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Improper Integrals Involving the Incomplete Aleph–Functions

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Abstract: In the present paper, we study and explore the general infinite integral involving the incomplete Aleph–functions. We also give some corollaries in terms of other incomplete special functions by specializing the parameters of incomplete Aleph-functions (for example, incomplete I-function, incomplete H-function). The results presented in this paper are unified and general in nature.

Keywords: Incomplete Gamma function, incomplete Aleph–function, Mellin-Barnes contour integrals, infinite integral.

1 Introduction and Preliminaries

Srivastava et al. [1] have studied the incomplete Gamma–function and incomplete hypergeometric function. Recently, Srivastava et al. [2] have introduced and studied the incomplete H–function and the incomplete $\bar{H}$–function. Several authors, Bansal et al. [3], Bansal and Kumar [4] and Bansal et al. [5] have introduced and studied the incomplete Aleph–function, the incomplete I–function and calculated the integrals about the incomplete H–function respectively. More recently, Kumar et al. [6] have studied the Boros integral with three parameters involving the incomplete I–functions and generalized multi-index Mittag-Leffler function. In this paper, we study the generalized finite integral concerning the product of the incomplete Aleph–function, defined here and the elliptic integrals of the first kind.
The incomplete Gamma functions $\gamma(\alpha,x)$ and $\Gamma(\alpha,x)$ are defined as follows:

$$\gamma(\alpha,x)=\int_0^x u^{\alpha-1}e^{-u}\,\mathrm{d}u \quad (\Re(\alpha)>0;x\geqslant 0). \tag{1}$$

$$\Gamma(\alpha,x)=\int_x^\infty u^{\alpha-1}e^{-u}\,\mathrm{d}u, \tag{2}$$

where, $x\geqslant 0$; $\Re(\alpha)>0$ when $x=0$.

We have the following relation:

$$\gamma(\alpha,x)+\Gamma(\alpha,x)=\Gamma(\alpha) \quad (\Re(\alpha)>0). \tag{3}$$

Now, we give the expression of the incomplete Aleph–functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,r}(z)$ defined by Bansal et al. [3], as follows:

$$\begin{aligned}{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)&={}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left(z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\\ &=\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}\mathrm{d}s,\end{aligned} \tag{4}$$

and

$$\begin{aligned}{}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)&={}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left(z\left|\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\\ &=\frac{1}{2\pi\omega}\int_L\frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}\mathrm{d}s.\end{aligned} \tag{5}$$

The incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,r}(z)$ defined above exits for $x\geq 0$ and the following validities conditions.

The contour L is in the s–plane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ if is a real number with loop, if necessary

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to ensure that the poles of $\Gamma(1-a_j-A_js), j=2,\cdots,n$ to the right of the contour L and the poles of $\Gamma(g_j+G_js), j=1,\cdots,m$ to the left of the contour L. The parameters τ_i, m, n, p_i, q_i are positive numbers satisfying $0\le n\le p_i$, $0\le m\le q_i$ and a_j, g_j, a_{ji}, g_{ji} are complex numbers. These poles of the integrand are assumed to be simple. We have the following conditions:

$$\Omega_i>0,\ |\arg(z)|<\frac{\pi}{2}\Omega_i,\ i=1,\cdots,r \tag{6}$$

$$\Omega_j\ge 0,\ |\arg(z)|<\frac{\pi}{2}\Omega_i \ \text{ and }\ \Re(\zeta_i)+1<0, \tag{7}$$

where

$$\Omega_i=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}G_j-\tau_i\max_{1\le i\le r}\left(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}G_{ji}\right), \tag{8}$$

and

$$\zeta_i=\sum_{j=1}^{m}g_j-\sum_{j=1}^{n}a_j+\tau_i\left(\sum_{j=m+1}^{q_i}g_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right)+\frac{p_i-q_i}{2},\quad i=1,\cdots,r. \tag{9}$$

We have the following relation:

$$^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)+{}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)=\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z). \tag{10}$$

$\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ is the Aleph–function defined by Südland et al. [7] and studied by many authors, for eg, Choi and Kumar [8,9], Kumar et al. [10], Kumar and Choi [11], Ram and Kumar [12], Saxena and Kumar [13], Südland et al. [14], and etc.

