Soft Fuzzy Syntopogenous Spaces

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Abstract: In this paper, we introduce the concepts of semi-topogenous (resp. topogenous) soft fuzzy order and the syntopogenous soft fuzzy structure and study many of their properties and show that there is a one-one corresponds between perfect topogenous soft fuzzy structures and soft fuzzy topological structures.

Keywords: soft set - soft fuzzy set - soft fuzzy topological space - soft fuzzy topogenous order - soft fuzzy syntopogenous structure.

1 Introduction

The soft set theory of Molodstov [6] offers a general mathematical tool for dealing with uncertainty and vagueness of objects. In the last three years many structures using the soft set theory are progressing rapidly [1, 2, 4, 11, 13]. Maji et al. [8, 9] proposed the concept of soft fuzzy set and developed some of their properties. In recent years, the researchers have contributed a lot towards the fuzzyfication of the soft set theory. In 1963 Csaszar [3] introduced the syntopogenous structures which are a unified theory of topologies, proximities and uniformities, and in 1983 Katsaras and Petalas [4, 5] used the ideas of Csaszar and the concept of fuzzy set to introduce the fuzzy syntopogenous structure which is a generalization of the fuzzy topology, fuzzy proximity and fuzzy uniformity structures. In this paper, we study the semi-topogenous (resp. topogenous) soft fuzzy orders which is a generalization of the ordinary semi-topogenous (resp. topogenous) fuzzy orders and also study of its properties continuous functions, images and inverse images of semi-topogenous (resp. topogenous) soft fuzzy order under the continuous functions are also studied. We show that any topogenous (resp. perfect, biperfect) soft fuzzy order on \( U \) is a parameterized collection of topogenous (resp. perfect, biperfect) fuzzy orders on \( U \). Also, we show that there is a one to one correspondence between soft fuzzy topological structures and perfect topogenous soft fuzzy structures.

2 Preliminaries

In this section, we recall the basic definitions and results of soft set and soft fuzzy set theory which may be found in \([1, 2, 7, 8, 9]\).

Definition 1. [7, 8] Let \( U \) be a universal set, \( E \) be a set of parameters and let \( A \subset E \). A pair \((F_A, E)\) is said to be a soft fuzzy subset of \( U \) with support \( A \), if \( F_A \) is a mapping \( F_A : E \rightarrow I^U \) for which \( F_A(e) \neq \emptyset \) only for every \( e \in A \). In other words, \( A \) soft fuzzy set is a parameterized collection of fuzzy sets. The collection of all soft fuzzy sets over \((U, E)\) is denoted by \( SFS(U, E) \). The soft set \( \phi = (\phi_E, E) \) defined by

\[
\phi_E : E \rightarrow I^U, \quad \phi_E(e) = \emptyset \forall e \in E
\]

is called the null soft fuzzy set on \( U \) also, the universal soft fuzzy set denoted \( \emptyset = (\emptyset_E, E) \) is defined by \( \emptyset_E(e) = \bigcup \forall e \in E \).

Definition 2. [2, 8, 9] Let \( F_A, G_B \) be two soft fuzzy sets. \( F_A \) is said to be a soft fuzzy subset of \( G_B \) denoted by \( F_A \preceq G_B \) if \( F_A(e) \subseteq G_B(e) \forall e \in E \). Also, \( F_A \) and \( G_B \) are called equals denoted by \( F_A \equiv G_B \) if \( F_A \preceq G_B \) and \( G_B \preceq F_A \).

Union, intersection and difference between soft fuzzy sets are given as follows.

Definition 3. [7, 8] Let \( F_A, G_B \in SFS(U, E) \).

1. The union \( F_A \) and \( G_B \) denoted by \( F_A \vee G_B \) is the soft fuzzy set \( H_C \) denoted by \( H_C(e) = F_A(e) \vee G_B(e) \forall e \in E \), where \( C = A \cup B \).
2. The intersection of \( F_A \) and \( G_B \) denoted by \( F_A \wedge G_B \) is the soft fuzzy set \( H_C \) denoted by \( H_C(e) = F_A(e) \wedge G_B(e) \forall e \in E \) where \( C = A \cap B \).

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The difference $F_A - G_B$ is the soft fuzzy set $H_C$ defined by

$$H_C(e) = F_A(e) \land (\bigcup_{i} G_B(e)) \quad \forall e \in E.$$  

(4) The complement of a soft fuzzy set $F_A$ denoted by $F_A^c$ and defined by $F_A^c = U_E - F_A$, i.e. $F_A^c(e) = 1 - F_A(e) \quad \forall e \in E$.

Note that the support of $F_A^c$ equals $A^c$. The complement of a fuzzy topology $\tau$ is denoted by $\tau^c$.

Theorem 1. [8,7] $(SFS(U,E), \forall, \land, c)$ is a deMorgan algebra.

Definition 4. [2,6,7] A soft fuzzy point is a soft fuzzy set $F_{e_0}$ with singleton support $\{e_0\}$ and fuzzy point image $\{x_t\}$ i.e. $F_{e_0}(x_t)$ where

$$F_{e_0}(x_t) = \left\{ \begin{array}{ll}
\{x_t\} & j = e_0 \\
\emptyset & e \neq e_0
\end{array} \right.$$  

for every $t \in (0,1], e \in E, x \in U$. $F_{e_0}$ is sometimes denoted by $(x_t)e_0$. Also, the soft fuzzy point $(x_t)e_0$ is called belongs to a soft fuzzy set $F_A$ denoted by $(x_t)e_0 \in F_A$ if $x_t \in F_A(e_0) \Leftrightarrow F_A(e_0)(x) \geq t$.

Definition 5. [1,2,6] Let $SFS(U,E)$ and $SFS(V,K)$ be the collections of all soft fuzzy sets over $(U,E)$ and $(V,K)$, respectively.

