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New Common Coupled Fixed Point Results of Integral Type Contraction in Generalized Metric Spaces

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Abstract: In the paper, we introduce the concept of integral type contraction with respect to generalized metric space and prove some new common coupled coincidence fixed point results of integral type contractive mappings in generalized metric space. Finally, we present some examples.

Keywords: Generalized metric space; common coupled coincidence fixed point; common fixed point; integral type contractive mappings.
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1 Introduction

The study of common fixed points of mappings which satisfies certain contractive conditions has been studied by numerous researchers due to its valuable applications in Mathematics as well as in other sciences. To carried out a survey in metric as well for cone metric spaces of a common fixed point theory, we refer the reader to [1,2,3,4,8,9,10,11,12,15,17,18]. In 2006 Mustafa and Sims [16], introduced the concept of G-metric space and presented some fixed point theorems in G-metric space. The concept of a coupled coincidence point of mapping was introduced by V. Lakshmikantham [5,14], they also studied some fixed point theorems in partially ordered metric spaces. In 2010 Shatanawi [20] gave the proof of coupled coincidence fixed point theorems in generalized metric spaces. In 2013 Feng and Yun [7], presented a common coupled fixed point theorem in generalized metric space and give some applications to integral equations. Moreover, In 2002, Branciari [6] presented the notion of integral type contractive mappings in complete metric spaces and study the existence of fixed points for mappings which is defined on complete metric space satisfies integral type contraction. Also F. Khojasteh et al. [13], introduced the idea of integral type contraction in cone metric spaces and proved some fixed point theorems in such spaces. In this paper we used the idea of Branciari [6] and presented some common coupled coincidence fixed point results of integral type contractive mappings in setting of generalized metric spaces. We recommend some other references for reader see [19,21]. Also we give suitable examples that support our results.

2 Preliminaries

We needs the following definitions and results in this paper.

Definition 2.1[*16*] *Let Y be a non-empty set and $G : Y\times Y\times Y \to R^+$ is a function that satisfies the following conditions:*
$(G1)$ $G(a,b,c) = 0$ *if* $a = b = c$,
$(G2)$ $G(a,a,b) > 0$ *for all* $a,b \in Y$ *with* $a \neq b$,
$(G3)$ $G(a,a,b) \leq G(a,b,c)$, *for all* $a,b,c \in Y$ *with* $c \neq b$
$(G4)$ $G(a,b,c) = G(a,c,b) = G(b,c,a) = \dots$, *symmetry in all variables,*
$(G5)$ $G(a,b,c) \leq G(a,s,s) + G(s,b,c)$ *for all* $a,b,c,s \in Y$.
Then the function G is called a generalized metric and the pair (Y,G) is called a G-metric space.

Example 2.2[*16*] *Let $Y = \{x,y\}$. Define G on $Y\times Y\times Y$ by*

$$G(x,x,x) = G(y,y,y) = 0, G(x,x,y) = 1, G(x,y,y) = 2$$

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and extend G to $Y \times Y \times Y$ by using the symmetry in the variables. Then it is clear that (Y,G) is a G-metric space.

Definition 2.3[16] *Let (Y,G) be a G-metric space and (a_n) a sequence of points of Y. A point $a \in Y$ is said to be the limit of the sequence (a_n), if $\lim_{n,m\to+\infty} G(a,a_n,a_m) = 0$ and we say that the sequence (a_n) is G-convergent to a.*

Proposition 1.[16] *Let (Y,G) be a G-metric space. Then the following are equivalent:*
(1) (a_n) is G-convergent to a.
(2) $G(a_n,a_n,a) \to 0$ as $n \to +\infty$.
(3) $G(a_n,a,a) \to 0$ as $n \to +\infty$.
(4) $G(a_n,a_m,a) \to 0$ as $n,m \to +\infty$.

Definition 2.4[15] *Let (Y,G) be a G-metric space. A sequence (a_n) is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbf{N}$ such that $G(a_n,a_m,a_l) < \varepsilon$, for all $n,m,l \geq k$; that is $G(a_n,a_m,a_l) \to 0$ as $n,m,l \to +\infty$.*

Proposition 2.[16] *Let (Y,G) be a G-metric space. Then the following are equivalent:*
(1) The sequence (a_n) is G-Cauchy.
(2) For every $\varepsilon > 0$, there is $k \in \mathbf{N}$ such that $G(a_n,a_m,a_m) < \varepsilon$, for all $n,m \geq k$.

