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Optimizing Public Transportation Management through Differential Gaming to Align Demand Control with Service Efficiency

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Abstract: Public transportation systems face the dual challenge of improving service quality while managing demand to prevent overuse or under-utilization. In this paper, we develop a differential game model to represent the interactions between two key players: the public transportation management (service provider) and demand regulators. The service provider aims to enhance service quality and system efficiency, while demand regulators focus on controlling excessive demand. Each player uses control strategies that influence the state of the system over time, represented by congestion levels or service utilization. We derive optimal strategies for both players using Hamiltonian functions and evaluate their performance through numerical simulations. Our results reveal how balancing these conflicting objectives can lead to a more efficient and cost-effective public transportation system.

Keywords: Differential Game, Demand Control, Hamiltonian Function, Optimal Control, Public transportation, Performance Indices, Service Performance, Simulation

1 Introduction

Effective management of public transportation systems is essential for urban mobility, environmental sustainability, and economic efficiency. These systems serve as the backbone of daily commutes, reducing congestion and pollution, and providing reliable service to millions of users. However, public transportation systems often face challenges such as fluctuating demand, limited capacity, and operational inefficiencies [1,2,3,4]. Balancing these factors to ensure both service quality and optimal utilization is a key concern for urban planners and policymakers, requiring innovative strategies such as those provided by differential game theory [5,6,7,8,9].

Public transportation systems operate in a dynamic environment, where both service providers and passengers' behaviors influence system performance [10, 11]. Service providers must constantly optimize routes, schedules, and capacity to meet varying levels of demand [12, 13]. At the same time, they must regulate the overuse of resources, ensuring that services are neither underutilized nor overwhelmed by excessive demand, a balance that is essential to maintaining the efficiency of the system [1,4]. This creates a delicate balancing act, where the dual objectives of improving service quality and controlling demand must be dynamically managed [12,14].

In this paper, we propose a novel approach using differential game theory to model the interactions between two key players in a public transportation system: the public transportation management (or service provider) and the demand regulators [3,9]. The service provider's main objective is to improve the efficiency and quality of the transportation service, while the demand regulators aim to control excessive demand that can lead to system overloading [5,15]. This interaction is modeled as a non-cooperative, non-zero-sum differential game, where each player seeks to optimize their respective performance indices, leading to a dynamic system where the control strategies of both players directly influence overall system behavior [2,16].
By applying optimal control theory through Hamiltonian

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functions, we derive strategies that enable both the service provider and the demand regulator to achieve their objectives [17,18]. Our model highlights the inherent conflict between enhancing service performance and controlling demand, showing how an optimal balance can be achieved [4,18]. We explore the impact of these strategies through numerical simulations, offering insights into how public transportation systems can be better managed to achieve higher efficiency with minimal resource use [12,19,20].

This work is organized as follows: Section 2 introduces the mathematical model of public transportation management, detailing the dynamics between the service provider and demand regulator. Section 3 presents the Hamiltonian approach for deriving optimal control strategies. Section 4 show cases the results of our numerical simulations and the impact of some of the factors on this study. Section 5 we discuss the optimization of the effective parameters to get the best results. Finally, Section 6 concludes with key insights and suggestions for future research, including the integration of real-time data and cooperative game theory to further enhance transportation system management.

2 Public Transportation Management Models

In the management of public transportation systems, a dynamic model is needed to represent how service and demand interact over time. The public transportation system is influenced by two main control inputs: one aimed at enhancing the service performance, and the other focused on regulating demand. These inputs affect the state of the system, which can be thought of as a representation of service utilization or congestion levels.

The evolution of the state variable $x(t)$, which may represent the level of congestion or the overall performance of the system, is governed by the following differential equation:

$$\dot{x} = u_1(1-x) - u_2 x \tag{1}$$

In this equation:
• $u_1(t)$ represents the efforts of the transportation management to enhance service (e.g., increasing fleet size, reducing wait times, or adding more routes).
• $u_2(t)$ represents the efforts to manage demand (e.g., through pricing strategies, service restrictions during peak times, or other regulatory measures).
• $x(t)$ is the state variable representing the degree of congestion or the level of system performance at time t.
• $\dot{x}$ is the rate of change of $x(t)$, capturing how quickly the system responds to the control efforts of the players.

The model assumes that as the service provider increases effort u_1, the state variable x decreases, indicating an improvement in the system's performance (e.g., reduced congestion). Conversely, the effort u_2 by demand regulators aims to reduce demand by decreasing the level of service utilization, particularly during peak times.

To evaluate the effectiveness of these control strategies, we define two performance indices, J_1 and J_2, which measure the outcomes of the service provider's and demand regulator's efforts, respectively. These indices capture the balance between improving service, controlling demand, and minimizing control costs.