If we set $\tau_i\to 1$, then the incomplete $\aleph$–functions $^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and $^{(\gamma)}\aleph^{m,n}_{p_i,q_i,r}(z)$ reduce respectively to incomplete I-functions $^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)$ and $^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)$ (see also, [6]), given as follows:

$$^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)={}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_i}\\ (g_j,G_j)_{1,m},(g_{ji},G_{ji})_{m+1,q_i}\end{matrix}\right)$$
$$=\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}ds, \tag{11}$$

and

$$^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)={}^{(\gamma)}I^{m,n}_{p_i,q_i,r}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_i}\\ (g_j,G_j)_{1,m},(g_{ji},G_{ji})_{m+1,q_i}\end{matrix}\right)$$
$$=\frac{1}{2\pi\omega}\int_L\frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}ds, \tag{12}$$

under the same conditions that above with $\tau_i\to 1$. Now, we suppose $r=1$, the incomplete I-functions $^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)$ and $^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)$ reduce respectively to incomplete H-functions $^{(\Gamma)}H^{m,n}_{p,q}(z)$ and $^{(\gamma)}H^{m,n}_{p,q}(z)$, given as follows:

$$^{(\Gamma)}H^{m,n}_{p,q}(z)={}^{(\Gamma)}H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q}\end{matrix}\right)$$
$$=\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}z^{-s}ds, \tag{13}$$

and

$$^{(\gamma)}H^{m,n}_{p,q}(z)={}^{(\gamma)}H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q}\end{matrix}\right)$$
$$=\frac{1}{2\pi\omega}\int_L\frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}z^{-s}ds, \tag{14}$$

under the same conditions verified by the incomplete I-functions with $r=1$. By using the formula (5), we have the following relations:

$$^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(z)+{}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)=I^{m,n}_{p_i,q_i,r}(z), \tag{15}$$

the function $I^{m,n}_{p_i,q_i,r}(z)$ being the function defined by Saxena [15], and

$$^{(\Gamma)}H^{m,n}_{p,q}(z)+{}^{(\gamma)}H^{m,n}_{p,q}(z)=H^{m,n}_{p,q}(z). \tag{16}$$

2 Required integral

In this section, we give an generalized infinite integral, see Prudnikov et al. [16, Ch 2.2.11, page 314].

Lemma 1.

$$\int_0^\infty\frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^{\upsilon}}dx=2^{1\mp\alpha}\,\upsilon\,z^{\alpha-\upsilon}\left\{\begin{matrix}\frac{\Gamma(2\alpha)\Gamma(\upsilon-\alpha)}{\Gamma(1+\alpha+\upsilon)}\\ 2^{-\upsilon}\frac{\Gamma(2\upsilon-2\alpha)\Gamma(\alpha)}{\Gamma(1+2\upsilon-\alpha)}\end{matrix}\right\}, \tag{17}$$

where

$$|\arg(z)|<\pi,\ 0<\Re(\alpha)<\Re(\upsilon).$$

3 Main integrals

Now, we will give a general infinite integral.

Let

$$X_{\alpha,\upsilon}=\frac{x^{\alpha}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^{\upsilon}}. \tag{18}$$

Next, we define the unified infinite integral involving the incomplete Aleph–functions. the general result with an unified infinite integrals with several parameters.

Theorem 1.

$$\int_0^\infty\frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^{\upsilon}}{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(ZX_{a,b})\,dx=2^{1\mp\alpha}z^{\alpha-\upsilon}\left\{\begin{matrix}1\\2^{-\upsilon}\end{matrix}\right\}$$
$$\times\,{}^{(\Gamma)}\aleph^{m,n+3}_{p_i+3,q_i+2,\tau_i,r}\left(\left\{\begin{matrix}2^{-a}z^{a-b}Z\\2^{a-b}z^{a-b}Z\end{matrix}\right\}\middle|\begin{matrix}(a_1,A_1,x'),\left\{\begin{matrix}A_1\\A_2\end{matrix}\right\},\\(g_j,G_j)_{1,m},\end{matrix}\right.$$
$$\left.\begin{matrix}(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ [\tau_i(g_{ji},G_{ji})]_{m+1,q_i},\left\{\begin{matrix}B_1\\B_2\end{matrix}\right\}\end{matrix}\right). \tag{19}$$

