 10. *Assume that $(\mathscr{X},\mathbb{A},d)$ be a $\mathbf{C}^*$-algebra $Vb-MSs$. A mapping $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\to\mathscr{P}\cup\mathscr{Q}$ is called a cyclic Banach-Chatterjea type (**CBCT**) mapping if the following inequality verifies:*

$$
d(\mathscr{T}\nu,\mathscr{T}\varpi)\preceq\lambda\left[d(\nu,\varpi)+d(\varpi,\mathscr{T}\nu)+d(\nu,\mathscr{T}\varpi)\right], \tag{4}
$$

for all $\nu\in\mathscr{P}$, $\varpi\in\mathscr{Q}$, $c\geq 1$ and $\lambda\in\mathbb{A}_+$ such that $\|\lambda\|<\dfrac{1}{3\,\|b\|\,(1+2\,\|b\|)}$.

Corollary 2. *Assume that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty closed subset of a complete $\mathbf{C}^*$-algebra $Vb-MSs$ $(\mathscr{X},\mathbb{A},d)$ and $\mathscr{T}$ be a **CBCT** mapping. Then $\mathscr{T}$ possesses a **UFP** in $\mathscr{P}\cap\mathscr{Q}$.*

In the third part, we combine a continuous function $\mho$ with the results of Banach, Kannan and Chatterjea to obtain the following definition in $\mathbf{C}^*$-algebra $Vb-MSs$.

Definition 11. *Suppose that* $(\mathscr{X},\mathbb{A},d)$ *be a* $\mathbf{C}^*$*-algebra* $Vb-MSs$*. Then* $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\to\mathscr{P}\cup\mathscr{Q}$ *is called a* Δ*-cyclic Banach-Kannan-Chatterjea type (*Δ*-***CBKCT***) mapping if the following inequality satisfies: For all* $\nu\in\mathscr{P}$, $\varpi\in\mathscr{Q}$,

$$d(\mathscr{T}\nu,\mathscr{T}\varpi)\preceq\lambda\left[d(\nu,\varpi)+d(\nu,\mathscr{T}\nu)+d(\varpi,\mathscr{T}\varpi)+d(\varpi,\mathscr{T}\nu)+d(\nu,\mathscr{T}\varpi)\right]$$
$$+\beta\,\mho\left(d(\nu,\mathscr{T}\nu),\,d(\varpi,\mathscr{T}\varpi),\,d(\nu,\mathscr{T}\varpi),\,\frac{c\,d(\varpi,\mathscr{T}\nu)}{1+c^2}\right),\tag{5}$$

where $c\geq 1$ *and* $\lambda,\beta\in\mathbb{A}_+$ *such that* $\|\lambda\|<\dfrac{1}{3\,\|b\|\,(2+\|b\|)}$.

To discuss the existence and uniqueness of the **FP** for the mapping $\mathscr{T}$, we present the following theorem:

Theorem 6. *Suppose that* $\mathscr{P}$ *and* $\mathscr{Q}$ *are non-empty closed subset of a complete* $\mathbf{C}^*$*-algebra* $Vb-MS$ $(\mathscr{X},\mathbb{A},d)$ *and* $\mathscr{T}$ *be a* Δ*-***CBKCT** *mapping. Then,* $\mathscr{T}$ *has a* **UFP** *in* $\mathscr{P}\cap\mathscr{Q}$.

Proof. Let ν_0 be arbitrary element in $\mathscr{P}$. Since $\mathscr{T}$ is **CM**, we get $\mathscr{T}\nu_0\in\mathscr{Q}$ and $\mathscr{T}^2\nu_0\in\mathscr{P}$. Set $\nu_{n+1}=\mathscr{T}\nu_n=\mathscr{T}^{n+1}\nu_0,\ n\in\mathbb{N}$, and applying the condition (5), we have

$$\begin{aligned}
d(\nu_n,\nu_{n+1}) &= d(\mathscr{T}\nu_{n-1},\mathscr{T}\nu_n)\\
&\preceq \lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_{n-1},\mathscr{T}\nu_{n-1})+d(\nu_n,\mathscr{T}\nu_n)+d(\nu_n,\mathscr{T}\nu_{n-1})+d(\nu_{n-1},\mathscr{T}\nu_n)\right]\\
&\quad+\beta\,\mho\left(d(\nu_{n-1},\mathscr{T}\nu_{n-1}),\,d(\nu_n,\mathscr{T}\nu_n),\,d(\nu_{n-1},\mathscr{T}\nu_n),\,\frac{c\,d(\nu_n,\mathscr{T}\nu_{n-1})}{1+c^2}\right)\\
&= \lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_{n-1},\nu_n)+d(\nu_n,\nu_{n+1})+d(\nu_n,\nu_n)+d(\nu_{n-1},\nu_{n+1})\right]\\
&\quad+\beta\,\mho\left(d(\nu_{n-1},\nu_n),\,d(\nu_n,\nu_{n+1}),\,d(\nu_{n-1},\nu_{n+1}),\,0\right)\\
&\preceq 2\,\lambda\,d(\nu_{n-1},\nu_n)+\lambda\,d(\nu_n,\nu_{n+1})+b\,\lambda\left[d(\nu_{n-1},\nu_n)+d(\nu_n,\nu_{n+1})\right]\\
&= (2+b)\,\lambda\,d(\nu_{n-1},\nu_n)+(\mathscr{I}+b)\,\lambda\,d(\nu_n,\nu_{n+1}),
\end{aligned}$$