The first performance index, J_1, represents the objective of the service provider:

$$J_1 = \int_0^T \left(r_1 x(t) - k_1 u_1^2(t)\right) dt \tag{2}$$

Here:
• r_1 represents the weight given to improving service quality, with higher values indicating a greater emphasis on reducing congestion.
• k_1 penalizes the control effort u_1, reflecting the operational costs incurred in improving the service.
• The goal of the service provider is to maximize service quality (minimizing $x(t)$) while minimizing the control costs (penalizing excessive effort in u_1).

The second performance index, J_2, represents the goal of the demand regulator:

$$J_2 = \int_0^T \left(r_2(1-x(t)) - k_2 u_2^2(t)\right) dt \tag{3}$$

Here:
• r_2 represents the weight given to controlling demand, with higher values indicating a stronger emphasis on reducing service utilization (e.g., during peak periods).
• k_2 penalizes the control effort u_2, reflecting the costs associated with implementing demand control measures.
• The goal of the demand regulator is to decrease demand (minimizing $1-x(t)$) while minimizing the control effort (avoiding excessive u_2).

In summary, these performance indices capture the trade-offs between improving service performance, managing demand, and minimizing control costs. The dynamics of the system are driven by the efforts of both the service provider and the demand regulator, and the model helps to identify the optimal strategies for each player to achieve their respective objectives[5,6,7,8,9].

Next, we introduce the Hamiltonian functions, which are essential for deriving the optimal control laws:

$$H_1(t,x,u_1,u_2,\lambda_1) = (r_1 x - k_1 u_1^2) + \lambda_1(u_1(1-x) - u_2 x) \tag{4}$$

$$H_2(t,x,u_1,u_2,\lambda_2) = (r_2(1-x)) - k_2u_2^2 + \lambda_2(u_1(1-x) - u_2x) \tag{5}$$

To find the optimal controls u_1 and u_2, we take the partial derivatives of the Hamiltonian with respect to u_1 and u_2 and set them equal to zero:

$$\frac{\partial H_1}{\partial u_1} = -2k_1u_1 + \lambda_1(1-x) = 0 \quad \Rightarrow \quad u_1 = \frac{\lambda_1(1-x)}{2k_1} \tag{6}$$

$$\frac{\partial H_2}{\partial u_2} = -2k_2u_2 - \lambda_2 x = 0 \quad \Rightarrow \quad u_2 = -\frac{\lambda_2 x}{2k_2} \tag{7}$$

Substituting these values into the expression for $\dot{x}$ gives us:

$$\dot{x} = \frac{\lambda_1(1-x)}{2k_1}(1-x) + \frac{\lambda_2 x}{2k_2}x \tag{8}$$

To further analyze the system, we calculate the time derivatives of the co-state variables λ_1 and λ_2 by differentiating H_1 and H_2 with respect to x:

First, for λ_1, we compute:

$$H_1(t,x,u_1,u_2,\lambda_1) = r_1x - k_1u_1^2 + \lambda_1(u_1(1-x) - u_2x) \tag{9}$$

The partial derivative with respect to x is:

$$\frac{\partial H_1}{\partial x} = r_1 - \lambda_1u_1 - \lambda_1u_2 \tag{10}$$

Thus, the time derivative of λ_1 is:

$$\dot{\lambda}_1 = -\frac{\partial H_1}{\partial x} = -r_1 + \lambda_1u_1 + \lambda_1u_2 \tag{11}$$

Similarly, for λ_2, we start with:

$$H_2(t,x,u_1,u_2,\lambda_2) = r_2(1-x) - k_2u_2^2 + \lambda_2(u_1(1-x) - u_2x) \tag{12}$$

The partial derivative with respect to x is:

$$\frac{\partial H_2}{\partial x} = -r_2 + \lambda_2u_1 - \lambda_2u_2 \tag{13}$$

Therefore, the time derivative of λ_2 is:

$$\dot{\lambda}_2 = -\frac{\partial H_2}{\partial x} = r_2 - \lambda_2u_1 + \lambda_2u_2 \tag{14}$$

Finally, after substituting the values of u_1 and u_2 into the expressions for $\dot{\lambda}_1$ and $\dot{\lambda}_2$, we obtain:

$$\dot{\lambda}_1 = -r_1 + \frac{\lambda_1^2(1-x)}{2k_1} - \frac{\lambda_1\lambda_2 x}{2k_2} \tag{15}$$

$$\dot{\lambda}_2 = r_2 + \frac{\lambda_1\lambda_2(1-x)}{2k_1} - \frac{\lambda_2^2 x}{2k_2} \tag{16}$$

These final expressions represent the dynamics of the co-state variables within the model, taking into account the optimal control laws derived earlier.

