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An Analytical Approach to Time-Fractional Harry Dym Equation

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Abstract: In this paper, we consider the time-fractional Harry Dym equation. An analytical solution is obtained in series form using q-Homotopy Analysis Method (q-HAM). The presence of the auxiliary parameter h and the fraction-factor in this method gives it an edge over other existing analytical methods for non-linear differential equations. Comparisons are made with several other analytical methods. Our error analysis shows that the analytical solution converges very rapidly to the exact solution. Numerical results are obtained using Mathematica 9 and MATLAB R2012b.

Keywords: Harry Dym Equation; Fractional Derivative; q-Homotopy Analysis Method

1 Introduction

The commonly used analytical methods to solve non-linear equations have different restrictions and discretization of variables are involved in numerical techniques which leads to rounding off errors see [1].

The Harry Dym equation is a useful mathematical model (dynamical) which represents a system in which both dispersion and non-linearity are being coupled together with applications in several physical systems. The strong links between Harry Dym equation and Korteweg-de Vries equation allow it to find applications in the problems of hydrodynamics [2].

Generally, for the past three decades, fractional calculus has been considered with great importance due to its various applications in physics, fluid flow, control theory of dynamical systems, chemical physics, electrical networks, and so on. The quest of getting accurate methods for solving resulted non-linear model involving fractional order is of utmost concern of many researchers in this field today.

Various analytical methods have been put to use successfully to obtain solutions of classical Harry Dym Equation such as Adomian Decomposition Method (ADM), Direct Integration Method (DIM), Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM), Power Series Method, Sumudu Transform Method see [3,4] and the time-fractional type was considered using Homotopy Perturbation Method (HPM) in [5]. Recently, a modified HAM called q-Homotopy Analysis Method was introduced in [6], see also [7,8,9]. It was proven that the presence of fraction factor in this method enables a fast convergence better than the usual HAM which then makes it more reliable see also the following references [10].

To the best of our knowledge, no attempt has been made regarding analytical solution of time-fractional Harry Dym Equation using q-Homotopy Analysis Method. In this paper, we consider this equation subject to some appropriate initial condition. We compare our results with other analytical results for these problems and the exact solution. The numerical results of the problems are presented graphically.

2 Preliminaries

This section is devoted to some definitions and some known results. Caputo's fractional derivative is adopted in this work.

Definition 1.*The Riemann-Liouville's* (RL) *fractional integral operator of order* $\alpha \geq 0$, *of a function* $f \in L^1(a,b)$ *is given as*

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$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \; t>0, \; \alpha>0, \quad (1)$$

where Γ is the Gamma function and $I^0 f(t) = f(t)$.

Definition 2. *The fractional derivative in the Caputo's sense is defined as [11],*

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (2)$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t>0$.

Lemma 1. *[11] Let $t \in (a,b]$. Then*

$$\left[I^{\alpha}(t-a)^{\beta}\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}, \qquad \alpha \geqslant 0, \quad \beta>0. \quad (3)$$

Also,

$$\left[D^{\alpha}(t-a)^{\beta}\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}, \qquad \alpha \geqslant 0, \quad \beta>0. \quad (4)$$

3 q-Homotopy Analysis Method (q-HAM)

Differential equation of the form

$$N\left[D_t^{\alpha} u(x,t)\right] - g(x,t) = 0 \quad (5)$$

is considered, where N is a non-linear operator, D_t^{α} denotes the Caputo fractional derivative, (x,t) is a pair of independent variables, g is a known function and u is an unknown function. Generalizing the original homotopy method, the zeroth-order deformation equation is constructed as the zeroth-order deformation equation

$$(1-nq)L\left(\phi(x,t;q) - u_0(x,t)\right) = qhH(x,t)\left(N[D_t^{\alpha}\phi(x,t;q)] - g(x,t)\right), \quad (6)$$

where $n \geqslant 1$, $q \in \left[0, \frac{1}{n}\right]$ denotes the so-called embedded parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $H(x,t)$ is a non-zero auxiliary function.

It is clearly seen that when $q=0$ and $q=\frac{1}{n}$, equation (6) becomes

$$\phi(x,t;0) = u_0(x,t) \quad \text{and} \quad \phi(x,t;\frac{1}{n}) = u(x,t) \quad (7)$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

If $u_0(x,t)$, L, h, $H(x,t)$ are chosen appropriately, solution $\phi(x,t;q)$ of equation(6) exists for $q \in \left[0, \frac{1}{n}\right]$.