that is,

$$\left(\mathscr{I}-(\mathscr{I}+b)\,\lambda\right)d(\nu_n,\nu_{n+1})\ \preceq\ (2+b)\,\lambda\,d(\nu_{n-1},\nu_n),$$

where $\mathscr{I}+b,\ \lambda\in\mathbb{A}_+$. Taking the condition (2) of Lemma 1 in consideration, we find

$$(\mathscr{I}+b)\,\lambda\in\mathbb{A}_+\quad\text{and}\quad\|b\lambda\|<\|b\|\,\frac{1}{3\,\|b\|\,(2+\|b\|)}<\frac{1}{2}.$$

By the condition (1) of Lemma 1, we obtain

$$\left(\mathscr{I}-(\mathscr{I}+b)\,\lambda\right)^{-1}\in\mathbb{A}_+\quad\text{and}\quad(2+b)\,\lambda\left(\mathscr{I}-(\mathscr{I}+b)\,\lambda\right)^{-1}\in\mathbb{A}_+,$$

with

$$\left\|(2+b)\,\lambda\left(\mathscr{I}-(\mathscr{I}+b)\,\lambda\right)^{-1}\right\|<1.$$

Then, from the condition (3) of Lemma 1, we have

$$d(\nu_n,\nu_{n+1})\ \preceq\ (b+2)\,\lambda\left(\mathscr{I}-(\mathscr{I}+b)\,\lambda\right)^{-1}d(\nu_{n-1},\nu_n).$$

that is

$$\begin{aligned}
d(\nu_n,\nu_{n+1}) &\preceq \delta\,d(\nu_{n-1},\nu_n)\\
&\preceq \delta^2\,d(\nu_{n-2},\nu_{n-1})\\
&\ \ \vdots\\
&\preceq \delta^n\,d(\nu_0,\nu_1)=\delta^n\,\xi.
\end{aligned}$$

where $\delta = (\mathscr{I}+b)\,\lambda\,(\mathscr{I}-b\lambda)^{-1}$ and $\xi = d(\nu_0,\nu_1)$.

Next, we show that $\{\nu_m\}$ is Cauchy sequence w.r.t. $\mathbb{A}$. Assume that $m \geq n$, for every $m,n \in \mathbb{N}$, then we get

$$\begin{aligned} d(\nu_n,\nu_m) &\preceq b\,d(\nu_n,\nu_{n+1}) + b^2\,d(\nu_{n+1},\nu_{n+2}) + \ldots + b^{m-n}\,d(\nu_{m-1},\nu_m) \\ &\preceq b\,\delta^n\,\xi + b^2\,\delta^{n+1}\,\xi + \ldots + b^{m-n}\,\delta^{m-1}\,\xi \\ &= \sum_{k=n}^{m-1} b^{m-n}\delta^k\,\xi. \\ &= \sum_{k=n}^{m-1} \left| b^{\frac{k-n+1}{2}}\,\delta^{\frac{k}{2}}\,\xi^{\frac{1}{2}} \right|^2 \\ &\preceq \|\xi\| \sum_{k=n}^{m-1} \|b\|^{k-n+1}\,\|\delta\|^k\,\mathscr{I} \\ &\preceq \|\xi\| \sum_{k=n}^{m-1} \|b\|^k\,\|\delta\|^k\,\mathscr{I} \\ &= \|\xi\| \sum_{k=n}^{m-1} (\|b\|\,\|\delta\|)^k\,\mathscr{I} \\ &\preceq \|\xi\| \sum_{k=n}^{\infty} (\|b\|\,\|\delta\|)^k\,\mathscr{I} \\ &\preceq \|\xi\|\,\frac{(\|b\|\,\|\delta\|)^n}{1-(\|b\|\,\|\delta\|)}\,\mathscr{I}. \end{aligned}$$