Provided that $|\arg(z)|<\pi$, $0<\Re(\alpha)<\Re(\upsilon)$ *and we have the following conditions:* $0<a,b,x',2a-b,a-b,$

$0 < \Re(\alpha) + b\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) < \Re(\upsilon) + a\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) + \frac{1}{2}$, *$\Omega_i > 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$, $i = 1,\cdots,r$ or $\Omega_i \geq 0$, $|\arg(z)| < \frac{\pi}{2}\Omega_i$ and $\Re(\zeta_i)+1<0$, Ω_i and ζ_i is defined by (8) and (9) respectively, where*

$$A_1 = (-\upsilon;b),(1-2\alpha;2a),(1+\alpha-\upsilon;b-a);$$
$$B_1 = (1-\upsilon;b),(-\alpha-\upsilon;a+b); \tag{20}$$

$$A_2 = (-\upsilon;b),(1-2\upsilon+2\alpha;2b-2a),(1-\alpha;a);$$
$$B_2 = (1-\upsilon;b),(\alpha-2\upsilon;2b-a). \tag{21}$$

Proof. To prove the (19), first we express the modified incomplete Aleph–function in Mellin-Barnes contour integral with the help of (4) and interchange the order of integrations which is acceptable due to absolute convergence of the integral involved. Next, expressing the quantity $X_{\{.\}}$ in function of x and collecting the power of x, then we have $\mathscr{I}$ (say), as given by

$$\mathscr{I} = \int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^\upsilon} {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(ZX_{a,b})\,\mathrm{d}x$$
$$= \int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^\upsilon}$$
$$\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^n\Gamma(1-a_j-A_jt)\prod_{j=1}^m\Gamma(g_j+G_jt)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}$$
$$\times Z^{-t}X_{a,b}^{-t}\,\mathrm{d}t\,\mathrm{d}x. \tag{22}$$

On changing the order of integration, we arrive at

$$\mathscr{I} = \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-b_1t,x')\prod_{j=2}^n\Gamma(1-a_j-A_jt)\prod_{j=1}^m\Gamma(g_j+G_jt)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{jit}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+b_{ji}t)\right]}$$
$$\times \int_0^{+\infty}\frac{x^{\alpha-at-1}}{(x+z+\sqrt{A+2xz})^{\upsilon-bt}}Z^{-t}\,\mathrm{d}x\,\mathrm{d}t. \tag{23}$$

By replacing α by $(\alpha-at)$ and υ by $(\upsilon-bt)$ respectively then applying lemma 1, this gives

$$\mathscr{I} = \int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^\upsilon} {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(ZX_{a,b})\,\mathrm{d}x$$
$$= 2^{1\mp\alpha}z^{\alpha-\upsilon}\frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^n\Gamma(1-a_j-A_jt)\prod_{j=1}^m\Gamma(g_j+G_jt)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}$$
$$\times\left\{\begin{matrix}\frac{2^{at}z^{-at+bt}\Gamma(2\alpha-2at)\Gamma(\upsilon-\alpha+at-bt)}{\Gamma(1+\alpha+\upsilon-at-bt)}\\ \frac{z^{-at+bt}2^{-\upsilon-at+bt}\Gamma(2\upsilon-2\alpha+2at-2bt)\Gamma(\alpha-at)}{\Gamma(1+2\upsilon-\alpha+at-2bt)}\end{matrix}\right\}Z^{-t}(\upsilon-bt)\,\mathrm{d}t. \tag{24}$$

By applying the definition of the Gamma incomplete Aleph–function by the contour integral (see the equation (4), we have desired result (18).