A soft mapping $(\phi, \psi)$ from $(U,E)$ to $(V,K)$ is an ordered pair of mappings $\phi : U \rightarrow V$ and $\psi : E \rightarrow K$. The image of any soft fuzzy set $F_A$ over $(U,E)$ under $(\phi, \psi)(F_A)$ is the soft fuzzy set over $(V,K)$, defined by:

$$(\phi, \psi)(F_A)(k) = \left\{ \begin{array}{ll}
\bigvee_{e \in A} \psi^{-1}(k) \phi(F(e)) & A \cap \psi^{-1}(k) \neq \phi \\
\emptyset & \text{otherwise}
\end{array} \right.$$  

where $\phi : I^{U} \rightarrow I^{V}$ is the fuzzy mapping induced by $\phi : U \rightarrow V$ as usual.

The preimage of a soft fuzzy set $G_B$ over $(V,K)$ under $(\phi, \psi)$, denoted by $(\phi, \psi)^{-1}(G_B)$ is the fuzzy soft set over $(U,E)$, defined by

$$(\phi, \psi)^{-1}(G_B)(e) = \left\{ \begin{array}{ll}
\phi^{-1}(G_B(\psi(e))) & \forall e \in \psi^{-1}(B) \\
\emptyset & \text{otherwise}
\end{array} \right.$$  

Definition 6. [2,10] A soft fuzzy topology $\tau$ on $(U,E)$ is a family of soft fuzzy sets over $(U,E)$ satisfies:

1. $\phi, \psi \in \tau$
2. $F_A, G_B \in \tau \Rightarrow F_A \land G_B \in \tau$
3. $F_A^\alpha \in \tau, \alpha \in \Gamma \Rightarrow \forall \alpha \in \Gamma, F_A^\alpha \in \tau$

The triple $(U,E,\tau)$ is called a soft fuzzy topological space, members of $\tau$ are called open soft fuzzy sets and their complements are called closed soft fuzzy sets.

Theorem 2. A soft fuzzy topological space is a collection of parameterized fuzzy topological spaces. And also any parameterized collection of fuzzy topological spaces is a soft fuzzy topological space.

Proof: Straightforward

Definition 7. [3,5,6] A semi-topogenous order on a non-empty set $X$ is a binary relation $R$ on $P(X)$ satisfies the following conditions:

1. $\phi R \phi XRX$
2. $ARB \Rightarrow A \leq B$
3. $A \leq ARB \leq B \Rightarrow A_1RB_1$

A semi-topogenous order $R$ on $P(X)$ is called

1. topogenous (or top. for short) if it satisfies $A_iRB_i \forall i \in \{1,2,\ldots,n\} \Rightarrow (\bigcup_{i=1}^n A_i)R(\bigcup_{i=1}^n B_i)$ and
2. perfect semi-topogenous if

$$A_iRB_i \forall i \in \Delta \Rightarrow (\bigcup_{i \in \Delta} A_i)R(\bigcup_{i \in \Delta} B_i), \text{ for any index set } \Delta.$$  

3. biperfect topogenous if

$$A_iRB_i \forall i \in \Delta \Rightarrow (\bigcup_{i \in \Delta} A_i)R(\bigcup_{i \in \Delta} B_i) \text{ and } (\bigcap_{i \in \Delta} A_i)R(\bigcap_{i \in \Delta} B_i),$$  

for any index set $\Delta$.

Definition 8. [3,6] The complement of a semi-topogenous (resp. top., perfect semi-top., biperfect top.) order $R$ on $P(X)$ denoted by $R^c$ and is defined by

$$AR^cB \Leftrightarrow B^cRA^c$$  

where $A^c$ is the complement of $A$, and $B^c$ is the complement of $B$.

Proposition 3. [3] The complement of a semi-topogenous order on $X$ is also a semi-topogenous order on $X$.

Definition 9. [3,6,12] A syntopogenous structure on a set $X \neq \phi$ is a non-empty family $S$ of topogenous orders on $X$ satisfies the following conditions

(S1) $\forall R_1, R_2 \in S \rightarrow R_1 \leq R_2, R_2 \leq R_1$
(S2) $\forall R \in S \rightarrow R^c \subseteq S \rightarrow R \leq R^c \leftrightarrow R^c$

The pair $(X,S)$ is called a syntopogenous space. In case $S$ consists of a single topogenous order, it is called a topogenous structure. If all topogenous orders on a syntopogenous structure $S$ are perfect (resp. biperfect), it is called perfect (resp. biperfect) syntopogenous structure.

Definition 10. [3,5] A syntopogenous structure $S$ on a set $X$ is called finer than another one $S_2$ on the same set $X$ if for each $R \in S_2$ there exists a member of $S_1$ finer than $R$.

3 Soft fuzzy topogenous orders

In this section the soft fuzzy topogenous orders are introduced as a generalization of the ordinary fuzzy topogenous orders and many of their properties are given.

Definition 11. A relation $R$ on $SFS(U,E)$ is said to be a semi-topogenous soft fuzzy order (s.t.sfo. for short) if it satisfies the following condition; for any $F_A, G_B, H_C$ and $K_D \in SFS(U,E)$

1. $\phi R \phi U \rightarrow U^R$
2. $F_A R G_B \Rightarrow F_A \leq G_B$
(3) $H_C \subseteq F_A R_G B \subseteq K_D \Rightarrow H_C R_K D$

Also, a semi-topogenous soft fuzzy order $R$ is called a topogenous soft fuzzy order (t.sfo. for short) if it satisfies the condition:

(i) $F_A R_G B$ and $H_C R_K D \Rightarrow (F_A \vee R_C H) R (G_B \vee K_D)$ and $(F_A \wedge R_C H) R (G_B \wedge K_D)$, semi-topogenous soft fuzzy order $R$ is called

(ii) Perfect if

$$(F_A) R (G_B), i \in J \Rightarrow (\vee_i \in J)(F_A) R (\vee_i \in J)(G_B))$$

(iii) biperfect if

$$(F_A) R (G_B), i \in J \Rightarrow (\vee_i \in J)(F_A) R (\vee_i \in J)(G_B)) \text{ and } (\wedge_i \in J)(F_A) R (\wedge_i \in J)(G_B))$$

Definition 12. Let $R_1$ and $R_2$ be two semi-topogenous soft fuzzy orders on $(U, E)$, then we say $R_2$ is finer than $R_1$ or $R_1$ is coarser than $R_2$, denoted by $R_1 \subseteq R_2$ if

$$F_A R_1 G_B \Rightarrow F_A R_2 G_B \forall F_A, G_B \in SFS(U, E).$$

In the following theorem a generation of a semi-topogenous soft fuzzy order is given using a collection of soft fuzzy sets.

