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Pricing Formula for Power Options with Jump-Diffusion

Siti Nur Iqmal Ibrahim[1,2,*], *John G. O'Hara*[3] *and Muhammad Syazwan Mohd Zaki*[1]

[1] Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
[2] Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
[3] Centre for Computational Finance and Economic Agents, University of Essex, Colchester CO4 3SQ, United Kingdom

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Abstract: Payoff of a power option is typified by its underlying share price raised to a constant power. Also known as leveraged option, a minor change in its underlying may lead to a significant change in its price. In this study, we derive pricing formula for power options using the martingale approach when the underlying asset follows a jump-diffusion process.

Keywords: power option, leveraged option, Black-Scholes, geometric Brownian motion, jump-diffusion

1 Introduction

Anything that derives its value from an underlying asset is a derivative, and the underlying asset for stock options is the stock. Since stock options were first introduced, they have gained popularity and have been used widely. According to [5], a call (put) option gives the buyer the right, but not the obligation, to purchase (sell) the underlying share at a strike price. At the time of purchase, the maturity date is agreed upon. Hence, both options are subject to expiration and can expire worthless. Other than the call and put options, there are European options which is exercisable only at its maturity date, and American options which is exercisable at any time between its purchase date and its maturity date.

The standard call and put options are referred to as vanilla options. Many non-standard options emerge from the extension of vanilla options, and these are called exotic options. Exotic options are typified by the different form of the payoffs to that of vanilla options, and power options, also known as leveraged option is one example.

Power options are options whose payoffs are based on the underlying asset raised to a power [11,12]. This additional feature gives the buyer of the option the potential to receiving a much higher payoff than that from a vanilla option. A small change in the value of the underlying of a power option may lead to a significant change in the option's price [13,14]. Moreover, [4] and [12] show that when the dynamics of the underlying share price is a geometric Brownian motion (GBM), a package of vanilla options can be used to hedge a single power option.

Given the underlying asset price S, strike price K, and maturity time T. Then, under the Black-Scholes [2] environment, the price of a power option is given as follows (see [4,12]):

$$PC = S_0^{\beta} \exp\left[(\beta-1)\left(r+\frac{\beta\sigma^2}{2}\right)T\right] N(d_{1,\beta}) - Ke^{-rT} N(d_{2,\beta}), \tag{1}$$

where r is the risk-free rate, σ is the volatility, β is the constant power, and:

$$d_{1,\beta} = \frac{\ln\left(S_0^{\beta}/K\right) + \beta\left(r - \frac{\sigma^2}{2} + \beta\sigma^2\right)T}{\beta\sigma\sqrt{T}},$$

$$d_{2,\beta} = d_{1,\beta} - \beta\sigma\sqrt{T}.$$

Reference [6] shows that the closed-form pricing formulas for power options as given by Equation (1) can be obtained from the closed-form pricing formulas for vanilla options via a transformation on the underlying asset price and the volatility. Moreover, [3] refers to

$$S_0^{\beta} e^{(\beta-1)\left(r+\frac{1}{2}\beta\sigma^2\right)T}$$

as a power contract with unit strike.

* Corresponding author e-mail: iqmal@upm.edu.my

In the Black-Scholes model, the asset price process follows a GBM with drift:

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t,$$

where W_t is a standard Brownian motion. There exists evidence in the market which shows that GBM with drift is an inaccurate process to model the movements of asset prices. In addition, the Black-Scholes model assumes constant volatility. However, this assumption contradicts with the fact that market prices exhibit volatility smile and heavy tails distribution. Stochastic volatility models and models with jumps are two famous extensions to the Black-Scholes model which try to capture these shortcomings.