Theorem 2. *By the similar method as done in (19), we prove the following formula about the incomplete gamma Aleph-function:*

$$\int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^\upsilon} {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(ZX_{a,b})\,\mathrm{d}x = 2^{1\mp\alpha}z^{\alpha-\upsilon}\left\{\begin{matrix}1\\2^{-\upsilon}\end{matrix}\right\}$$
$$\times {}^{(\gamma)}\aleph^{m,n+3}_{p_i+3,q_i+2,\tau_i,r}\left(\left\{\begin{matrix}2^{-a}z^{a-b}Z\\2^{a-b}z^{a-b}Z\end{matrix}\right\}\middle|\begin{matrix}(a_1,A_1,x'),\left\{\begin{matrix}A_1\\A_2\end{matrix}\right\},\\(g_j,G_j)_{1,m},\end{matrix}\right.$$
$$\left.\begin{matrix}(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ [\tau_i(g_{ji},G_{ji})]_{m+1,q_i},\left\{\begin{matrix}B_1\\B_2\end{matrix}\right\}\end{matrix}\right) \tag{25}$$

with the same conditions and notations that (19).

4 Special cases

In this section, we give special cases of our main results concerning the incomplete I–functions and incomplete H–function, We have

Corollary 1.

$$\int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^\upsilon} {}^{(\Gamma)}I^{m,n}_{p_i,q_i,r}(ZX_{a,b})\,\mathrm{d}x = 2^{1\mp\alpha}z^{\alpha-\upsilon}\left\{\begin{matrix}1\\2^{-\upsilon}\end{matrix}\right\}$$
$$\times {}^{(\Gamma)}I^{m,n+3}_{p_i+3,q_i+2,r}\left(\left\{\begin{matrix}2^{-a}z^{a-b}Z\\2^{a-b}z^{a-b}Z\end{matrix}\right\}\middle|\begin{matrix}(a_1,A_1,x'),\left\{\begin{matrix}A_1\\A_2\end{matrix}\right\},\\(g_j,G_j)_{1,m},\end{matrix}\right.$$
$$\left.\begin{matrix}(a_j,A_j)_{2,n},[(a_{ji},A_{ji})]_{n+1,p_i}\\ [(g_{ji},G_{ji})]_{m+1,q_i},\left\{\begin{matrix}B_1\\B_2\end{matrix}\right\}\end{matrix}\right), \tag{26}$$

provided that, $|\arg(z)| < \pi, 0 < \Re(\alpha) < \Re(\upsilon)$ and we have the following conditions: $0 < a,b,x',b-2a,b-a$, $0 < \Re(\alpha) - b\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) < \Re(\upsilon) - a\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) + \frac{1}{2}, \Omega_i > 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$ for $i = 1,\cdots,r;$ $\Omega_i > 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$, $i = 1,\cdots,r$ or $\Omega_i \geq 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$ and $\Re(\zeta_i)+1<0$, where

$$\Omega_i = \sum_{j=1}^n A_j + \sum_{j=1}^m G_j - \max_{1\leq i\leq r}\left(\sum_{j=n+1}^{p_i}A_{ji} + \sum_{j=m+1}^{q_i}G_{ji}\right), \tag{27}$$

and

$$\zeta_i = \sum_{j=1}^m g_j - \sum_{j=1}^n a_j + \left(\sum_{j=m+1}^{q_i}g_{ji} - \sum_{j=n+1}^{p_i}a_{ji}\right) + \frac{p_i-q_i}{2},\ i=1,\cdots,r. \tag{28}$$

Corollary 2.

$$\int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^\upsilon} {}^{(\gamma)}I^{m,n}_{p_i,q_i,r}(ZX_{a,b})\,\mathrm{d}x = 2^{1\mp\alpha}z^{\alpha-\upsilon}\left\{\begin{matrix}1\\2^{-\upsilon}\end{matrix}\right\}$$
$$\times {}^{(\gamma)}I^{m,n+3}_{p_i+3,q_i+2,r}\left(\left\{\begin{matrix}2^{-a}z^{a-b}Z\\2^{a-b}z^{a-b}Z\end{matrix}\right\}\middle|\begin{matrix}(a_1,A_1,x'),\left\{\begin{matrix}A_1\\A_2\end{matrix}\right\},\\(g_j,G_j)_{1,m},\end{matrix}\right.$$
$$\left.\begin{matrix}(a_j,A_j)_{2,n},[(a_{ji},A_{ji})]_{n+1,p_i}\\ [(g_{ji},G_{ji})]_{m+1,q_i},\left\{\begin{matrix}B_1\\B_2\end{matrix}\right\}\end{matrix}\right), \tag{29}$$

under the same conditions that the corollary 1.