Theorem 4. Let $\mathcal{D}$ be an arbitrary family of soft fuzzy sets on $(U, E)$, such that $\mathcal{F}, \mathcal{U} \in \mathcal{D}$, and let $R_{\mathcal{D}}$ be the binary relation on $SFS(U, E)$ defined by,

$$F_A R_{\mathcal{D}} G_B \text{ if } \exists H_C \in \mathcal{D} \Rightarrow F_A R_{\mathcal{D}} G_B \forall F_A, G_B \in SFS(U, E)$$

then:

(I) the binary relation $R_{\mathcal{D}}$ is a semi-topogenous soft fuzzy order on $SFS(U, E)$ which is called generated by $\mathcal{D}$.

(II) The relation $R_{\mathcal{D}}$ is a topogenous soft fuzzy order on $SFS(U, E)$ if $\mathcal{D}$ is closed under finite union and finite intersection i.e.

$$F_A, G_B \in \mathcal{D} \Rightarrow F_A \vee G_B \in \mathcal{D} \text{ and } F_A \wedge G_B \in \mathcal{D}$$

(III) The relation $R_{\mathcal{D}}$ is a perfect semi-topogenous soft fuzzy order if $\mathcal{D}$ is closed under arbitrary union i.e.

$$(\exists (F_A) \in \mathcal{D} \Rightarrow (\exists (F_A) \in \mathcal{D}) \in \mathcal{D})$$

(iv) The relation $R_{\mathcal{D}}$ is a biperfect topogenous soft fuzzy order if $\mathcal{D}$ is closed under arbitrary union and arbitrary intersection, i.e.

$$(\exists (F_A) \in \mathcal{D} \Rightarrow (\exists (F_A) \in \mathcal{D}) \in \mathcal{D})$$

Proof

(I) (1) Since $\mathcal{F}, \mathcal{U} \in \mathcal{D}$, then $\mathcal{F} R_{\mathcal{D}} \mathcal{F}$ and $\mathcal{U} R_{\mathcal{D}} \mathcal{U}$

(2) $F_A R_{\mathcal{D}} G_B \Rightarrow \exists H_C \in \mathcal{D} \text{ s.t. } F_A \subseteq H_C \subseteq G_B \Rightarrow F_A \subseteq G_B$

(3) If $K_D \subseteq F_A R_{\mathcal{D}} G_B$, then $\exists L_M \in \mathcal{D} \Rightarrow H_C \subseteq F_A \subseteq G_B \Rightarrow F_A \subseteq G_B$

(II) If $F_A R_{\mathcal{D}} G_B$ and $H_C R_{\mathcal{D}} K_M$, then $\exists L_M, Q_P \in \mathcal{D} \Rightarrow F_A \subseteq F_A \subseteq G_B \subseteq G_B$ and $F_A \subseteq F_A \subseteq K_M$. so by the given condition $F_A \wedge H_C \subseteq F_A \wedge Q_P \subseteq G_B \wedge K_M$ and $F_A \wedge H_C \subseteq F_A \wedge Q_P \subseteq G_B \wedge K_M$, which implies that $F_A \wedge H_C \subseteq F_A \wedge Q_P \subseteq G_B \wedge K_M$ and also $F_A \wedge H_C \subseteq F_A \wedge Q_P \subseteq G_B \wedge K_M$.

The proofs of, III and IV are similar.

Example 1. (1) The subset relation on $SFS(U, E)$ is a biperfect topogenous soft fuzzy order and is called discrete and defined by

$$R_C = \{(F_A, G_B) : F_A, G_B \in SFS(U, E), F_A \subseteq G_B\}.$$

(2) The relation $R_{\mathcal{D}}$ generated by the collection $\mathcal{D} = \{\mathcal{F}, \mathcal{U}\}$ which defined by

$$R_{\mathcal{D}} = \{(F_A, G_B) : F_A = \mathcal{F} \text{ or } G_B = \mathcal{U}\}$$

is a topogenous soft fuzzy order and is called indiscrete.

Lemma 5. The composition of two $s.t.sfo.\mathcal{S}$ (respectively $t.sfo.\mathcal{S}$, $p.sfo.\mathcal{S}$ and $b.sfo.\mathcal{S}$) $R_1$ and $R_2$ on $SFS(U, E)$ as a relation $R_1 \circ R_2$ is also $s.t.sfo.$ (respectively, t.sfo., p.sfo. and b.sfo.) where $R_1 \circ R_2$ is defined as follows for every $F_A, G_B \in SFS(U, E)$,

$$F_A R_1 G_B \Rightarrow \exists H_C \in SFS(U, E) \Rightarrow F_A R_1 H_C \text{ and } H_C R_2 G_B.$$

Proof

Lemma 6. Let $\{R_{\alpha} : \alpha \in \Delta\}$ be a family of semi-topogenous (resp. topogenous, perfect topogenous, biperfect topogenous) soft fuzzy orders on $SFS(U, E)$. Then

(1) $R = \bigcap_{\alpha} R_{\alpha}$ is also a semi-topogenous (resp. topogenous, perfect topogenous, biperfect topogenous) soft fuzzy order on $SFS(U, E)$, where

$$F_A R_{\mathcal{D}} G_B \text{ iff } F_A R_{\alpha} G_B \forall \alpha \in \Delta$$

(2) $R = \bigcup_{\alpha} R_{\alpha}$ is also a semi-topogenous soft fuzzy order on $SFS(U, E)$, where

$$F_A R_{\mathcal{D}} G_B \text{ iff } F_A R_{\alpha} G_B$$

for some $\alpha_0 \in \Delta$

Proof: Straightforward.