After reducing the equation of the system, we obtain:

$$\dot{x} = \frac{\lambda_1(1-x)}{2k_1}(1-x) + \frac{\lambda_2 x}{2k_2}x \tag{17}$$

$$\dot{\lambda}_1 = -r_1 + \frac{\lambda_1^2(1-x)}{2k_1} - \frac{\lambda_1\lambda_2 x}{2k_2} \tag{18}$$

$$\dot{\lambda}_2 = r_2 + \frac{\lambda_1\lambda_2(1-x)}{2k_1} - \frac{\lambda_2^2 x}{2k_2} \tag{19}$$

$$\dot{x}(0) = x_0, \quad \lambda_1(T) = 0, \quad \lambda_2(T) = 0 \tag{20}$$

By integrating the differential equations (17), (18) and (19), we have the following system:

$$\begin{aligned} x(t) &= x_0 + \int_0^t \frac{\lambda_1(1-x)^2}{2k_1} + \frac{\lambda_2 x^2}{2k_2}\,dt \\ &= x_0 + \int_0^t \frac{\lambda_1}{2k_1} - \frac{\lambda_1}{k_1}x + \frac{\lambda_1}{2k_1}x^2 + \frac{\lambda_2}{2k_2}x^2\,dt \end{aligned} \tag{21}$$

$$\begin{aligned} \lambda_1(t) &= \int_T^t -r_1 + \frac{\lambda_1^2(1-x)}{2k_1} - \frac{\lambda_1\lambda_2 x}{2k_2}\,dt \\ &= \int_T^t -r_1 + \frac{\lambda_1^2}{2k_1} - \frac{\lambda_1^2}{2k_1}x - \frac{\lambda_1\lambda_2}{2k_2}x\,dt \end{aligned} \tag{22}$$

$$\begin{aligned} \lambda_2(t) &= \int_T^t r_2 + \frac{\lambda_1\lambda_2}{2k_1}(1-x) - \frac{\lambda_2^2}{2k_2}x\,dt \\ &= \int_T^t r_2 + \frac{\lambda_1\lambda_2}{2k_1} - \frac{\lambda_1\lambda_2}{2k_1}x - \frac{\lambda_2^2}{2k_2}x\,dt \end{aligned} \tag{23}$$

By applying the Picard method to equations (21), (22) and (23), we have:

$$\begin{aligned} x_n(t) = x_0 + \int_0^t &\frac{(\lambda_1)_{n-1}}{2k_1} - \frac{(\lambda_1)_{n-1}}{k_1}x_{n-1} \\ &+ \frac{(\lambda_1)_{n-1}}{2k_1}x_{n-1}^2 + \frac{(\lambda_2)_{n-1}}{2k_2}x_{n-1}^2\,dt \end{aligned} \tag{24}$$

$$\begin{aligned} (\lambda_1(t))_n = \int_T^t &-r_1 + \frac{(\lambda_1)_{n-1}^2}{2k_1} - \frac{(\lambda_1)_{n-1}^2}{2k_1}x_{n-1} \\ &- \frac{(\lambda_1)_{n-1}(\lambda_2)_{n-1}}{2k_2}x_{n-1}\,dt \end{aligned} \tag{25}$$

$$\begin{aligned} (\lambda_2(t))_n = \int_T^t &r_2 + \frac{(\lambda_1)_{n-1}(\lambda_2)_{n-1}}{2k_1} - \frac{(\lambda_1)_{n-1}(\lambda_2)_{n-1}}{2k_1} \\ &x_{n-1} - \frac{(\lambda_2)_{n-1}^2}{2k_2}x_{n-1}\,dt \end{aligned} \tag{26}$$

$$x(0) = x_0, \quad (\lambda_1)_0 = 0, \quad (\lambda_2)_0 = 0 \tag{27}$$

For $n=1$, we have:

$$x_1(t) = x_0 + \int_0^t \frac{(\lambda_1)_0}{2k_1} - \frac{(\lambda_1)_0}{k_1}x_0 + \frac{(\lambda_1)_0}{2k_1}x_0^2 + \frac{(\lambda_2)_0}{2k_2}x_0^2\,dt \tag{28}$$

$$(\lambda_1(t))_1 = \int_T^t -r_1 + \frac{(\lambda_1)_0^2}{2k_1} - \frac{(\lambda_1)_0^2}{2k_1}x_0 - \frac{(\lambda_1)_0(\lambda_2)_0}{2k_2}x_0\,dt \tag{29}$$