Corollary 3.

$$\int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^{\upsilon}}\, {}^{(\Gamma)}H_{p,q}^{m,n}(ZX_{a,b})\,\mathrm{d}x = 2^{1\mp\alpha} z^{\alpha-\upsilon}\left\{\begin{matrix}1\\2^{-\upsilon}\end{matrix}\right\}$$

$$\times\, {}^{(\Gamma)}H_{p+3,q+2}^{m,n+3}\left(\left\{\begin{matrix}2^{-a}z^{a-b}Z\\2^{a-b}z^{a-b}Z\end{matrix}\right\}\left|\begin{matrix}(a_1,A_1,x'),\left\{\begin{matrix}A_1\\A_2\end{matrix}\right\},(a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q}\left\{\begin{matrix}B_1\\B_2\end{matrix}\right\}\end{matrix}\right.\right). \tag{30}$$

Provided that, $|\arg(z)| < \pi, 0 < \Re(\alpha) < \Re(\upsilon)$ *and we have the following conditions:* $0 < a,b,x',b-2a,b-a,$

$$0 < \Re(\alpha) + a \min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) < \Re(\upsilon) + b \min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) + \frac{1}{2}$$

, $\Omega \geq 0$, $|\arg(z)| < \frac{\pi}{2}\Omega$, *and* $\Re(\zeta)+1<0$, *where*

$$\Omega = \sum_{j=2}^{n} A_j + \sum_{j=1}^{m} G_j - \left(\sum_{j=n+1}^{p} A_j + \sum_{j=m+1}^{q} G_j\right), \tag{31}$$

and

$$\zeta = \sum_{j=1}^{m} g_j - \sum_{j=2}^{n} a_j + \left(\sum_{j=m+1}^{q} g_j - \sum_{j=n+1}^{p} a_j\right) + \frac{p-q}{2}. \tag{32}$$

Corollary 4.

$$\int_0^\infty \frac{x^{\alpha-1}}{\left(x+z+\sqrt{\left\{\begin{matrix}x^2\\z^2\end{matrix}\right\}+2xz}\right)^{\upsilon}}\, {}^{(\gamma)}H_{p,q}^{m,n}(ZX_{a,b})\,\mathrm{d}x = 2^{1\mp\alpha} z^{\alpha-\upsilon}\left\{\begin{matrix}1\\2^{-\upsilon}\end{matrix}\right\}$$

$$\times\, {}^{(\gamma)}H_{p+3,q+2}^{m,n+3}\left(\left\{\begin{matrix}2^{-a}z^{a-b}Z\\2^{a-b}z^{a-b}Z\end{matrix}\right\}\left|\begin{matrix}(a_1,A_1,x'),\left\{\begin{matrix}A_1\\A_2\end{matrix}\right\},(a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q}\left\{\begin{matrix}B_1\\B_2\end{matrix}\right\}\end{matrix}\right.\right). \tag{33}$$

*Remark.*We have the same generalized finite integral with the $\aleph$–function [12,7], the I-function [17,15] and the Fox's H-function [18,19].

5 Conclusions and Future work

The significance of our given results lies in their manifold generality. First, by specializing the various parameters as well as variable in the incomplete Aleph-functions ${}^{(\Gamma)}\aleph_{p_i,q_i,\tau_i,r}^{m,n}(z)$ and ${}^{(\gamma)}\aleph_{p_i,q_i,\tau_i,r}^{m,n}(z)$, we can obtain a large number of results involving remarkably broad variety of useful special functions which are explicable in terms of I-function defined by Saxena [15], H-function, Meijer's G-function, E-function, and hypergeometric function of one variable. Secondly, by specializing the parameters of unified infinite integral given in this paper, we can also get plenty of integrals involving the incomplete Aleph–functions, the incomplete I–functions and the incomplete H-functions.

Declarations

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Availability of data and materials

No data were used for this study.

Competing interests

The authors declare that they have no competing interests.

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Authors contributions

DK made significant contributions to the creation of the work. FY contributed to the design of the work and handled the analysis. FY and MSB conceptualized and double-checked the Analysis part. DK and FY were involved in the manuscript's drafting or critical revision for important intellectual content. All authors read and approved the final version of manuscript.

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