Remark 7. The union of a family of topogenous soft fuzzy orders is not in general a topogenous soft fuzzy order, as we show in the following example.

Example 2. Let

$$U = \{a, b, c\}, E = \{e_1, e_2, e_3, e_4\}$$

$$D_1 = \{\mathcal{F}, \mathcal{U}, E_A\}, D_2 = \{\mathcal{F}, \mathcal{U}, G_B\},$$

$$F_A = \begin{cases} F(e_1) = (a, 0.4), (b, 0.1), (c, 0) \\ F(e_2) = (a, 0.6), (b, 0.5), (c, 0.8) \\ F(e_3) = (a, 0.2), (b, 0.1), (c, 0) \\ F(e_4) = (a, 0), (b, 0), (c, 0) \end{cases}$$

$$G_B = \begin{cases} G(e_1) = (a, 0), (b, 0), (c, 0) \\ G(e_2) = (a, 0.7), (b, 0.2), (c, 0.1) \\ G(e_3) = (a, 0), (b, 0), (c, 0) \\ G(e_4) = (a, 0.5), (b, 0.3), (c, 0.9) \end{cases}$$

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Let \( R_d = \{(\emptyset, \emptyset), (U, U)\} \cup \{(H_D, K_E): H_D \leq F_A \leq K_E\} \)
\( R_d = \{(\emptyset, \emptyset)(U, U)\} \cup \{(H_D, K_E): H_D \leq G_B \leq K_E\} \)
\( R_{d1} \cup R_{d2} = \{(\emptyset, \emptyset), (U, U)\} \cup \{(H_D, K_E): H_D \leq F_A \leq K_E \text{ or } H_D \leq G_B \leq K_E\} = R_{d1} \cup R_{d2}\)

It is clear that \( F_A R_{d1} U \cup D A \) and \( G_B R_{d1} U \cup D B \) but
\( \{(F_A \cap G_B), (F_A \cap G_B)\} \not\in R_{d1} \cup R_{d2} \)

also
\( \{(F_A \cup G_B), (F_A \cup G_B)\} \not\in R_{d1} \cup R_{d2} \)

then \( R_{d1} \cup R_{d2} \) is not a topogenous soft fuzzy order.

**Definition 13.** The complement of a soft fuzzy order \( R \) on \( SFS(U, E) \) denoted by \( R^c \) is defined by

\[
F_A R^c B \iff G_B R^c F_A, \forall F_A, G_B \in SFS(U, E).
\]

\( R \) is called symmetric iff \( R = R^c \).

**Theorem 8.** Let \( R_1 \) and \( R_2 \) be two semi-topogenous (respectively topogenous, perfect, biperfect) soft fuzzy orders on \( SFS(U, E) \), then,

1. \( R_1^c \) is a semi-topogenous (respectively topogenous, perfect, biperfect) soft fuzzy order.
2. \( R_1 \cap R_2 \) is a semi-topogenous soft fuzzy order.
3. \( R_1 \cap R_2 \) is a semi-topogenous soft fuzzy order.

**Proof** (1) Let \( R_1 \) be a semi-topogenous soft fuzzy order on \( SFS(U, E) \).

(i) \( \emptyset \emptyset \rightarrow \emptyset \emptyset \emptyset \emptyset \emptyset \)

(ii) \( F_A R_1^c G_B \Rightarrow G_B R_1 F_A \Rightarrow G_B \leq F_A \Rightarrow F_A \leq G_B, \forall F_A, G_B \in SFS(U, E) \)

(iii) Let \( K_D \leq F_A R_1^c G_B \leq H_C \Rightarrow H_C \leq G_B R_1 F_A \leq K_D \Rightarrow H_C R_1 K_D \Rightarrow K_D R_1 H_C \forall K_D, F_A, G_B, H_C \in SFS(U, E) \)

then \( R_1^c \) is a semi-topogenous soft fuzzy order. The rest of (1) is similar.

(2) For any \( F_A, G_B \in SFS(U, E) \),

\[
F_A R_1^c G_B \Leftrightarrow G_B R_1 F_A \Leftrightarrow F_A R_1 G_B, \text{i.e } R_1^c = R_1.
\]

(3) For any \( F_A, G_B \in SFS(U, E) \), let \( R_1 \cap R_2 \). So,

\[
F_A R_1^c G_B \Rightarrow G_B R_1 F_A \Rightarrow G_B R_2 F_A \Rightarrow F_A R_2 G_B.
\]

Then \( R_1^c \cap R_2^c \).

(4) For any \( F_A, G_B \in SFS(U, E) \),

\[
F_A (R_1 \cap R_2)^c G_B \Leftrightarrow G_B (R_1 \cap R_2)^c \Leftrightarrow \exists \exists H_D \exists H_D (H_D R_D R_2 F_A^c \Leftrightarrow \exists \exists H_D \exists H_D R_D R_2 F_A^c \Leftrightarrow \exists \exists H_D \exists H_D R_D R_2 F_A^c \Leftrightarrow \exists \exists H_D \exists H_D R_D R_2 F_A^c \Leftrightarrow \exists \exists H_D \exists H_D R_D R_2 F_A^c.
\]

i.e \( (R_1 \cap R_2)^c = R_2^c \cap R_1^c \)

**Definition 14.** Let \( f: U \rightarrow V \) be a function between sets and let \( E \) be any set of parameters. Using \( f \) we can determine two mappings \((f^+, 1_E): (U, E) \rightarrow (V, E)\) and \((f^-, 1_E): (V, E) \rightarrow (U, E)\) as follows:

\[
\begin{align*}
\text{Let } f^+: I^U \rightarrow I^V, f^-: I^V \rightarrow I^U \text{ are defined by } f^+(\mu)(y) &= \cup_{x \in f^{-1}(y)} \mu(x) \forall x \in V, y \in V, \\
&f^-(\gamma)(x) = \gamma(f(x)) \forall y \in V, x \in U. \text{So, the image and the preimage of a soft set under a soft mapping is given as follows, for every soft fuzzy set } F_A \in SFS(U, E), (f^+, 1_E)(F_A) \in SFS(V, E) \text{ is given by}
\end{align*}
\]

\[
\{(f^+, 1_E)(F_A)(e)(y) = \cup_{x \in f^{-1}(y)} (F_A)(x) \}
\]

Also for every soft fuzzy set \( G_B \in SFS(V, E) \), the pre-image \((f^-, 1_E)(G_B)(e)(y) \in SFS(U, E) \) is given by \((f^-, 1_E)(G_B)(e)(y) = (G_B(e))(f(x)) \) Denote \((f^+, 1_E)\) by \( f^+ \) and \((f^-, 1_E)\) by \( f^- \). In the following we define the inverse image of a semi-topogenous soft fuzzy order \( R \) under a function \( f \).