$$(\lambda_2(t))_1 = \int_T^t r_2 + \frac{(\lambda_1)_0(\lambda_2)_0}{2k_1} - \frac{(\lambda_1)_0(\lambda_2)_0}{2k_1}x_0 - \frac{(\lambda_2)_0^2}{2k_2}x_0\,dt \tag{30}$$

$$x(0) = x_0, \quad (\lambda_1)_0 = 0, \quad (\lambda_2)_0 = 0 \tag{31}$$

Therefore, the first approximation for x, λ_1 and λ_2 is the following:

$$x_1(t) = x_0 \tag{32}$$

$$(\lambda_1(t))_1 = \int_T^t -r_1\,dt = -r_1(t-T) \tag{33}$$

$$(\lambda_2(t))_1 = \int_T^t r_2\,dt = r_2(t-T) \tag{34}$$

Hence the first approximation of u_1 and u_2 is the following:

$$(u_1)_1 = \frac{(\lambda_1)_1(1-x_1)}{2k_1} = \frac{-r_1(t-T)}{2k_1}(1-x_0) \tag{35}$$

$$(u_2)_1 = \frac{-(\lambda_2)_1 x_1}{2k_2} = \frac{-r_2(t-T)}{2k_2}x_0 \tag{36}$$

For $n=2$, we have:

$$x_2(t) = x_1 + \int_0^t \frac{(\lambda_1)_1}{2k_1} - \frac{(\lambda_1)_1}{k_1}x_1 + \frac{(\lambda_1)_1}{2k_1}x_1^2 + \frac{(\lambda_2)_1}{2k_2}x_1^2\,dt \tag{37}$$

$$(\lambda_1(t))_2 = \int_T^t -r_1 + \frac{(\lambda_1)_1^2}{2k_1} - \frac{(\lambda_1)_1^2}{2k_1}x_1 - \frac{(\lambda_1)_1(\lambda_2)_1}{2k_2}x_1\,dt \tag{38}$$

$$(\lambda_2(t))_2 = \int_T^t r_2 + \frac{(\lambda_1)_1(\lambda_2)_1}{2k_1} - \frac{(\lambda_1)_1(\lambda_2)_1}{2k_1}x_1 - \frac{(\lambda_2)_1^2}{2k_2}x_1\,dt \tag{39}$$

Therefore, the second approximation for x, λ_1 and λ_2 is the following:

$$x_2(t) = x_0 + \left[\left(x_0 - \frac{1}{2}\right)\frac{r_1}{k_1} - \frac{r_1 x_0^2}{2k_1} + \frac{r_2 x_0^2}{2k_2}\right]\left(\frac{t^2}{2} - Tt\right) \tag{40}$$

$$(\lambda_1(t))_2 = \left[-r_1 + \frac{r_1}{2k_1}(1-x_0) + \frac{r_1 r_2}{2k_2}x_0\right]\frac{(t-T)^3}{3} \tag{41}$$

$$(\lambda_2(t))_2 = \left[r_2 - \frac{r_1 r_2}{2k_1}(1-x_0) - \frac{r_2^2}{2k_2}x_0\right]\frac{(t-T)^3}{3} \tag{42}$$

Hence the second approximation of u_1 and u_2 is the following:

$$\begin{aligned}(u_1)_2 &= \frac{(\lambda_1)_2(1-x_2)}{2k_1}\\ &= \left[\frac{(t-T)^3}{6k_1}\right]\left[-r_1 + \frac{r_1}{2k_1}(1-x_0) + \frac{r_1 r_2}{2k_2}x_0\right][1 - x_0\\ &- \left(\left(x_0 - \frac{1}{2}\right)\frac{r_1}{k_1} - \frac{r_1 x_0^2}{2k_1} + \frac{r_2 x_0^2}{2k_2}\right)\left(\frac{t^2}{2} - Tt\right)]\end{aligned} \tag{43}$$

$$\begin{aligned}(u_2)_2 &= \frac{-(\lambda_2)_2 x_2}{2k_2}\\ &= -\left[\frac{(t-T)^3}{6k_2}\right]\left[r_2 - \frac{r_1 r_2}{2k_1}(1-x_0) - \frac{r_2^2}{2k_2}x_0\right][x_0\\ &+ \left(\left(x_0 - \frac{1}{2}\right)\frac{r_1}{k_1} - \frac{r_1 x_0^2}{2k_1} + \frac{r_2 x_0^2}{2k_2}\right)\left(\frac{t^2}{2} - Tt\right)]\end{aligned} \tag{44}$$

The following section provides an in-depth look at the numerical simulation process and the results derived from the public transportation management model.