The dynamics of a share price can be described via jump-diffusion processes. This is supported by empirical evidences, such as the work of [1] and [8], which support the existence of jumps in share prices. To the authors' best knowledge, [9] was the pioneer in introducing jumps in the share price process for option pricing problems. Merton [9] uses conditional normality of MJD model and presents the price of the option as a conditional Black-Scholes type solution. Moreover, [10] compares pure jump models with Merton jump-diffusion (MJD) model, and shows that the latter incorporates most of the share price's behavior.

In this paper, we study the pricing of European-style power options when the dynamics of the underlying share price follows a jump-diffusion process. The paper is organized as follows. Section 2 presents the model derivation for the asset price with jump-diffusion. Section 3 derives the pricing formulas for power options with jump-diffusion process using a martingale approach. Section 4 provides the numerical results, and Section 5 concludes the paper.

2 Merton Jump-Diffusion Model

This section describes the jump-diffusion model as provided in [9]. In a risk-neutral setting, the dynamics of the asset price process with jump-diffusion follows the following stochastic differential equation (SDE):

$$dS_t = (r-\lambda k)S_t\,dt + \sigma S_t\,dW_t + (y_t-1)S_t\,dN_t, \qquad (2)$$

where r is the risk-free rate, σ is the volatility of the asset return when no jumps occur, W_t is a Wiener process, N_t is a Poisson process with intensity λ, y_t is the absolute price jump size, and (y_t-1) is the relative price jump size [9, see].

Merton assumes that the logarithmic asset price jump sizes follows a normal distribution,

$$\ln y_t \sim \mathcal{N}(\mu_J, \delta^2),$$

where μ_J is the log-return jump-size and δ is the volatility of log-return jump, which are identically and independently distributed.

Hence,

$$(y_t-1) \sim Lognormal\left(e^{\mu_J+\frac{\delta^2}{2}}-1,\, e^{2\mu_J+\delta^2}\left(e^{\delta^2}-1\right)\right).$$

Using Itô formula for jump-diffusion process, Equation (2) can be solved as follows:

$$S_t = S_0 \exp\left[\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t\right]\prod_{i=1}^{N_t} y_i, \qquad (3)$$

or similarly:

$$S_t = S_0 \exp\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)t+\sigma W_t+\sum_{i=1}^{N_t}\ln y_i\right], \qquad (4)$$

where the expectation of the price change due to a jump is

$$k \equiv \mathbf{E}(y_t-1) = \exp\left[\mu_J+\frac{\delta^2}{2}\right]-1.$$

Equation (4) can be written as follows:

$$S_t = S_0 \exp(X_t),$$

where the share price process $\{S_t : 0 \le t \le T\}$ is presented as an exponential Lévy process $\{X_t : 0 \le t \le T\}$ which is a GBM with jumps. It follows that the logarithmic returns can also be modeled as Lévy process as such:

$$\ln\left(\frac{S_t}{S_0}\right) \equiv X_t = \left(r-\frac{\sigma^2}{2}-\lambda k\right)t+\sigma W_t+\sum_{i=1}^{N_t}\ln y_i. \qquad (5)$$

3 Pricing Power Options with Jump-Diffusion

Let $(\Omega,\mathscr{F},\mathbf{Q})$ be a probability space on which a Brownian motion W_t is defined, $\{\mathscr{F}_t : 0 \le t \le T\}$ is the filtration generated by W_t, and $\mathbf{Q}$ is a risk-neutral measure. The dynamics of the MJD process is given by Equation (4) as such:

$$S_T = S_t \exp\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)t+\sigma W_t+\sum_{i=1}^{N_t}\ln y_i\right]. \qquad (6)$$

A power option has a payoff function of

$$H\left(S_T^\beta\right)$$

where its price is the discounted risk-neutral conditional expectation of the payoff at r as follows:

$$PC_{JD}(t,S_t) = \exp[-r(T-t)]\mathbf{E}^{\mathbf{Q}}\left[H\left(S_T^\beta\right)|\mathscr{F}_t\right]. \qquad (7)$$