**Definition 15.** Let \( f: U \rightarrow V \) be a function and let \( R \) be a semi-topogenous (respectively, topogenous, perfect, biperfect) soft fuzzy order on \( (V, E) \). The inverse image of \( R \) under \( f \) denoted by \( f^{-1}(R) \) defined by

\[
F_A f^{-1}(R) G_B \iff f^+(F_A) R(f^-(G_B))^c
\]

**Proposition 9.** The inverse image of a semi-topogenous soft fuzzy order on \( (V, E) \) is a semi-topogenous soft fuzzy order on \( (U, E) \).

**Proof Straightforward**

**Proposition 10.** Let \( f: U \rightarrow V \) be a function and let \( R \) be a semi-topogenous (topog., prefect, biperfect) soft fuzzy order on \( (V, E) \). Then for every \( F_A, G_B \in SFS(U, E) \),

\[
F_A f^{-1}(R) G_B \iff \exists H_C, K_D \in SFS(V, E) \text{ s.t. } H_C R_K D \\ 
F_A \leq f^+(H_C), f^- (K_D) \leq G_B. \text{ Also, } H_C R_K D \Rightarrow f^+(H_C)f^{-1}(R) f^{-1}(K_D) \).
\]

**Proof** In fact, \( F_A f^{-1}(R) G_B \) if \( f^+(F_A) R(f^-(G_B))^c \), let \( H_C = f^+(F_A), K_D = (f^-(G_B))^c \), then we get

\[
H_C R_K D, H_C, K_D \in SFS(V, E)
\]

and

\[
F_A \leq f^+(H_C), f^-(K_D) \Rightarrow f^+(H_C) f^- (K_D) \leq G_B. \text{ Conversely if } \exists H_C, K_D \in SFS(V, E)
\]

such that \( H_C R_K D, F_A \leq f^+(H_C) \text{ and } f^- (K_D) \leq G_B, \) then

\[
f^+(F_A) \leq H_C \text{ and } G_B \leq f^-(K_D)^c \Rightarrow f^+(F_A) \leq f^-(K_D)^c
\]

Consequently, \( f^+(G_B)^c \leq K_D \) which implies that \( K_D \leq (f^-(G_B))^c. \) So, \( H_C R_K D \) implies that \( f^+(F_A) \leq H_C R_K D \leq (f^-(G_B))^c. \) Which implies that \( f^+(F_A) R(f^-(G_B))^c \), and that \( \bar{F}_A(f^{-1}(R))G_B. \)
Theorem 11. Let $f : U \to V$ be a function, $R_1, R_2$ and $R$ be semi-topogenous (resp. topogenous perfect semi-topogenous, bi-perfect) soft fuzzy order on $(V, E)$. Then

(I) $R_1 \subseteq R_2$ implies $f^{-1}(R_1) \subseteq f^{-1}(R_2)$, and the converse is true if $f$ is surjective.

(II) $f^{-1}(R^c) = (f^{-1}(R))^c$.

Proof Let $f : U \to V$ be a function and $R_1 \subseteq R_2$ are two semi-topogenous soft fuzzy orders on $(V, E)$, and let $F_A, G_B \in SFS(U, E)$. Then

$$F_A(f^{-1}(R_1))G_B \Rightarrow f^c(F_A)R_1(f^c(G_B))^c$$

Conversely, let $f$ be a surjective function $f^{-1}(R_1) \subseteq f^{-1}(R_2)$, and let $F_A, G_B$ for some $F_A, G_B \in SFS(V, E)$, so $f^c(F_A)(f^{-1}(R_1))(f^c(G_B)) \Rightarrow (f^c(F_A)(f^{-1}(R_1)))(f^c(G_B)) \Rightarrow \exists H, K, D \in SFS(V, E)$

$$H_C R_2 K, f^c(f^c(F_A))(f^c(G_B)) \leq f^c(f^c(H_C), f^c(K_D) \leq f^c(G_B)$$

then $f^c(f^c(F_A))(f^c(G_B))$.

Theorem 12. Let $f : U \to V$ and $g : V \to W$ be two functions, $g \circ f : V \to W$ is the composition of $f, g$ then for any topogenous soft fuzzy order $R$ on $(W, E)$ we have

$$(g \circ f)^{-1}(R) = f^{-1}(g^{-1}(R))$$

Proof For any two soft fuzzy sets $F_A$ and $G_B$ of $(U, E)$, $F_A((g \circ f)^{-1})RG_B$

$$\Leftrightarrow (g \circ f)^c(F_A)RG_B(f^c(g \circ f))^c$$

$$\Leftrightarrow g^c(F^c_A)RG_B(f^c(g \circ f))^c$$

$$\Leftrightarrow f^c(F_A)(g^{-1}(R))(f^c(G_B))^c$$

Theorem 13. Let $f : U \to V$ be a function, $R_1$ and $R_2$ two semi-topogenous soft fuzzy orders on $(V, E)$ and $R = R_1 \circ R_2$ then $f^{-1}(R) \subseteq f^{-1}(R_1) \circ f^{-1}(R_2)$ and the equality holds if $f$ is surjective.

Proof Straightforward.