Definition 2.5[16] *A G-metric space (Y,G) is called G-complete if every G-Cauchy sequence in (Y,G) is G-convergent in (Y,G).*

Definition 2.6[5] *An element $(a,b) \in Y \times Y$ is called a coupled coincidence point of the mappings $F : Y \times Y \to Y$ and $g : Y \to Y$ if $F(a,b) = ga$ and $F(b,a) = gb$.*

Definition 2.7[14] *let Y be a non-empty set. Then we say that the mappings $F : Y \times Y \to Y$ and $g : Y \to Y$ are commutative if $gF(a,b) = F(ga,gb)$.*

Definition 2.8[14] *An element $(a,b) \in Y \times Y$ is called a coupled fixed point of mapping $F : Y \times Y \to Y$ if $F(a,b) = a$ and $F(b,a) = b$.*

Proposition 3.[16] *Let (Y,G) be a G-metric space. Then for any $a,b,c,e \in Y$, it follows that*
(i) if $G(a,b,c) = 0$, then $a = b = c$;
(ii) $G(a,b,c) \leq G(a,a,b) + G(a,a,c)$;
(iii) $G(a,b,b) \leq 2G(b,a,a)$;
(iv) $G(a,b,c) \leq G(a,e,c) + G(e,b,c)$;
(v) $G(a,b,c) \leq \frac{2}{3}(G(a,b,e) + G(a,e,c) + G(e,b,c))$;
(vi) $G(a,b,c) \leq G(a,e,e) + G(b,e,e) + G(c,e,e)$.

Proposition 4.[16] *Let (Y,G) be a G-metric space. Then the function $G(a,b,c)$ is jointly continuous in all three of its variables.*

Proposition 5.[16] *Let (Y,G) and (Y',G') be G-metric spaces, then the mapping $f : Y \to Y'$ is G-continuous at a point $a \in Y$ if and only if it is G-sequentially continuous at a; that is, whenever $\{a_n\}$ is G-convergent to a, $(f(a_n))$ is G-convergent to $f(a)$.*

In 2002, Branciari in [6] introduced a general contractive condition of integral type as follows.

Theorem 2.9[6] *Let (Y,d) be a complete metric space, $\alpha \in (0,1)$, and $f : Y \to Y$ is a mapping such that for all $x,y \in Y$,*

$$\int_0^{d(f(x),f(y))} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is nonnegative and Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,+\infty)$ such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$, then f has a unique fixed point $a \in Y$, such that for each $x \in Y$, $\lim_{n\to\infty} f^n(x) = a$.

In this manuscript we use the above idea of Branciari [6] and presented our results in generalized metric spaces.

3 Main Results

In this section we will prove some common coupled fixed point results in generalized metric space by using integral type contractive mappings. We will start our work by the following lemma.

Lemma 1. *Let (Y,G) be a G-metric space. Suppose $H_1,H_2,H_3 : Y \times Y \to Y$ and $h : Y \to Y$ be four mappings such that*

$$\begin{aligned}
&\int_0^{G(H_1(a,b),H_2(p,q),H_3(r,c))} \varphi(t)dt \leq \\
&\alpha_1 \int_0^{G(ha,hp,hr)} \varphi(t)dt + \alpha_2 \int_0^{G(hb,hq,hc)} \varphi(t)dt \\
&+ \alpha_3 \int_0^{G(ha,hp,hp)} \varphi(t)dt + \alpha_4 \int_0^{G(hb,hq,hq)} \varphi(t)dt \\
&+ \alpha_5 \int_0^{G(ha,hr,hr)} \varphi(t)dt + \alpha_6 \int_0^{G(hb,hc,hc)} \varphi(t)dt \\
&+ \alpha_7 \int_0^{G(hr,ha,ha)} \varphi(t)dt + \alpha_8 \int_0^{G(hc,hb,hb)} \varphi(t)dt
\end{aligned} \tag{3.1}$$

for all $a,b,c,p,q,r \in Y$, where $\alpha_i \geq 0, i = 1,2,\ldots,8$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8 < 1$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Assume that (a,b) is a common coupled coincidence point of the mappings pair (H_1,h), (H_2,h) and (H_3,h). Then $H_1(a,b) = H_2(a,b) = H_3(a,b) = ha = hb = H_1(b,a) = H_2(b,a) = H_3(b,a)$.