Consider that

$$\begin{aligned} \|b\|\,\|\delta\| &= \|b\| \left\| (2+b)\,\lambda \left(\mathscr{I}-(\mathscr{I}+b)\,\lambda\right)^{-1} \right\| \;\leq\; \|b\|\,\|(2+b)\|\,\|\lambda\| \left\| \left(\mathscr{I}-(\mathscr{I}+b)\,\lambda\right)^{-1} \right\| \\ &\leq \|b\|\,(2+\|b\|)\,\|\lambda\| \left\| \sum_{j=0}^{\infty} \left((\mathscr{I}+b)\,\lambda\right)^j \right\| \\ &< \|b\|\,(2+\|b\|) \left(\frac{1}{3\,\|b\|\,(2+\|b\|)} \right) \sum_{j=0}^{\infty} \|(\mathscr{I}+b)\,\lambda\|^j \\ &< \frac{1}{2}\,\frac{1}{1-\|(\mathscr{I}+b)\,\lambda\|} < \frac{1}{2}\,\frac{1}{1-\frac{1}{2}} = 1. \end{aligned}$$

Thus

$$\|\xi\|\,\frac{(\|b\|\,\|\delta\|)^n}{1-(\|b\|\,\|\delta\|)}\,\mathscr{I} \longrightarrow 0_{\mathbb{A}} \qquad \text{as} \quad n \longrightarrow \infty.$$

Consequently, $\{\nu_m\}$ is Cauchy sequence w.r.t. $\mathbb{A}$. From the completeness of $(\mathscr{X},\mathbb{A},d)$, there exists an element $\nu \in \mathscr{X}$ such that $\lim_{m\to\infty} \nu_m = \nu$.

Since $\{\nu_{2m}\}$ is a sequence in $\mathscr{P}$ and $\{\nu_{2m-1}\}$ in $\mathscr{Q}$, we find that both sequences converge to the same limit ν. Furthermore, since $\mathscr{P}$ and $\mathscr{Q}$ are closed sets, this leads to $\nu \in \mathscr{P} \cap \mathscr{Q}$.

Now, consider

$$
\begin{aligned}
d(\mathscr{T}\nu,\nu) &\preceq b\left[d(\mathscr{T}\nu,\mathscr{T}\nu_{2m})+d(\mathscr{T}\nu_{2m},\nu)\right]\\
&= b\,d(\mathscr{T}\nu,\mathscr{T}(\mathscr{T}\nu_{2m-1}))+b\,d(\mathscr{T}\nu_{2m},\nu)\\
&\preceq b\,\lambda\left[d(\nu,\mathscr{T}\nu_{2m-1})+d(\nu,\mathscr{T}\nu)+d(\mathscr{T}\nu_{2m-1},\mathscr{T}(\mathscr{T}\nu_{2m-1}))\right.\\
&\qquad\left.+d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu)+d(\nu,\mathscr{T}(\mathscr{T}\nu_{2m-1}))\right]\\
&\quad+\ b\,\beta\,\mho\left(d(\nu,\mathscr{T}\nu),\ d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu_{2m}),\ d(\nu,\mathscr{T}\nu_{2m}),\ \frac{c\,d(\mathscr{T}\nu_{2m-1},\mathscr{T}\nu)}{1+c^2}\right)\\
&\quad+b\,d(\mathscr{T}\nu_{2m},\nu)\\
&\preceq b\,\lambda\,d(\nu,\nu_{2m})+b\,\lambda\,d(\nu,\mathscr{T}\nu)+b\,\lambda\,d(\nu_{2m},\nu_{2m+1})+b\,\lambda\,d(\nu_{2m},\mathscr{T}\nu)+b\,\lambda\,d(\nu,\nu_{2m+1})\\
&\quad+\ b\,\beta\,\mho\left(d(\nu,\mathscr{T}\nu),\ d(\nu_{2m},\nu_{2m+1}),\ d(\nu,\nu_{2m+1}),\ \frac{c\,d(\nu_{2m},\mathscr{T}\nu)}{1+c^2}\right)\\
&\quad+b\,d(\nu_{2m+1},\nu).
\end{aligned}
$$

Taking $m\longrightarrow\infty$, we obtain

$$
\begin{aligned}
d(\mathscr{T}\nu,\nu) &\preceq 2\,b\,\lambda\,d(\mathscr{T}\nu,\nu)\\
&\preceq 2\,\|b\|\,\|\lambda\|\,d(\mathscr{T}\nu,\nu)\\
&\prec 2\,\|b\|\left(\frac{1}{3\,\|b\|\,(2+\|b\|)}\right)d(\mathscr{T}\nu,\nu)\\
&\prec 2\left(\frac{1}{2}\right)d(\mathscr{T}\nu,\nu)=d(\mathscr{T}\nu,\nu),
\end{aligned}
$$