4 The relation between soft fuzzy orders and ordinary fuzzy orders

It is clear that a soft set is a parametrized collection of sets. And a soft topological structure on a set is a parametrized collection of ordinary topological structures on the same set. Also a similar result will be proved for the soft fuzzy orders.

Theorem 14. Let $R$ be a semi-topogenous (respectively, topogenous, perfect topogenous, bi-perfect topogenous) soft fuzzy order on $(U, E)$. For every $e \in E$, consider the relation $R_e$ on $I^U$ given by, for $\mu, v \in I^U$, $\mu R_e v$ if $\exists F_A, G_B \in SFS(U, E)$ such that $\mu = F_A(e), v = G_B(e)$ and $F_A R G_B$. Then $R_e$ is semi-topogenous (respectively, topogenous, perfect topogenous, bi-perfect topogenous) soft fuzzy order on $U$, for every $e \in E$.

Proof (1) $\Phi_R \Phi \Rightarrow \Phi_R \Phi$, also $\Phi_R \Phi \Rightarrow \Phi_R \Phi$ for every $e \in E$.

(2) Let $\mu R_e v \Rightarrow \exists F_A, G_B \in SFS(U, E) \ni \mu = F_A(e), v = G_B(e)$ and $F_A \circ R G_B \Rightarrow F_A \circ G_B \Rightarrow F_A(e) \leq G_B(e) \Rightarrow \mu \leq v$.

(3) Let $\eta \leq \mu R_e v \leq \xi$ implies $\exists F_A, G_B \in SFS(U, E) \ni \mu \leq F_A(e), v = G_B(e) \Rightarrow \exists F_A, G_B \in SFS(U, E)$ defined by

$$F_A(e) = \eta, F_A(t) = 0 \forall t \in E - \{e\}$$

$$G_B(e) = \xi, G_B(t) = 1 \forall t \in G - \{e\}.$$

Consequently $F_A \leq F_A R G_B \leq G_B \Rightarrow F_A \circ R G_B \Rightarrow \eta R_e \xi$. The previous theorem show that that any semi-topogenous soft fuzzy order $R$ on $(U, E)$ generates a parametrized collection of semi-topogenous fuzzy orders $\{R_e : e \in E\}$ on $U$.

Theorem 15. Any topogenous (respectively, perfect topogenous, bi-perfect topogenous) soft fuzzy order on $(U, E)$ is a parametrized collection of topogenous (respectively, perfect topogenous, bi-perfect topogenous) soft fuzzy orders $\{R_e : e \in E\}$ on $U$.

Proof Let $R$ be a topogenous soft order on $(U, E)$. For every $e \in E$ consider the semi-topogenous fuzzy order $R_e$ given in the previous theorem.

Let $\mu R_e v$ and $\xi R_e \zeta$ for some fuzzy sets $\mu, v, \xi, \zeta \in I^U$. Then, there exist $F_A, G_B, M_C, N_D \in SFS(U, E)$ such that $F_A(e) = \mu, G_B(e) = v, M_C(e) = \xi, N_D(e) = \zeta, F_A R G_B$ and $M_C R N_D$. This implies that $(F_A \vee M_C)R(G_B \vee N_D)$. Consequently

$$(F_A \vee M_C)(e) = F_A(e) \vee M_C(e) = \mu \vee \xi,$$

and $$(G_B \vee N_D)(e) = G_B(e) \vee N_D(e) = v \vee \zeta.$$ So, $(\mu \vee \xi) R_e (v \vee \zeta)$. The rest of the proof is similar.
Theorem 16. Every parameterized collection \( \{ R_e : e \in E \} \) of semi-topogenous (respectively, perfect topogenous, biperfect topogenous) fuzzy orders on a set \( U \) generate in a canonical correspondence a unique semi-topogenous (respectively, perfect topogenous, biperfect topogenous) soft fuzzy order \( R \) on \( (U,E) \).

Proof Let \( \{ R_e : e \in E \} \) be a collection of semi-topogenous fuzzy orders on a set \( U \). Consider the relation \( R \) on \( SFS(U,E) \) given as follows, for every two soft fuzzy sets \( F_A, G_B \in SFS(U,E) \) \( F_A \preceq GB \) if \( F_A(e) \subseteq G_B(e) \forall e \in E \).

1. Since \( R_e \) is a semi-topogenous order \( \forall e \in E \), so, \( 0R_0 \) and \( 1R_1 \forall e \in E \Rightarrow \Phi \Phi \Phi \) and \( \overline{UR} \).

2. Let \( F_A \preceq GB \Rightarrow F_A(e) \subseteq G_B(e) \forall e \in E \Rightarrow F_A(e) \leq G_B(e) \) for every \( e \in E \Rightarrow F_A \leq GB \).

3. Let \( M_C \subseteq F_A \leq GB \subseteq N_D \Rightarrow M_C(e) \leq F_A(e) \subseteq G_B(e) \leq N_D(e) \forall e \in E \Rightarrow M_C(e) \subseteq N_D(e) \forall e \in E \Rightarrow M_C \sqsubseteq RN_D \).

The rest of the proof is by the same argument.

Remark 17. It is clear that from theorem (2,12,13), both notions soft topogenous order and topogenous soft order are the same, also soft topological space and topological soft space are the same.

5 The syntopogenous soft fuzzy structures

Definition 16. A syntopogenous soft fuzzy structure on a set \( (U,E) \) is a non-empty family \( S \) of topogenous soft fuzzy orders on \( (U,E) \) having the following two properties:

1. If \( R_1, R_2 \in S \exists R \in S \ s.t \ R_1 \subseteq R, R_2 \subseteq R \)
2. \( \forall R_1 \in S \exists R_2 \in S \ s.t \ R_1 \subseteq R_2 \circ R_2 \).

The pair \( (U,E,S) \) is called a syntopogenous soft fuzzy space. In case \( S \) consists of a single topogenous soft fuzzy order, it is called a simple syntopogenous soft fuzzy structure (or topogenous structure). If all orders of the space \( (U,E,S) \) are perfect or biperfect, then it is called perfect (or biperfect) syntopogenous soft fuzzy space.