Proof. Since (a,b) is a common coupled coincidence point of the mappings pair (H_1,h), (H_2,h) and (H_3,h), we have $ha = H_1(a,b) = H_2(a,b) = H_3(a,b)$ and $hb = H_1(b,a) = H_2(b,a) = H_3(b,a)$. Suppose $ha \neq hb$. Then by (3.1), we get

$$\int_0^{G(ha,hb,hb)}\varphi(t)dt = \int_0^{G(H_1(a,b),H_2(b,a),H_3(b,a))}\varphi(t)dt$$
$$\leq \alpha_1\int_0^{G(ha,hb,hb)}\varphi(t)dt + \alpha_2\int_0^{G(hb,ha,ha)}\varphi(t)dt$$
$$+ \alpha_3\int_0^{G(ha,hb,hb)}\varphi(t)dt + \alpha_4\int_0^{G(hb,ha,ha)}\varphi(t)dt$$
$$+ \alpha_5\int_0^{G(hb,hb,hb)}\varphi(t)dt + \alpha_6\int_0^{G(ha,ha,ha)}\varphi(t)dt$$
$$+ \alpha_7\int_0^{G(hb,ha,ha)}\varphi(t)dt + \alpha_8\int_0^{G(ha,hb,hb)}\varphi(t)dt$$
$$= (\alpha_1+\alpha_3+\alpha_8)\int_0^{G(ha,hb,hb)}\varphi(t)dt$$
$$+ (\alpha_2+\alpha_4+\alpha_7)\int_0^{G(hb,ha,ha)}\varphi(t)dt.$$

Also by (3.1), we have

$$\int_0^{G(hb,ha,ha)}\varphi(t)dt = \int_0^{G(H_1(b,a),H_2(a,b),H_3(a,b))}\varphi(t)dt$$
$$\leq \alpha_1\int_0^{G(hb,ha,ha)}\varphi(t)dt + \alpha_2\int_0^{G(ha,hb,hb)}\varphi(t)dt$$
$$+ \alpha_3\int_0^{G(hb,ha,ha)}\varphi(t)dt + \alpha_4\int_0^{G(ha,hb,hb)}\varphi(t)dt$$
$$+ \alpha_5\int_0^{G(ha,ha,ha)}\varphi(t)dt + \alpha_6\int_0^{G(hb,hb,hb)}\varphi(t)dt$$
$$+ \alpha_7\int_0^{G(ha,hb,hb)}\varphi(t)dt + \alpha_8\int_0^{G(hb,ha,ha)}\varphi(t)dt$$
$$= (\alpha_1+\alpha_3+\alpha_8)\int_0^{G(hb,ha,ha)}\varphi(t)dt$$
$$+ (\alpha_2+\alpha_4+\alpha_7)\int_0^{G(ha,hb,hb)}\varphi(t)dt.$$

Therefore

$$\int_0^{G(ha,hb,hb)}\varphi(t)dt + \int_0^{G(hb,ha,ha)}\varphi(t)dt$$
$$\leq (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_7+\alpha_8)$$
$$\int_0^{(G(ha,hb,hb)+G(hb,ha,ha))}\varphi(t)dt.$$

Since $0 \leq \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_7+\alpha_8 < 1$, we get

$$\int_0^{G(ha,hb,hb)}\varphi(t)dt + \int_0^{G(hb,ha,ha)}\varphi(t)dt < \int_0^{G(ha,hb,hb)}\varphi(t)dt + \int_0^{G(hb,ha,ha)}\varphi(t)dt,$$

which is contradiction. So $ha = hb$, hence

$H_1(a,b) = H_2(a,b) = H_3(a,b) = ha = hb = H_1(b,a) = H_2(b,a) = H_3(b,a)$.