which is contradiction, then $d(\mathscr{T}\nu,\nu)=0_{\mathbb{A}}$ and so $\nu=\mathscr{T}\nu$. This means ν is a **FP** of $\mathscr{T}$. Now, let ϖ is another **FP** of $\mathscr{T}$ such that $\varpi\neq\nu$, then

$$
\begin{aligned}
0_{\mathbb{A}} \preceq d(\nu,\varpi) = d(\mathscr{T}\nu,\mathscr{T}\varpi) &\preceq \lambda\left[d(\nu,\varpi)+d(\nu,\mathscr{T}\nu)+d(\varpi,\mathscr{T}\varpi)+d(\varpi,\mathscr{T}\nu)+d(\nu,\mathscr{T}\varpi)\right]\\
&\quad+\beta\,\mho\left(d(\nu,\mathscr{T}\nu),\ d(\varpi,\mathscr{T}\varpi),\ d(\nu,\mathscr{T}\varpi),\ \frac{c\,d(\varpi,\mathscr{T}\nu)}{1+c^2}\right)\\
&= 3\,\lambda\,d(\nu,\varpi)\\
&\preceq 3\,\|\lambda\|\,d(\nu,\varpi)\\
&\prec 3\left(\frac{1}{3\,\|b\|\,(2+\|b\|)}\right)d(\nu,\varpi)\ \prec\ d(\nu,\varpi)
\end{aligned}
$$

which lead to contradiction. Thus, $\nu=\varpi$ then we completed the proof.

If we combine the results of Banach, Kannan and Chatterjea only, we will obtain the following definition and corollary in $\mathbf{C}^*$-algebra $Vb-MSs$.

Definition 12. *Suppose that $(\mathscr{X},\mathbb{A},d)$ be a $\mathbf{C}^*$-algebra $Vb-MSs$. Then $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\to\mathscr{P}\cup\mathscr{Q}$ is called a cyclic Banach-Kannan-Chatterjea type (**CBKCT**) mapping if the following inequality satisfies: For all $\nu\in\mathscr{P}$, $\varpi\in\mathscr{Q}$,*

$$
d(\mathscr{T}\nu,\mathscr{T}\varpi)\preceq\lambda\left[d(\nu,\varpi)+d(\nu,\mathscr{T}\nu)+d(\varpi,\mathscr{T}\varpi)+d(\varpi,\mathscr{T}\nu)+d(\nu,\mathscr{T}\varpi)\right], \tag{6}
$$

where $c\geq 1$ and $\lambda\in\mathbb{A}_+$ such that $\|\lambda\|<\dfrac{1}{3\,\|b\|\,(2+\|b\|)}$.

Corollary 3. *Suppose that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty closed subset of a complete $\mathbf{C}^*$-algebra $Vb-MS$ $(\mathscr{X},\mathbb{A},d)$ and $\mathscr{T}$ be a **CBKCT** mapping. Then, $\mathscr{T}$ has a **UFP** in $\mathscr{P}\cap\mathscr{Q}$.*

Now, we present nontrivial examples to support Corollary 2 as follow:

Example 5. Consider $\mathscr{X}=[0,4]$ be a Banach space, and the mapping $d:\mathscr{X}\times\mathscr{X}\longrightarrow\mathbb{A}^+$ is defined by

$$d(\nu,\varpi)=\|\nu-\varpi\|^p.\mathscr{I}$$

where $p>1$ and $\mathscr{I}$ is the identity mapping. Then $(\mathscr{X},\mathbb{A}^+,d)$ is a $\mathbf{C}^*$-algebra $Vb-MS$ with $b=2^p$.

Define the mapping $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\longrightarrow\mathscr{P}\cup\mathscr{Q}$ by $\mathscr{T}\nu=\dfrac{\nu}{4}$ where $\mathscr{P}=[0,2]$ and $\mathscr{Q}=[0,1]$. First of all, we prove that $\mathscr{T}$ is a **CM**:

$$\text{Assume}\quad \nu\in\mathscr{P}\Longrightarrow 0\le\nu\le 2 \quad\Longrightarrow\quad 0\le\frac{\nu}{4}\le\frac{1}{2}$$
$$\Longrightarrow 0\le\mathscr{T}\nu\le\frac{1}{2}\quad\Longrightarrow\quad\mathscr{T}\nu\in\mathscr{Q}.$$

$$\text{Suppose}\quad \varpi\in\mathscr{Q}\Longrightarrow 0\le\varpi\le 1 \quad\Longrightarrow\quad 0\le\frac{\varpi}{4}\le\frac{1}{4}$$
$$\Longrightarrow 0\le\mathscr{T}\varpi\le\frac{1}{4}\quad\Longrightarrow\quad\mathscr{T}\varpi\in\mathscr{P}.$$

Hence, $\mathscr{T}$ is cyclic.