3 Numerical Simulation

In this section, the results of the numerical simulation for the public transportation management model are presented. The model is governed by differential equations, which describe the evolution of the state variable $x(t)$ and the control inputs $u_1(t)$ and $u_2(t)$[8,9]. The control inputs aim to optimize both service performance and demand management. For the simulation, the following parameter values were utilized:

These parameters control the trade-offs between service optimization and demand management. Specifically, r_1 and r_2 determine the importance of service performance and demand control, respectively, while k_1 and k_2 penalize excessive control efforts. The total simulation time, T, defines the observation period for system behavior, and x_0 represents the initial condition of the state variable, potentially reflecting initial congestion levels. The simulation results illustrate how the system's state variable evolves over time, showcasing the effectiveness of the control strategies in balancing service enhancement and demand regulation.

Next, we explore the key results and performance metrics from the simulation.

Table 1: Parameter values used in the simulation.

Parameter	Description	Value
r_1	Coefficient for service performance impact	0.5
r_2	Coefficient for demand control impact	0.2
k_1	Weight for control effort of u_1	1.0
k_2	Weight for control effort of u_2	1.0
T	Total simulation time (seconds)	2.0
x_0	Initial condition for state variable $x(t)$	0.5

4 Simulation Results

This section explores the outcomes of the simulation, analyzing the behavior of the state variable and control inputs over time, as well as the performance indices that measure the trade-off between system optimization and control efforts. The following analysis provides insights into the effectiveness of control strategies in balancing service improvement and demand regulation.

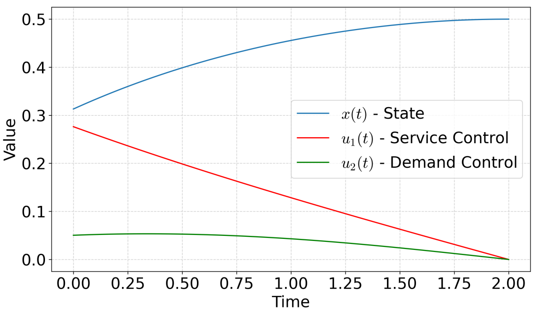

Fig. 1: Time evolution of the system state and control variables

The simulation outputs, visualized in Fig. 1, depict the time evolution of the state variable $x(t)$ and the control inputs $u_1(t)$ and $u_2(t)$. The state variable, $x(t)$, which might represent congestion or service utilization, decreases from an initial value of 0.5, indicating system improvement over time. This decline suggests that the control actions are effectively optimizing the system's performance. The control input $u_1(t)$, which is responsible for improving service, shows a decreasing trend, signifying increased efforts to enhance service, such as by adding more vehicles or reducing wait times. On the other hand, the control input $u_2(t)$, related to demand management, begins with a small negative value, decreases initially, and then rises again. This pattern reflects an initial intensification of demand control efforts, followed by a reduction as the system approaches equilibrium.

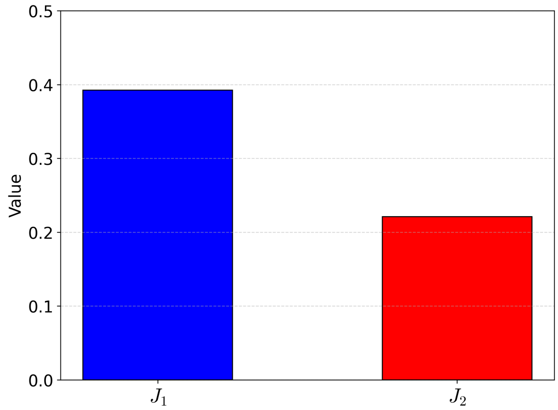

Fig. 2: Comparison of initial performance indices between the service provider's objective function J_1 and the demand regulator's objective function J_2 before optimization.

Fig. 2 presents the integrands for the performance indices J_1 and J_2, which measure the system's performance relative to the control efforts. The integrand for J_1, which balances service improvement and control effort, decreases over time, reflecting a reduction in congestion or enhanced service performance. However, as $u_1(t)$ becomes more negative, the control effort increases, leading to diminishing returns in performance improvement. The integrand for J_2, representing demand control, shows an increase over time, indicating that demand management becomes progressively more challenging as the simulation progresses. The system expends more effort on demand control, with diminishing returns toward the end of the simulation.

The final values of the performance indices are $J_1 = 0.39111$ and $J_2 = 0.22131$. These values provide insights into the trade-offs between service optimization and demand management. The higher value of J_1 indicates that improving service required more effort compared to demand control, as reflected in the lower value of J_2. Overall, the system requires less effort to manage demand than to enhance service, but both objectives necessitate significant control inputs.