Definition 17. A syntopogenous soft fuzzy structure \( S_1 \) on \( (U,E) \) is called finer than another one \( S_2 \) on the same space if \( \forall R \in S_2 \exists R' \in S_1 \) finer than \( R \), and is denoted by \( S_2 \sqsubseteq S_1 \).

Lemma 18. Let \( (U,E,S) \) be a syntopogenous soft fuzzy space, then \( S' = \{ R' : R \in S \} \) is a syntopogenous soft fuzzy structure, and is called the complement of \( S \). Also, \( S \) is called symmetric if \( S' = S \)

Proof Straightforward.

Proposition 19. If \( R \) is a topogenous soft fuzzy order on \( (U,E) \), then \( \{ R \} \) is a topogenous soft fuzzy structure if it satisfies the condition: for every \( F_A, G_B \in SFS(U,E) \) if \( F_A \preceq GB \), then there exists \( H_C \in SFS(U,E) \) \( F_A \preceq RH_C \preceq GB \)

Proof Straightforward.

Proposition 20. Let \( f \) be a function from \( (U,E) \) into \( (V,E) \), \( S \) be a syntopogenous soft fuzzy structure on \( (V,E) \). Then the family \( f^{-1}(S) = \{ f^{-1}(R) : R \in S \} \) is a syntopogenous soft fuzzy structure of \( (U,E) \) and it is called the inverse image of \( S \) by the mapping \( f \).

Proof (1) Let \( f^{-1}(R_1), f^{-1}(R_2) \in (f^{-1}(S)) \). Since \( S \) is syntopogenous soft fuzzy structure on \( (V,E) \) then \( \exists R \in S \) s.t \( R \sqsubseteq f^{-1}(R_1), R \sqsubseteq f^{-1}(R_2) \).

2. Let \( f^{-1}(R) \in (f^{-1}(S)) \Rightarrow \exists R' \in S \) such that \( R \sqsubseteq R' \circ R' \).

Thus, \( f^{-1}(R) \subseteq f^{-1}(R' \circ R') \subseteq f^{-1}(R') \circ f^{-1}(R') \).

Therefore, \( f^{-1}(S) \) is a syntopogenous soft fuzzy structure of \( (U,E) \).

Proposition 21. Let \( f \) be a function, \( (f,I_E) : (U,E) \rightarrow (V,E) \), and let \( S, S' \) be two syntopogenous soft fuzzy structures on \( (V,E) \)

1. If \( S \) is perfect (respectively biperfect, symmetric), then \( f^{-1}(S) \) is also perfect (respectively biperfect, symmetric).

2. If \( S \subseteq S' \), then \( f^{-1}(S) \subseteq f^{-1}(S') \).

Proof (i) It obvious

(ii) \( f^{-1}(S) \subseteq f^{-1}(R) \sqsubseteq \{ f^{-1}(R') : R' \in S \} \)

Definition 18. Let \( S \) and \( S' \) be two syntopogenous soft fuzzy structures on \( (U,E) \) and \( (V,E) \), respectively, and let \( f \) be a function from \( (U,E) \) into \( (V,E) \). Then \( f \) is said to be \( (S,S') \) continuous iff \( f^{-1}(S') \subseteq S \).

Theorem 22. Let \( (U,S_1,E), (V,S_2,E), (W,S_3,E) \) be syntopogenous soft fuzzy spaces. If \( (f, I_E) : (U,E) \rightarrow (V,E) \) is \( (S_1, S_2) \)-continuous and \( (g, I_E) : (V,E) \rightarrow (W,E) \) is \( (S_2, S_3) \)-continuous. Then \( (g \circ f, I_E) : (U,E) \rightarrow (W,E) \) is \( (S_1, S_3) \)-continuous.

Proof The continuity of \( f : (U,S_1,E) \rightarrow (V,S_2,E) \) and \( g : (V,S_2,E) \rightarrow (W,S_3,E) \) implies that for every \( R \in S_3 \), there exists \( R_1 \sqsubseteq R_2 \) such that \( g^{-1}(R_1) \sqsubseteq R_1 \). Also, there exists \( R_2 \sqsubseteq S_2 \) such that \( f^{-1}(R_1) \sqsubseteq R_2 \).

Consequently, \( f^{-1}(g^{-1}(R_1)) \sqsubseteq R_2 \). This implies that \( g \circ f \) is continuous.

It is well known that there exists a one to one correspondence between the collection of all topological structures on a set and the collection of all perfect topogenous structures on the same set [9,10]. The following theorem shows a similar result in the soft case.

Theorem 23. For any non-empty set \( U \), there exists a one-to-one and onto correspondence between the collection of all soft fuzzy topological structures on \( (U,E) \) and the collection of all perfect topogenous soft fuzzy structures on the same space \( (U,E) \) with any set of parameters \( E \).
Proof

For every perfect topogenous soft fuzzy structure \( \{ R \} \) on \((U, E)\), consider the collection \( \tau_R = \{ G_A : A \in (U, E), G_A \cap R_A \neq \emptyset, C \in \mathcal{C} \} \), so \( \Phi_E, \bar{U}_E \in \tau_R \).

If \( \{ G^\alpha_{A_a} : \alpha \in \Gamma \} \subset \tau_R \), \( R \) is perfect then \( \forall \alpha \in \Gamma G^\alpha_{A_a} \in \tau_R \).

Also if \( G^1_\alpha \cap G^2_\alpha \in \tau_R \), \( R \) is topogenous then \( G^1_\alpha \cap G^2_\alpha \in \tau_R \), i.e. \( \tau_R \) is a soft fuzzy topological structure on \((U, E)\).

Also for every soft fuzzy topological structure \( \tau \) on \((U, E)\) consider the following order \( \tau_t \) on \((U, E)\), defined by \( F_A R_t H_B \) if \( \exists G_c \in \tau \supset F_A \leq G_c \leq H_B \), \( \forall F_A, G_B \in SFS(U, E) \).