Now,

$$d(\nu,\varpi)+d(\nu,\mathscr{T}\varpi)+d(\varpi,\mathscr{T}\nu)\ \succeq\ \|\nu-\varpi\|^p.\mathscr{I}+\left\|\nu-\frac{\varpi}{4}\right\|^p.\mathscr{I}+\left\|\varpi-\frac{\nu}{4}\right\|^p.\mathscr{I},$$

from the fact $\|\kappa-\rho\|^p\preceq 2^p\left(\|\kappa\|^p+\|\rho\|^p\right)$, we get

$$\begin{aligned}
d(\nu,\varpi)+d(\nu,\mathscr{T}\varpi)+d(\varpi,\mathscr{T}\nu) &\succeq \|\nu-\varpi\|^p.\mathscr{I}+\frac{1}{2^p}\left\|\left(\nu-\frac{\varpi}{4}\right)-\left(\varpi-\frac{\nu}{4}\right)\right\|^p.\mathscr{I}\\
&=\left[1+\left(\frac{1}{2^p}\right)^2\left(\frac{5}{2}\right)^p\right].\mathscr{I}\ \left(\|\nu-\varpi\|^p.\mathscr{I}\right)\\
&=\left[1+\left(\frac{1}{2^p}\right)^2\left(\frac{5}{2}\right)^p\right].\mathscr{I}\ d(\nu,\varpi)\\
&\succeq\left(\frac{1}{2^p}\right)^2.\mathscr{I}\ d(\nu,\varpi)\ =\ \frac{1}{\lambda}\ d(\mathscr{T}\nu,\mathscr{T}\varpi).
\end{aligned}$$

Then

$$d(\mathscr{T}\nu,\mathscr{T}\varpi)\preceq\lambda\left[d(\nu,\varpi)+d(\nu,\mathscr{T}\varpi)+d(\varpi,\mathscr{T}\nu)\right],$$

where $\lambda=2^{2p}.\mathscr{I}$. Therefore all hypotheses of Corollary 2 are satisfied and 0 is a **UFP** of $\mathscr{T}$.

Example 6. Assume that all assumptions of Example 5 are satisfied. Let $\mathscr{Z}=(\check{\vartheta}_{ij})_{1\le i,j\le\le n}\in\mathscr{M}_n(\mathbb{C})$ and $\mathscr{W}=(w_{ij})_{1\le i,j\le\le n}\in\mathscr{M}_n(\mathbb{C})^+$, where $\mathscr{M}_n(\mathbb{C})^+$ represent the set of all $m\times n$-matrices with complex entries. Then the matrix equations be

$$\mathscr{Z}-\sum_{k=1}^{n}\mathscr{D}_k^*(\mathscr{Z})\mathscr{D}_k=\mathscr{W}.$$

Define the metric $d:\mathscr{X}\times\mathscr{X}\longrightarrow\mathbb{A}^+$ as follow

$$d(\mathscr{Z},\mathscr{Y})=\|\mathscr{Z}-\mathscr{Y}\|^p.\alpha,$$

where $p>1$ and $\alpha\in\mathbb{A}^+$. Then $(\mathscr{X},\mathbb{A}^+,d)$ is a $\mathbf{C}^*$-algebra $Vb-MS$ with $b=2^p$.

Also, we construct the mapping $\mathscr{T}:\mathscr{P}\cup\mathscr{Q}\longrightarrow\mathscr{P}\cup\mathscr{Q}$ such that

$$\mathscr{T}\mathscr{Z}=\frac{1}{4}\left(\sum_{k=1}^{n}\mathscr{D}_k^*(\mathscr{Z})\mathscr{D}_k+\mathscr{W}\right),$$

where $\mathscr{D}_1, \mathscr{D}_2, ..., \mathscr{D}_n \in \mathscr{M}_n(\mathbb{C})$ such that $\sum_{k=1}^{n} \|\mathscr{D}_k\|^2 \leq 1$ and $\|\mathscr{W}\| = 1$. Now, we show that $\mathscr{T}$ is a **CM**:

$$
\begin{aligned}
\text{Let} \quad \mathscr{Z} \in \mathscr{P} &\Longrightarrow 0 \leq \mathscr{Z} \leq 2 \\
&\Longrightarrow 0 \leq \mathscr{D}_k^* \mathscr{Z} \leq 2\,\mathscr{D}_k^* \\
&\Longrightarrow 0 \leq \mathscr{D}_k^* (\mathscr{Z}) \mathscr{D}_k \leq 2\,\mathscr{D}_k^* \mathscr{D}_k \\
&\Longrightarrow 0 \leq \frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Z}) \mathscr{D}_k\right) \leq \frac{1}{4}\left(2\sum_{k=1}^{n} \mathscr{D}_k^* \mathscr{D}_k\right) \\
&\Longrightarrow \frac{1}{4}\mathscr{W} \leq \frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Z}) \mathscr{D}_k + \mathscr{W}\right) \leq \frac{1}{4}\left(2\sum_{k=1}^{n} \mathscr{D}_k^* \mathscr{D}_k + \mathscr{W}\right) \\
&\Longrightarrow \frac{1}{4}\|\mathscr{W}\| \leq \|\mathscr{T}\mathscr{Z}\| \leq \frac{1}{4}\left(\left\|2\sum_{k=1}^{n} \mathscr{D}_k^* \mathscr{D}_k + \mathscr{W}\right\|\right) \\
&\Longrightarrow \frac{1}{4}\|\mathscr{W}\| \leq \|\mathscr{T}\mathscr{Z}\| \leq \left(\frac{1}{2}\left\|\sum_{k=1}^{n} \mathscr{D}_k^* \mathscr{D}_k\right\| + \frac{1}{4}\|\mathscr{W}\|\right) \\
&\Longrightarrow \frac{1}{4}\|\mathscr{W}\| \leq \|\mathscr{T}\mathscr{Z}\| \leq \left(\frac{1}{2}\sum_{k=1}^{n} \|\mathscr{D}_k\|^2 + \frac{1}{4}\|\mathscr{W}\|\right) \\
&\Longrightarrow \frac{1}{4} \leq \mathscr{T}\mathscr{Z} \leq \frac{3}{4} \quad \Longrightarrow \quad \mathscr{T}\mathscr{Z} \in \mathscr{Q}.
\end{aligned}
$$

Similarly, if $\mathscr{Y} \in \mathscr{Q}$, we can show that $\mathscr{T}\mathscr{Y} \in \mathscr{P}$. Thus $\mathscr{T}$ is a cyclic.
Now,

$$
\begin{aligned}
d(\mathscr{T}\mathscr{Z}, \mathscr{T}\mathscr{Y}) &= \left\|\frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Z}) \mathscr{D}_k + \mathscr{W}\right) - \frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Y}) \mathscr{D}_k + \mathscr{W}\right)\right\|^p . \alpha \\
&= \Bigg\|\left\{\frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Z}) \mathscr{D}_k + \mathscr{W}\right) - \mathscr{Y}\right\} \\
&\qquad + \left\{\mathscr{Z} - \frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Y}) \mathscr{D}_k + \mathscr{W}\right)\right\} + \left\{\mathscr{Y} - \mathscr{Z}\right\}\Bigg\|^p . \alpha \\
&\preceq 2^{p-1}\Bigg\{\left\|\frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Z}) \mathscr{D}_k + \mathscr{W}\right) - \mathscr{Y}\right\|^p . \alpha \\
&\qquad + \left\|\mathscr{Z} - \frac{1}{4}\left(\sum_{k=1}^{n} \mathscr{D}_k^* (\mathscr{Y}) \mathscr{D}_k + \mathscr{W}\right)\right\|^p . \alpha + \|\mathscr{Y} - \mathscr{Z}\|^p . \alpha\Bigg\} \\
&= 2^{p-1}\left\{\|\mathscr{T}\mathscr{Z} - \mathscr{Y}\|^p . \alpha + \|\mathscr{Z} - \mathscr{T}\mathscr{Y}\|^p . \alpha + \|\mathscr{Y} - \mathscr{Z}\|^p . \alpha\right\} \\
&= 2^{p-1}\left\{\|\mathscr{Z} - \mathscr{Y}\| + \|\mathscr{Y} - \mathscr{T}\mathscr{Z}\| + \|\mathscr{Z} - \mathscr{T}\mathscr{Y}\|\right\} . \alpha \\
&= 2^{p-1}\left\{d(\mathscr{Z}, \mathscr{Y}) + d(\mathscr{Y}, \mathscr{T}\mathscr{Z}) + d(\mathscr{Z}, \mathscr{T}\mathscr{Y})\right\} . \mathscr{I}.
\end{aligned}
$$

It follows that

$$
d(\mathscr{T}\mathscr{Z}, \mathscr{T}\mathscr{Y}) \preceq \lambda\left[d(\mathscr{Z}, \mathscr{Y}) + d(\mathscr{Y}, \mathscr{T}\mathscr{Z}) + d(\mathscr{Z}, \mathscr{T}\mathscr{Y})\right],
$$

where $\lambda = 2^{p-1} \,.\, \mathscr{I}$. The form of the previous inequality is equivalent to the next form:

$$d(\mathscr{T}\nu, \mathscr{T}\varpi) \;\preceq\; \lambda \left[d(\nu, \varpi) + d(\varpi, \mathscr{T}\nu) + d(\nu, \mathscr{T}\varpi) \right].$$

Therefore all assumptions of Corollary 2 are verified and 0 is a **UFP** of $\mathscr{T}$.