Equation (7) can be expressed as:

$$PC_{JD}(t,S_t) = \exp[-r(T-t)] \times \mathbf{E}^{\mathbf{Q}}\left[H\left(S_t^{\beta}\exp\left[\ln\left(\frac{S_T^{\beta}}{S_t^{\beta}}\right)\right]\right)\Big|\mathscr{F}_t\right]. \quad (8)$$

Replacing Equation (5) into Equation (8), and letting $\ln y_k = Y_k$ yields:

$$PC_{JD}(t,S_t) = e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\left[H\left(S_t^{\beta}e^{\beta\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)(T-t)+\sigma W_t+\sum_{k=1}^{N_{T-t}}Y_k\right]}\right)\right]. \quad (9)$$

The distribution of the compound Poisson process is:

$$\sum_{k=1}^{N_{T-t}} Y_k \sim \mathscr{N}\left(i\mu_J, i\delta^2\right),$$

for i number of jumps:

$$i \equiv N_{T-t} = 0,1,2,3,\ldots$$

Using the law of iterated expectations, we have the following:

$$PC_{JD}(t,S_t) = e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\left\{\mathbf{E}^{\mathbf{Q}}\left[H\left(S_t^{\beta}e^{\beta\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)(T-t)+\sigma W_t+\sum_{k=1}^{N_{T-t}}Y_k\right]}\right)\Big|N_{T-t}=i\right]\right\} \quad (10)$$

Therefore, by conditioning on i and letting $t=0$ yields:

$$PC_{JD}(t,S_t) = e^{-rT}\sum_{i=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^i}{i!} \times \mathbf{E}^{\mathbf{Q}}\left[H\left(S_0^{\beta}e^{\beta\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)T+\sigma W_T+\sum_{k=1}^{i}Y_k\right]}\right)\right], \quad (11)$$

where

$$\beta\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)T+\sigma W_T+\sum_{k=1}^{i}Y_k\right] \sim \mathscr{N}\left(\beta\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)T+i\mu\right];\beta^2\left(\sigma^2T+i\delta^2\right)\right). \quad (12)$$

Suppose we consider the payoff of a power option as follows:

$$H\left(S_T^{\beta}\right) = \max\left(S_T^{\beta}-K,0\right).$$

Let

$$m = \beta\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)T+i\mu\right]$$

and

$$s = \beta\sqrt{\sigma^2T+i\delta^2}.$$

Therefore,

$$\ln\left(\frac{S_T^{\beta}}{S_t^{\beta}}\right) \sim \mathscr{N}\left(m,s^2\right) \sim m+sz.$$

On that account, the price of a power option in Equation (11) is provided as follows:

$$\begin{aligned}PC_{JD}(t,S_t) &= e^{-rT}\sum_{i=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^i}{i!}\mathbf{E}^{\mathbf{Q}}\left[\max\left(S_0^{\beta}e^{\ln\left(S_T^{\beta}/S_0^{\beta}\right)}-K,0\right)\right]\\ &= e^{-rT}\sum_{i=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^i}{i!}\\ &\quad\times\left\{\mathbf{E}^{\mathbf{Q}}\left[S_0^{\beta}e^{\ln\left(S_T^{\beta}/S_0^{\beta}\right)}\mathbf{I}_{\left\{S_0^{\beta}>K\right\}}\right]-K\mathbf{E}^{\mathbf{Q}}\left[\mathbf{I}_{\left\{S_0^{\beta}>K\right\}}\right]\right\}\\ &= e^{-rT}\sum_{i=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^i}{i!}\\ &\quad\times\left\{\mathbf{E}^{\mathbf{Q}}\left[S_0^{\beta}e^{m+sz}\mathbf{I}_{\left\{z>\frac{\ln\left(K/S_0^{\beta}\right)-m}{s}\right\}}\right]\right.\\ &\quad\left.-K\mathbf{E}^{\mathbf{Q}}\left[\mathbf{I}_{\left\{z>\frac{\ln\left(K/S_0^{\beta}\right)-m}{s}\right\}}\right]\right\}. \end{aligned} \quad (13)$$