It is clear that \( \Phi, \bar{U} \in \tau_t \), implies that \( \Phi R_t \Phi \) and \( \bar{U} R_t \bar{U} \). Also \( F_A R_t H_B \) implies that \( F_A \leq H_B \). If \( F^\alpha_{A_a} R_t H^\alpha_{B_a} \) \( \alpha \in \Gamma \), then \( \exists G^\alpha_{C_a} \in \tau \) such that \( F^\alpha_{A_a} \succeq G^\alpha_{C_a} \succeq H^\alpha_{B_a} \), so \( \forall \alpha \in \Gamma G^\alpha_{C_a} \in \tau \) and,

\[
\forall \alpha \in \Gamma G^\alpha_{C_a} \succeq \forall \alpha \in \Gamma F^\alpha_{A_a} \succeq \forall \alpha \in \Gamma H^\alpha_{B_a}
\]

consequently, \( \forall \alpha \in \Gamma F^\alpha_{A_a} R_t \forall \alpha \in \Gamma H^\alpha_{B_a} \), i.e. \( \tau_t \) is a perfect order.

Also, if \( F^1_{A_1} R_t H^1_{B_1}, i = 1,2 \), then \( \exists G^1_{C_1} \in \tau \supset F^1_{A_1} \succeq G^1_{C_1} \succeq H^1_{B_1}, i = 1,2 \). This implies that \( G^1_{C_1} \in \tau \) and \( (F^1_{A_1} \cup F^2_{A_2}) R_t (H^1_{B_1} \cup H^2_{B_2}) \). Consequently \( \tau_t \) is a perfect topogenous soft fuzzy order on \((U, E)\). Also it is clear that by the definition of \( \tau_t \), we have \( \tau_t \circ R_t \) is coarser than \( R_t \), which implies that \( \tau_t \) is a perfect topogenous soft fuzzy structure.

Now, consider any soft fuzzy topological structure \( \tau \) on \((U, E)\) and consider the order \( R_t \) and the topology \( \tau_t \).

For every \( G_A \in \tau_t \), it follows that \( G_A R_t G_A \), which implies that \( G_A \in \tau_t \). Also if \( R \) is any perfect topogenous soft fuzzy order on \((U, E)\), consider \( \tau_t \) and \( R_t \). If \( F_A R_t H_B \) for some \( F_A, H_B \in SFS(U, E) \), then \( \exists G_c \in \tau_t \) such that \( F_A \leq G_c \leq H_B \), so \( G_c \in \tau_t \), which implies that \( F_A R_t H_B \). Consequently the correspondence in the Theorem is one to one and onto.

**Proposition 24.** For any syntopogenous soft fuzzy structure \( \bar{S} \) on the space \((U, E)\), the collection \( S' = \{ R_S \} \) is a topogenous soft fuzzy structure on \((U, E)\), where \( R_S = \bigcup \{ R : R \in S \} \).

**Proof** Straightforward.

**Corollary 25.** Using the last theorem and proposition we can determine in a canonical correspondence, for any syntopogenous soft fuzzy structure \( \bar{S} \) on a space \((U, E)\), a soft fuzzy topological structure denoted \( \tau_{\bar{S}} \) which is \( \tau_{\bar{S}} \) or indeed it is \( \tau_{R_S} \), where \( R_S \) is the coarsest perfect topogenous order finer than \( R_S \).

**Proposition 26.** If \( S_1, S_2 \) are two syntopogenous soft fuzzy structures on \((U, E)\) and \( S_1 \subset S_2 \), then \( \tau_{S_1} \subset \tau_{S_2} \).

**Proof** Straightforward.

**Theorem 27.** Consider two collections of all soft fuzzy subsets \((U, E)\) and \((V, E)\), and let \( f \) be any surjective function \( (f, I_E) : (U, E) \to (V, E) \). If \( S \) is a syntopogenous soft fuzzy structure on \((V, E)\), then \( \tau_{f^{-1}(S)} = f^{-1}(\tau_S) \).

**Proof** Let \( F_A \in \tau_{f^{-1}(S)} \), so \( F_A R^p F_A \), where \( R^p \) is the coarsest perfect topogenous structure finer than \( R \) and \( R = (f^{-1}(S))^c \). Consequently \( f^p(F_A)R^p(f^p(F_A))^c \) for some \( R_0 \), consequently, there exists \( G_B \in \tau_S \) for which \( f^p(F_A) \leq G_B \leq f^p(f^p(F_A))^c \). This implies that \( f^p(F_A) \lhd (f^p(F_A))^c \). Let \( G_B \) be \( f^p(F_A) \) in \( f^{-1}(\tau_S) \), i.e. \( \tau_{f^{-1}(S)} = f^{-1}(\tau_S) \).

Conversely, let \( F_A \in \tau_{f^{-1}(S)} \), so \( f^p(F_A) \in \tau_S \). Consequently, there exists \( R_0 \), such that \( f^p(F_A)R^p_0 \), since \( f \) is surjective, then \( f^p(F_A) = (f^p(F_A))^c \). Consequently \( f^p(F_A)R^p_0(f^p(F_A))^c \) which implies that \( F_A(f^{-1}(R_0)^p F_A \) i.e. \( F_A(f^{-1}(R_0)^p F_A \) consequently, \( \tau_{f^{-1}(S)} \) and this means that \( F_A \in \tau_{f^{-1}(S)} \).

**Theorem 28.** Let \((U, S_1, E), (V, S_2, E)\) be two syntopogenous soft fuzzy spaces \( f : U \to V \) be a function. If \( (f, I_E) \) is \((S_1, S_2)\)-continuous, then \( (f, I_E) \) is \( \tau_{S_1} \subset \tau_{S_2} \) continuous.

**Proof** Let \( F_A \in \tau_{S_2} \), so \( F_A R^p_2 F_A \). So \( (f^p_2(F_A)) f^{-1}(R^p_2) \). \( f \) is \((S_1, S_2)\) continuous implies that \( f^{-1}(R^p_2) \subset S_1 \). Consequently \( (f^p_2(F_A)) \subset S_1 \) implies that \( \tau_{S_1} \).

**References**


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