4 Solving nonlinear integral equation

In this part, assume $\mathscr{X} = C([0,1], \mathbb{A})$ where $C[0,1]$ denotes the set of all real continuous functions on $[0,1]$, $\mathbb{A}$ be a Banach algebra and the nonlinear integral equation

$$\nu(t) = \left(\int_0^1 K(t,s,\nu(s))\, ds \right) . \mathscr{I}_{\mathbb{A}} \qquad \forall\, t \in [0,1], \tag{7}$$

where $\mathscr{I}_{\mathbb{A}}$ is the identity of $\mathbb{A}$, $K : [0,1] \times [0,1] \times \mathbb{A} \longrightarrow \mathbb{A}$ is a continuous function and let

$$\mathscr{T}\nu(t) = \left(\int_0^1 K(t,s,\nu(s))\, ds \right) . \mathscr{I}_{\mathbb{A}}. \tag{8}$$

Define the metric as follows:

$$d(\nu, \varpi) = \|\nu(t) - \varpi(t)\|^p \,.\, \mathscr{I}_{\mathbb{A}} \tag{9}$$

Clearly, $(\mathscr{X}, \mathbb{A}, d)$ is a $\mathbf{C}^*$-algebra valued b-metric spaces with $b = 2^p$.
Let $a, b \in \mathscr{X}$ and $a_0, b_0 \in \mathbb{A}$ such that

$$a_0 \;\preceq\; a(t) \;\preceq\; b(t) \;\preceq\; b_0. \tag{10}$$

For all $t \in [0,1]$, Consider

$$a(t) \preceq \left(\int_0^1 K(t,s,b(s))\, ds \right) . \mathscr{I}_{\mathbb{A}}; \tag{11}$$

$$b(t) \succeq \left(\int_0^1 K(t,s,a(s))\, ds \right) . \mathscr{I}_{\mathbb{A}}. \tag{12}$$

Assume that for all $t \in [0,1], K(t,s,.)$ be a decreasing function, that is,

$$\forall\;\; \nu, \varpi \in \mathbb{A}, \quad j \succeq k \qquad \Longrightarrow \qquad K(t,s,\nu(s)) \,.\, \mathscr{I}_{\mathbb{A}} \preceq K(t,s,\varpi(s)) \,.\, \mathscr{I}_{\mathbb{A}}. \tag{13}$$

Also, Let

$$|K(t,s,\nu(s)) - K(t,s,\varpi(s))| \leq \mu\; |\nu(s) - \varpi(s)|. \tag{14}$$

Theorem 7. *Under the conditions from (9) to (14), the equation (7) has a unique solution.*

*Proof.*Let A_1 and A_2 be a closed subsets of $\mathscr{X}$ such that

$$A_1 = \{\nu(t) \in \mathscr{X} : \quad \nu(t) \preceq b(t)\} \subseteq \mathscr{X} \tag{15}$$

$$A_2 = \{\nu(t) \in \mathscr{X} : \quad \nu(t) \succeq a(t)\} \subseteq \mathscr{X} \tag{16}$$

Define the map $\mathscr{T} : \mathscr{X} \longrightarrow \mathscr{X}$ by

$$\mathscr{T}\nu(t) = \left(\int_0^1 K(t,s,\nu(s))\, ds \right) . \mathscr{I}_{\mathbb{A}}, \tag{17}$$

We shall prove that

$$\mathscr{T}(A_1) \subseteq A_2, \qquad \mathscr{T}(A_2) \subseteq A_1 \tag{18}$$

Let $\nu(t) \in A_1$, that is

$$\nu(t) \preceq b(t). \tag{19}$$

Using condition (13), we get

$$K(t,s,\nu(s)) \,.\, \mathscr{I}_{\mathbb{A}} \succeq K(t,s,b(s)) \,.\, \mathscr{I}_{\mathbb{A}} \qquad \forall\, t,s \in [0,1]. \tag{20}$$

By integration and using condition (11), we obtain

$$\left(\int_0^1 K(t,s,\nu(s))\,ds\right) . \mathscr{I}_{\mathbb{A}} \succeq \left(\int_0^1 K(t,s,b(s))\,ds\right) . \mathscr{I}_{\mathbb{A}} \succeq a(t) \tag{21}$$

Then, we obtain $\mathscr{T}\nu \in A_2$.