Expansion of $\phi(x,t;q)$ in Taylor series gives

$$\phi(x,t;r) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \quad (8)$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \quad (9)$$

Assume that the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series 8 converges at $q=\frac{1}{n}$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \left(\frac{1}{n}\right)^m. \quad (10)$$

Let the vector u_n be defined as follows:

$$\mathbf{u}_n = \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}. \quad (11)$$

Differentiating equation (6) m-times with respect to the (embedding) parameter q, then evaluating at $q=0$ and finally dividing them by $m!$, we have what is known as the m^{th}-order deformation equation, [12], as

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hH(x,t)\mathscr{R}_m\left(\mathbf{u}_{m-1}\right). \quad (12)$$

with initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0,1,2,\ldots,m-1. \quad (13)$$

where

$$\mathscr{R}_m\left(\mathbf{u}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\left(N[D_t^{\alpha}\phi(x,t;q)] - g(x,t)\right)}{\partial q^{m-1}}\bigg|_{q=0} \quad (14)$$

and

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1 \\ n & otherwise, \end{cases} \quad (15)$$

Remark. It should be emphasized that $u_m(x,t)$ for $m \geqslant 1$, is governed by the linear operator (12) with the linear boundary conditions that come from the original problem. The existence of the factor $\left(\frac{1}{n}\right)^m$ gives more chances for better convergence, faster than the solution obtained by the standard Homotopy method. Off course, when $n=1$, we are in the case of the standard homotopy method.

4 The Time-fractional Harry Dym Equation

We consider the time-fractional Harry Dym Equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = u^3 \frac{\partial^3 u}{\partial x^3}, \qquad 0<x<1, \quad 0<t\leq 1, \quad 0<\alpha\leq 1 \quad (16)$$

subjects to the initial condition

$$u(x,0) = \left(a - \frac{3\sqrt{b}}{2} x\right)^{\frac{2}{3}}. \quad (17)$$

The exact solution to this problem, $\alpha=1$, is

$$u(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x+bt)\right)^{\frac{2}{3}}. \quad (18)$$

Many authors have worked on this problem when $\alpha=1$, using various methods see [3,4].

4.1 Application of q-HAM

In order to use q-HAM to solve the problem considered in (16), we choose the linear operator

$$L[\phi(x,t;q)] = D_t^{\alpha}\phi(x,t;q) \tag{19}$$

with property that $L[c_1] = 0$, c_1 is constant.

We use initial approximation $u_0(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}}$. We can then define the non-linear operator as

$$N[\phi(x,t;q)] = D_t^{\alpha}\phi(x,t;q) - (\phi(x,t;q))^3\,\phi_{xxx}(x,t;q). \tag{20}$$

We construct the zeroth order deformation equation

$$(1-nq)L\left[\phi(x,t;q) - u_0(x,t)\right] = qhH(x,t)N\left[D_t^{\alpha}\phi(x,t;q)\right]. \tag{21}$$

We choose $H(x,t) = 1$ to obtain the mth-order deformation equation to be

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = h\mathscr{R}_m\left(\mathbf{u}_{m-1}\right), \tag{22}$$

with initial condition for $m \geqslant 1$, $u_m(x,0) = 0$, χ_m^* is as defined in (15) and

$$\mathscr{R}_m\left(\mathbf{u}_{m-1}\right) = D_t^{\alpha}u_{m-1} - \sum_{k=0}^{m-1}\sum_{i=0}^{k}\sum_{j=0}^{i} u_j u_{i-j} u_{k-i} u_{(m-1-k)xxx}. \tag{23}$$

So, the solution to the equation (16) for $m \geqslant 1$ becomes

$$u_m(x,t) = \chi_m^* u_{m-1} + hI^{\alpha}\left[\mathscr{R}_m\left(\mathbf{u}_{m-1}\right)\right]. \tag{24}$$

We therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned} u_1(x,t) &= \chi_1^* u_0 + hI^{\alpha}\left[D_t^{\alpha}u_0 - u_0^3(u_0)_{xxx}\right] \\ &= b^{\frac{3}{2}}h\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{1}{3}}\frac{t^{\alpha}}{\Gamma(1+\alpha)} \end{aligned} \tag{25}$$

$$\begin{aligned} u_2(x,t) &= \chi_2^* u_1 + hI^{\alpha}\left[D_t^{\alpha}u_1 - u_0^3(u_1)_{xxx} - 3u_0^2u_1(u_0)_{xxx}\right] \\ &= b^{\frac{3}{2}}(n+h)h\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{1}{3}}\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad + \frac{9b^3h^2}{4}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{4}{3}}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \end{aligned} \tag{26}$$

In the same way, $u_m(x,t)$ for $m = 3,4,\cdots$ can be obtained using Mathematica.