Let $\frac{\ln\left(K/S_0^{\beta}\right)-m}{s} = d_1$. Then, we have:

$$\begin{aligned}PC_{JD}(t,S_t) &= e^{-rT}\sum_{i=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^i}{i!}\\ &\quad\times\left[S_0^{\beta}e^{m}\int_{d_1}^{\infty}e^{sz}\left[\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\right]dz-K\int_{d_1}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\right]\\ &= e^{-rT}\sum_{i=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^i}{i!}\\ &\quad\times\left[S_0^{\beta}e^{m+\frac{s^2}{2}}\int_{d_1}^{\infty}\frac{e^{-\frac{(z-s)^2}{2}}}{\sqrt{2\pi}}dz-K\int_{d_1}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\right]. \end{aligned} \quad (14)$$

Let $w = z - s$. Then,

$$
\begin{aligned}
&PC_{JD}(t,S_t)\\
&= e^{-rT}\sum_{i=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^i}{i!}\\
&\quad\times\left[S_0^{\beta} e^{m+\frac{s^2}{2}}\int_{d_1-s}^{\infty}\frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}\,dw - K\int_{d_1}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz\right]\\
&= e^{-rT}\sum_{i=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^i}{i!}\\
&\quad\times\left\{S_0^{\beta} e^{m+\frac{s^2}{2}}[1-N(d_1-s)] - K[1-N(d_1)]\right\}\\
&= e^{-rT}\sum_{i=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^i}{i!}\left[S_0^{\beta} e^{m+\frac{s^2}{2}}N(-d_1+s) - KN(-d_1)\right].
\end{aligned}
$$

This completes the derivation, hence leads us to the following proposition.

Proposition 1 The price of a power call option with maturity T and strike price K, and whose underlying asset price follows the dynamics in (2) where jumps are normally distributed is given by:

$$
\begin{aligned}
&PC_{JD}(t,S_t)\\
&= e^{-rT}\sum_{i=0}^{\infty} e^{-\hat{\lambda} T}\frac{(\hat{\lambda} T)^i}{i!}\\
&\quad\times\left\{S_0^{\beta} e^{(\beta-1)\left(r+\frac{\beta\sigma^2}{2}\right)T} e^{\beta\left(-\lambda kT+i\mu+\frac{\beta i\delta^2}{2}\right)} N(d_{1,\beta,JD})\right.\\
&\quad\left. - KN(d_{2,\beta,JD})\right\},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
\hat{\lambda} &= \lambda\left(e^{\mu_J+\frac{\delta^2}{2}}\right),\\
k &= e^{\mu_J+\frac{\delta^2}{2}} - 1,\\
d_{1,\beta,JD} &= \left[\frac{\ln(S_0^{\beta}/K) + \beta\left[\left(r-\frac{\sigma^2}{2}-\lambda k\right)T + i\mu_J\right]}{\beta\sqrt{\sigma^2 T + i\delta^2}}\right]\\
&\quad + \beta\sqrt{\sigma^2 T + i\delta^2},\\
d_{2,\beta,JD} &= d_{1,\beta,JD} - \beta\sqrt{\sigma^2 T + i\delta^2}.
\end{aligned}
$$

This proposition is the main result of this paper.

4 Numerical Examples

In this section, we present numerical results for the prices of a power option under the MJD model.

Given the following parameters of $S = 5, r = 0.05, \tau = 0.25, \sigma = 0.2, \mu_J = -0.1$ and $\delta = 0.1$, Figure 1 shows Merton prices for vanilla options are higher than Black-Scholes prices. For power options, the parameters remain the same but we choose $S = 3$. It can be seen in Figures 2 and 3, where the former we have $\beta = 1.5$, and $\beta = 2$ for the latter, that the Merton prices are also greater than the Black-Scholes prices. However, the underlying asset of a power option is raised to a power. Therefore, the difference between the Merton and Black-Scholes prices are much higher for the power options than the vanilla options.