5 Sensitivity and Optimization

Sensitivity analysis is the study of how various sources of uncertainty in a mathematical model's or system's inputs can be assigned to the uncertainty in the model's or system's output, whether it be numerical or otherwise. This entails calculating sensitivity indices, which measure how much an input or set of inputs affects the result. Uncertainty and sensitivity analysis should ideally be conducted together. Uncertainty analysis is a related activity that focuses more on quantifying and propagating uncertainty.

Sensitivity analysis is a crucial component of model construction and quality control in models with numerous input variables. It can be helpful to ascertain the influence of an uncertain variable for a variety of reasons, such as [21]:

- Evaluating a model or system's ability to produce reliable outcomes when uncertainty is present.
- Improved comprehension of how input and output variables relate to one another in a system or model.
- Identifying model inputs that significantly raise output uncertainty and should thus be the focus of attention in order to boost robustness is one way to reduce uncertainty.
- Looking for model flaws (by finding unexpected connections between inputs and outputs).
- Fixing model input that has no bearing on the output or locating and eliminating unnecessary components from the model structure are examples of model simplification.
- Improving communication between decision makers and modelers (e.g., by making proposals more persuasive, compelling, comprehensible, or credible).
- Locating areas in the input factor space where the model's output satisfies some optimal condition or is either the maximum or the lowest (see optimization and Monte Carlo filtering).
- By concentrating on the sensitive parameters, models with a lot of parameters can be calibrated.
- To find significant relationships between observations, model inputs, and forecasts or predictions in order to improve models.

In this section, we study the effect of changing r_1, r_2, k_1, and k_2 on j_1 and j_2. To conduct this study, we will discuss the sensitivity of these factors as shown

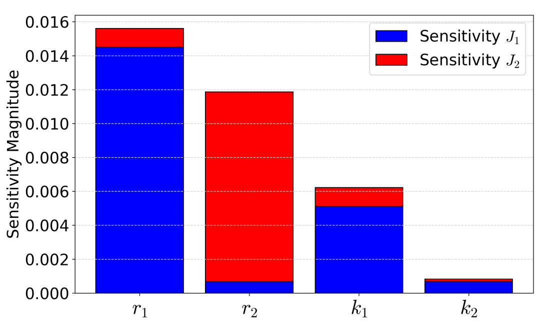

Fig. 3: Sensitivity analysis of performance indices for the impact of parameter variations r_1, r_2, k_1 and k_2 on the objective functions J_1 and J_2.

Based on the study of sensitivity and from Fig. 3, we noticed that the most effective factors on j_1 and j_2 are r_1 and r_2 as shown in Fig. 3. Since, the goal of an optimization problem is to maximize or minimize a function in relation to a set, which frequently represents the range of options accessible in a given circumstance. Comparing the many options to see which would be "best" is made possible by the function. For this reason, we will optimize these two factors to get the best value for them using the genetic algorithm as shown in the fig. 4

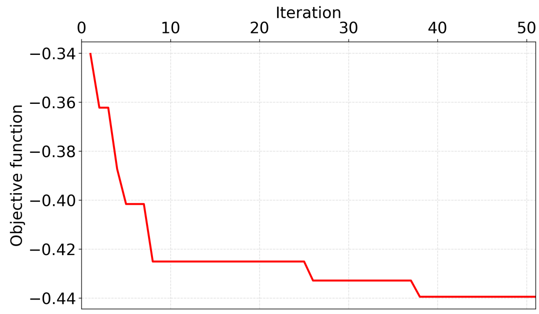

Fig. 4: Convergence of the Genetic Algorithm for finding the optimal values for r_1 and r_2 that minimize the combined cost function.

this Figure shows that the accuracy of the factors r_1 and r_2 which is increased significantly and the error rate decreased greatly with the iteration.

Therefore, we repeated the all previous calculations in the section 4, but by using the optimal values of r_1 and r_2, then we got the control inputs $u_1(t)$ and $u_2(t)$ as

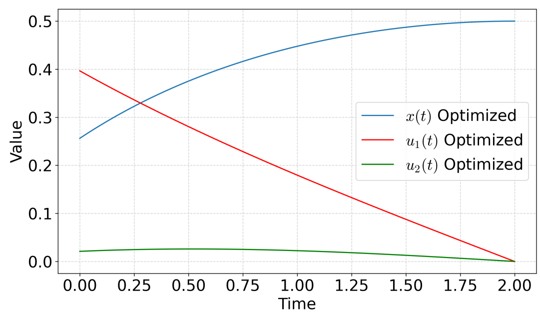

Fig. 5: Time evolution of the optimized system state and control variables after the optimization of weighting parameters r_1 and r_2 via the Genetic Algorithm.

Moreover, after applying this study, we noticed the effect on both j_1 and j_2, where j_1 increased significantly and j_2 decreased, as shown in the following figure.