By a similar way, let $\nu \in A_2$, that is,

$$\nu(t) \succeq a(t). \tag{22}$$

Using condition (13), we obtain

$$K(t,s,\nu(s)) \,.\, \mathscr{I}_{\mathbb{A}} \preceq K(t,s,a(s)) \,.\, \mathscr{I}_{\mathbb{A}} \qquad \forall\, t,s \in [0,1]. \tag{23}$$

By integration and using condition (12), we find

$$\left(\int_0^1 K(t,s,\nu(s))\,ds\right) . \mathscr{I}_{\mathbb{A}} \preceq \left(\int_0^1 K(t,s,a(s))\,ds\right) . \mathscr{I}_{\mathbb{A}} \preceq b(t). \tag{24}$$

Then, we have $\mathscr{T}\nu \in A_1$.

Now,

$$\begin{aligned}
d(\mathscr{T}\nu,\mathscr{T}\varpi) &= \|\mathscr{T}\nu(t) - \mathscr{T}\varpi(t)\|^p \,.\, \mathscr{I}_{\mathbb{A}} \\
&= \left\|\int_0^1 K(t,s,\nu(s))\,ds - \int_0^1 K(t,s,\varpi(s))\,ds\right\|^p . \mathscr{I}_{\mathbb{A}} \\
&\preceq \int_0^1 \|K(t,s,\nu(s)) - K(t,s,\varpi(s))\|^p\,ds \,.\, \mathscr{I}_{\mathbb{A}} \\
&\preceq \int_0^1 \mu^p\, \|\nu(s) - \varpi(s)\|^p\,ds \,.\, \mathscr{I}_{\mathbb{A}} \\
&\preceq \left[\mu^p\, \|\nu(s) - \varpi(s)\|^p \int_0^1 ds\right] . \mathscr{I}_{\mathbb{A}} \\
&\preceq \mu^p \left[\left\|\frac{\nu(s)-\varpi(s)}{3}\right\|^p . \mathscr{I}_{\mathbb{A}} + \left\|\frac{\nu(s)-\varpi(s)}{3}\right\|^p . \mathscr{I}_{\mathbb{A}} + \left\|\frac{\nu(s)-\varpi(s)}{3}\right\|^p . \mathscr{I}_{\mathbb{A}}\right] \\
&= \left(\frac{\mu}{3}\right)^p \left[\|\nu-\varpi\|^p . \mathscr{I}_{\mathbb{A}} + \|\mathscr{T}\nu-\varpi\|^p . \mathscr{I}_{\mathbb{A}} + \|\nu-\mathscr{T}\varpi\|^p . \mathscr{I}_{\mathbb{A}}\right]
\end{aligned}$$

Then,

$$d(\mathscr{T}\nu,\mathscr{T}\varpi) \preceq \lambda \left[d(\nu,\varpi) + d(\varpi,\mathscr{T}\nu) + d(\nu,\mathscr{T}\varpi)\right]$$

where $\lambda = \left(\frac{\mu}{3}\right)^p$. Therefore all conditions of Corollary 2 are satisfied and 0 is a unique fixed point of $\mathscr{T}$.

5 Conclusions

In fact, fixed point theory (**FP**) with its practical application is a dynamic, exciting, and crucial topic in the overall context of functional analysis. Due to its positive implications, Several authors have used this theory to explain and illustrate many generalizations, improvements and extensions to a variety of distance spaces with beneficial applications. In this manuscript, we interested in the results of Banach, Kannan and Chatterjea in the setting of $\mathbf{C}^*$-algebra valued b-metric spaces ($\mathbf{C}^*$-algebra $Vb-MSs$). We introduced some new definitions in $\mathbf{C}^*$-algebra $Vb-MSs$ such as Δ-cyclic Banach-Kannan type (Δ-**CBKT**) mapping, Δ-cyclic Banach-Chatterjea type (Δ-**CBCT**) mapping, and Δ-cyclic Banach-Kannan-Chatterjea type (Δ-**CBKCT**) mapping in the main theorems. Also, we dealt with cyclic Banach-Kannan type (**CBKT**) mapping, cyclic Banach-Chatterjea type (**CBCT**) mapping, and cyclic Banach-Kannan-Chatterjea type

(Δ-**CBKCT**) mapping in the objective corollaries. These definitions generalize and extend the mathematical results in the scientific literature. On the other hand, we introduced some nontrivial examples and application to the result related to cyclic Banach-Chatterjea type mapping for solving a nolinear integral equation.As a future work, our results can be utilized in other spaces such as $\mathbf{C}^*$-algebra valued metric spaces ($\mathbf{C}^*$-algebra $VMSs$), $\mathbf{C}^*$-algebra partial valued metric spaces ($\mathbf{C}^*$-algebra $PVMSs$), and $\mathbf{C}^*$-algebra valued bipolar metric spaces ($\mathbf{C}^*$-algebra $BI-VMSs$). Also, contraction condition can be improved, generalized, and extended to contain ϕ-contractions according to Berinde[54] and ζ-contractions in the context of Khojasteh et al.[55].

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