We can also see that in Figures 1, 2 and 3, as λ (i.e. intensity of the jumps per unit of time) increases from 1 to 5, the difference between Merton price and Black-Scholes price becomes larger. This is expected to occur since we set $\sigma_{BS} = \sigma_{MJD}$ where there exists λ, μ_J, and δ. Hence, higher Merton prices than Black-Scholes prices.

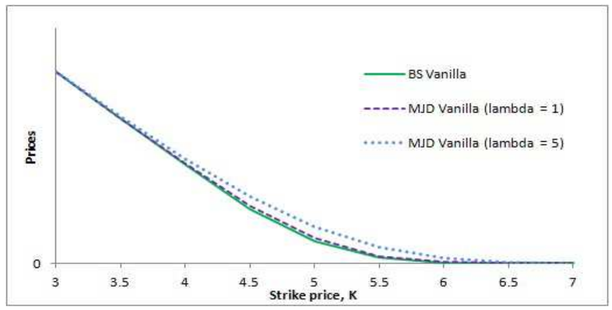

Fig. 1: Vanilla option prices: Black-Scholes vs Merton

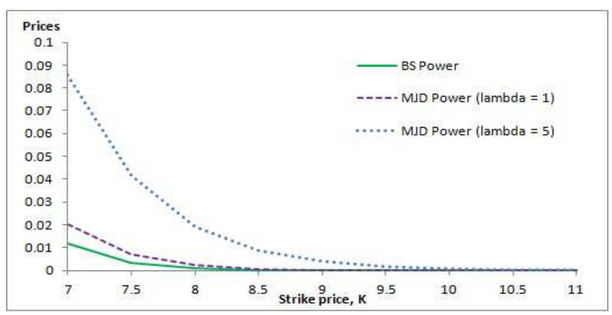

Fig. 2: Power option prices: Black-Scholes vs Merton ($\beta = 1.5$)

5 Conclusion

This paper prices power option under Merton's jump-diffusion model using a martingale approach. A pricing formula for power options with jump-diffusion is derived.

Numerical results display that vanilla option prices with jump-diffusion are higher than that without jumps. This is also the case for power options. However the difference between the Merton prices and the

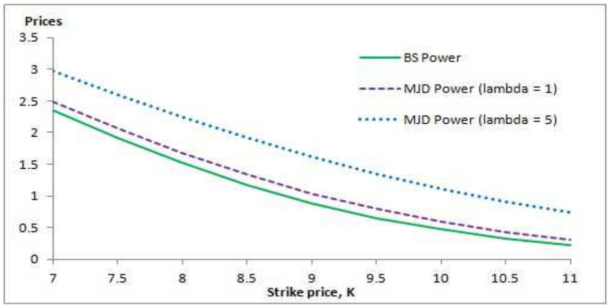

Fig. 3: Power option prices: Black-Scholes vs Merton ($\beta = 2.0$)

Black-Scholes prices are much higher for power options than that of vanilla options. This is expected because of the leverage feature of the power option where a small change in the underlying of a power option may lead to a significant change in the price of a power option.

A possible future work is to apply this work to price power barrier options [7] with jump-diffusion process.

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Siti Nur Iqmal Ibrahim received her PhD in Computational Finance from the University of Essex. Her research interests are in the areas of applied mathematics and mathematical finance.

John G. O'Hara is the Deputy Director of CCFEA and a Senior Lecturer at University of Essex, United Kingdom. He received his PhD degree from University of the Witwatersrand. His research interests are in the areas of applied mathematics and mathematical finance.

Muhammad Syazwan Mohd Zaki is a Research Assistant at Universiti Putra Malaysia. He holds a degree from Universiti Teknologi MARA. His research interests are in the areas of applied mathematics.