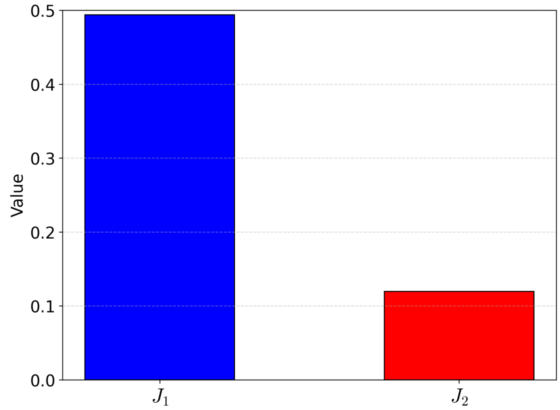

Fig. 6: Comparison of performance indices J_1 and J_2 after applying the optimal parameters derived from the Genetic Algorithm.

6 Conclusion

This paper presented a differential game model to optimize public transportation management by balancing service improvement and demand control. The model, based on a non cooperative, nonzero-sum framework, allowed us to derive optimal control strategies using Hamiltonian functions. Numerical simulations showed how the strategies effectively reduce congestion while maintaining system efficiency.

For future research, several areas could be explored to enhance the model's applicability. Real-time data integration would allow for dynamic adjustments in control strategies, making the system more responsive to changing conditions. Cooperative game theory could also be investigated, encouraging collaboration between service providers and demand regulators for more efficient outcomes. Additionally, incorporating dynamic pricing and multi-agent systems—such as different transportation modes—could provide a more holistic view of urban transportation networks. Finally, adding stochastic elements would improve the model's robustness, accounting for uncertainties in passenger demand and system disruptions.

In conclusion, this study lays the groundwork for a comprehensive approach to public transportation management, with future work offering pathways to refine and enhance the model's practical applications.

Appendix

SET T = 2.0
SET $x_0 = 0.5$
SET parameter ranges = {
"r_1": LINSPACE(0.3, 0.7, 5),
"r_2": LINSPACE(0.1, 0.5, 5),
"k_1": LINSPACE(0.8, 1.2, 5),
"k_2": LINSPACE(0.8, 1.2, 5),
"T": LINSPACE(1.0, 3.0, 5) }
SET t eval = LINSPACE(0, T, 500)

Functions:

1. **Function: System Dynamics**
FUNCTION system dynamics$(t, y, r_1, r_2, k_1, k_2)$ {
COMPUTE control inputs u_1, u_2
COMPUTE derivatives $dx/dt, d\lambda_1/dt, d\lambda_2/dt$
RETURN $[dx/dt, d\lambda_1/dt, d\lambda_2/dt]$}

2. **Function: Compute Performance Indices**
FUNCTION compute performance indices (r_1, r_2, k_1, k_2) {
INITIALIZE state vector y_0
SOLVE system dynamics over t eval
EXTRACT state variable x and control inputs u_1, u_2
COMPUTE performance indices J_1, J_2
RETURN J_1, J_2, x, u_1, u_2}

3. **Function: Sensitivity Analysis**
FUNCTION sensitivity analysis(base parameters, perturbation = 0.05) { COMPUTE baseline J_1, J_2
FOR each parameter IN base parameters {
PERTURB parameter by +perturbation and compute J_1, J_2
PERTURB parameter by -perturbation and compute J_1, J_2
CALCULATE sensitivity for J_1, J_2}
RETURN sensitivities}

4. **Function: Optimize r_1 and r_2**
FUNCTION optimize $r_1 \& r_2$ (r_1 bounds, r_2 bounds, k_1, k_2) {
DEFINE objective function
USE optimization algorithm to minimize objective function
RETURN optimized r_1, r_2 }

Main Execution:

1. **Initialize Parameters**
SET $r_1 = 0.5, r_2 = 0.2, k_1 = 1.0, k_2 = 1.0$

2. **Calculate Initial Performance Indices**
CALL compute performance indices (r_1, r_2, k_1, k_2)
PLOT state variable and control inputs
PLOT J_1 and J_2 as bar chart

3. **Perform Sensitivity Analysis**
CALL sensitivity analysis $([r_1, r_2, k_1, k_2])$
PLOT sensitivity results

4. **Optimize r1 and r2**
SET bounds for $r1$ and $r2$

CALL optimizerr_1&r_2(r_1 bounds, r_2 bounds, k_1, k_2)
CALL compute performance indices with optimized r_1, r_2
PLOT state variable and control inputs after optimization
PLOT optimized J_1 and J_2 as bar chart

Data Availability

The authors declare that the data supporting the findings of this study are available within the paper