Then the series solution expression by q-HAM can be written in the form

$$\begin{aligned} u(x,t;n;h) &= \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}} + \sum_{i=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \\ &= \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}} + b^{\frac{3}{2}}h\left(\frac{2n+h}{n^2}\right)\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{1}{3}}\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad + \frac{9b^3h^2}{4n^2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{4}{3}}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots. \end{aligned} \tag{27}$$

Equation (27) is an appropriate solution to the problem (16) in terms of convergence parameter h and n.

5 Numerical Results and Discussion

5.1 Exact Solution and Numerical Solution

In this section, we present graphs of solutions to equation (16) obtained by q-HAM using Mathematica 8 with appropriate values of h, a, b, α see Figure(1) and Figure(2). It should be noted that only the 2-term approximate series solution obtained by q-HAM is used for these plots.

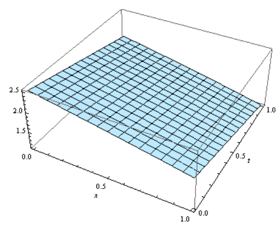

Fig. 1: q-HAM solution plot of u for $h = -1$, $n = 1$, $a = 4$, $b = 1$ and $\alpha = 1$

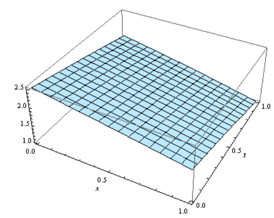

Fig. 2: Exact solution plot of u for $a = 4$, $b = 1$ and $\alpha = 1$

5.2 Effect of Auxiliary Parameter h

The effects of auxiliary parameter h and that of time, t is given through the Figures [(3)-(6)]. The impact of these parameters are greatly felt and this is one of the reasons why this method is considered over other analytical method where many terms are required to get very close to the exact solution.

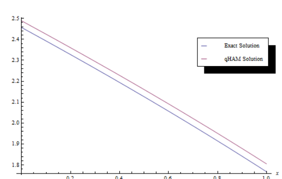

Fig. 3: The plot of u against x for $t = 0.1$, $h = -0.5$, $n = 1$, $a = 4$, $b = 1$ and $\alpha = 1$

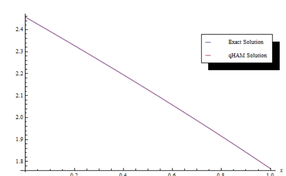

Fig. 4: The plot of u against x for $t = 0.1$, $h = -1$, $n = 1$, $a = 4$, $b = 1$ and $\alpha = 1$

5.3 Effect of α

We also generate some plots in Figure(7) and Figure(8) to display the effect of α on the solution obtained by q-HAM.

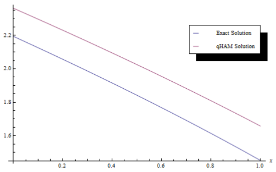

Fig. 5: The plot of u against x for $t = 0.5$, $h = -0.5$, $n = 1$, $a = 4$, $b = 1$ and $\alpha = 1$

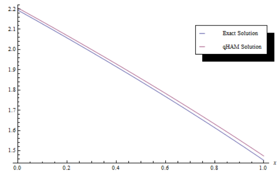

Fig. 6: The plot of u against x for $t = 0.5$, $h = -1$, $n = 1$, $a = 4$, $b = 1$ and $\alpha = 1$

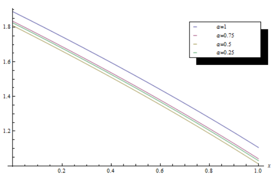

Fig. 7: The plot of u for different values of α and fixed $t = 1$

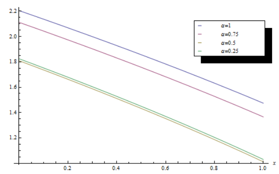

Fig. 8: The plot of u for different values of α and fixed $t = 0.5$

*Remark.*The result obtained here by q-HAM does not only give a good approximation to exact solution by taking only 2-term of the series but is in good agreement to other methods such as ADM, Power Series Method, VIM, HPM and so on used in the literature. These other methods have to use many terms of their series solution in other to get as closer as possible.

6 Conclusion

The major achievement of this paper is the demonstration of the successful application of the q-HAM to obtain analytical solutions of the nonlinear Harry Dym equation of time-fractional derivatives. Our results confirm that the method is really effective for handling solutions of a class of nonlinear PDEs of fractional order system The comparison made with other analytical methods such as HPM, VIM, ADM, DTM etc., enables us to see clearly the accuracy of q-HAM in the sense that just two terms are needed in our approximation to get close to the exact solution unlike other methods.

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