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Conflicts of Interest

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References

[1] **Dockner, E.J.; Jorgensen, S.; Van Long, N.; Sorger, G.**, *Differential Games in Economics and Management Science*, Cambridge University Press, Cambridge, UK, (2000).
[2] **Erickson, G.M.**, *Dynamic Models of Advertising Competition*, Kluwer, Boston, MA, USA, (2003).
[3] **Yeung, D.W.K.; Petrosjan, L.**, *Cooperative Stochastic Differential Games*. Springer, Berlin/Heidelberg, Germany, (2005).
[4] **Friedman, A.**, *Differential Games*, John Wiley & Sons, Hoboken, New Jersey, USA, (1971).
[5] **Awad Talal Alabdala, Essam EL-Seidy, Waleed Adel**, *Applying the Tau Technique in Multi-Player Differential Games: A Novel Approach in Economic Strategy Analysis*, Partial Differential Equations in Applied Mathematics, 10, 100724, (2024).
[6] **Maan T. A, Essam El-Seidy, Eliwa M. R, Muner M. A.**, *Innovative Stochastic Finite Difference Methods in Mean Field Games: A Wiener Process Approach*, Nonlinear Functional Analysis and Applications, (2024).
[7] **Shahd H. A, Essam El-Seidy, Eliwa M. R, Muner M. A.**, *Strategic Market Dynamics: Exploring Differential Games in Marketing and Pricing Strategies*, Journal of Game Theory, 13(1), pp. 1-6, (2024).
[8] **Maan T. A, Essam El-Seidy, Eliwa M. R, Muner M. A.**, *Optimizing Bioeconomic Models: A Comprehensive Approach Using the Jacobi Tau Method*, Journal of Game Theory, 13(1), pp. 7-14, (2024).
[9] **Shahd H. A, Essam El-Seidy, Eliwa M. R, Muner M. A.**, *Optimal Strategies in Bioeconomic Differential Games: Insights from Chebyshev Tau Method*, Nonlinear Functional Analysis and Applications, 29(2), pp. 527-543, (2024).
[10] **Bressan, A.**, *Noncooperative Differential Games: A Tutorial*, Milan Journal of Mathematics, 79, 357-427, (2011).
[11] **Canuto, C.; Hussaini, M.Y.; Quarteroni, A.; Zang, T.A.**, *Spectral Methods: Fundamentals in Single Domains*, Springer, New York, NY, USA, (2006).
[12] **Bhrawy, A.H.; Zaky, M.A.; Baleanu, D.**, *New Numerical Approximations for Space-Time Fractional Burgers' Equations via a Legendre Spectral-Collocation Method*. Romanian Reports in Physics, 67, 2, (2015).
[13] **Engwerda, J.C.**, *On the Open-Loop Nash Equilibrium in LQ Games*, Journal of Economic Dynamics and Control, 22(6), 729-762, (1998).
[14] **Zhang, H.; Wei, Q.; Liu, D.**, *An Iterative Dynamic Programming Method for Solving a Class of Nonlinear Zero-Sum Differential Games*, Automatica, 47, 207-214, (2011).
[15] **Kossioris, G.; Plexousakis, M.; Xepapadeas, A.; de Zeeuw, A.; Maler, K.G.**, *Feedback Nash Equilibria for Nonlinear Differential Games in Pollution Control*, Journal of Economic Dynamics and Control, 32, 1312-1331, (2008).
[16] **Jiménez-Lizárraga, M.; Basin, M.; Rodríguez, V.; Rodríguez, P.**, *Open-Loop Nash Equilibrium in Polynomial Differential Games via State-Dependent Riccati Equation*, Automatica, 53, 155-163, (2015).
[17] **Shen, J.; Tang, T.; Wang, L.L.**, *Spectral Methods in Algorithms, Analysis and Applications*, Springer, Berlin/Heidelberg, Germany, (2011).
[18] **Friedman, A.**, *Stochastic Differential Games: Theory and Applications*, John Wiley & Sons, (2014).
[19] **Doha, E.H.; Abd-Elhameed, W.M.; Bhrawy, A.H.**, *Efficient Spectral Ultra spherical-Galerkin Algorithms for the Direct Solution of 2nth-Order Linear Differential Equations*, Applied Mathematics and Computation, 247, 30-46, (2014).
[20] **Zhang, X.; Liu, Z.; Tang, Y.**, *An Efficient Jacobi Pseudo spectral Approximation for Nonlinear Complex Generalized Zakharov System*, Applied Mathematics and Computation, 267, 49-56, (2016).
[21] **Pannell, D. J.**, *Sensitivity Analysis of Normative Economic Models: Theoretical Framework and Practical Strategies*, Agricultural Economics. 16 (2): 139–152, (1997).

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