Theorem 3.1*Let (Y,G) be a G-metric space. Suppose $H_1,H_2,H_3 : Y\times Y \to Y$ and $h : Y \to Y$ be four mappings such that*

$$\int_0^{G(H_1(a,b),H_2(p,q),H_3(r,c))}\varphi(t)dt \leq$$
$$\alpha_1\int_0^{G(ha,hp,hr)}\varphi(t)dt + \alpha_2\int_0^{G(hb,hq,hc)}\varphi(t)dt$$
$$+ \alpha_3\int_0^{G(ha,hp,hp)}\varphi(t)dt + \alpha_4\int_0^{G(hb,hq,hq)}\varphi(t)dt$$
$$+ \alpha_5\int_0^{G(ha,hr,hr)}\varphi(t)dt + \alpha_6\int_0^{G(hb,hc,hc)}\varphi(t)dt$$
$$+ \alpha_7\int_0^{G(hr,ha,ha)}\varphi(t)dt + \alpha_8\int_0^{G(hc,hb,hb)}\varphi(t)dt \quad (3.2)$$

for all $a,b,c,p,q,r \in Y$, where $\alpha_i \geq 0, i = 1,2,\dots,8$ with $\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8 < 1$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\varphi(t)dt > 0$. Assume that H_1,H_2,H_3 and h satisfies the following conditions:
(i) $H_1(Y\times Y) \subset h(Y), H_2(Y\times Y) \subset h(Y), H_3(Y\times Y) \subset h(Y)$;
(ii) $h(Y)$ is G-complete;
(iii) h is G-continuous and continuous with H_1,H_2,H_3.
Then there exist a unique $a \in Y$ such that $ha = H_1(a,a) = H_2(a,a) = H_3(a,a) = a$.

Proof. Let $a_0,b_0 \in Y$. Since $H_1(Y\times Y) \subset h(Y)$, $H_2(Y\times Y) \subset h(Y)$, $H_3(Y\times Y) \subset h(Y)$, we can choose $a_1,a_2,a_3,b_1,b_2,b_3 \in Y$ such that $ha_1 = H_1(a_0,b_0), hb_1 = H_1(b_0,a_0), ha_2 = H_2(a_1,b_1), hb_2 = H_2(b_1,a_1), ha_3 = H_3(a_2,b_2)$ and $hb_3 = H_3(b_2,a_2)$. Combining this process, we can construct two sequences $\{a_n\}$ and $\{b_n\}$ in Y such that

$ha_{3n} = H_3(a_{3n-1},b_{3n-1})$, $hb_{3n} = H_3(b_{3n-1},a_{3n-1})$, $n = 1,2,3,\dots,$

$ha_{3n+1} = H_1(a_{3n},b_{3n})$, $hb_{3n+1} = H_1(b_{3n},a_{3n})$, $n = 1,2,3,\dots,$

$ha_{3n+2} = H_2(a_{3n+1},b_{3n+1})$, $hb_{3n+2} = H_2(b_{3n+1},a_{3n+1})$, $n = 1,2,3,\dots.$

If $ha_{3n} = ha_{3n+1}$, then $ha = H_1(a,b)$, where $a = a_{3n}, b = b_{3n}$.

If $ha_{3n+1} = ha_{3n+2}$, then $ha = H_2(a,b)$, where $a = a_{3n+1}, b = b_{3n+1}$.

If $ha_{3n+2} = ha_{3n+3}$, then $ha = H_3(a,b)$, where $a = a_{3n+2}, b = b_{3n+2}$.

Also, If $hb_{3n} = hb_{3n+1}$, then $hb = H_1(b,a)$, where $b = b_{3n}, a = a_{3n}$.

If $hb_{3n+1} = hb_{3n+2}$, then $hb = H_2(b,a)$, where $b = b_{3n+1}, a = a_{3n+1}$.

If $hb_{3n+2} = hb_{3n+3}$, then $hb = H_3(b,a)$, where $b = b_{3n+2}, a = a_{3n+2}$.

Without loss of generality, we may assume that $ha_n \neq ha_{n+1}$ and $hb_n \neq hb_{n+1}$, for all $n = 0,1,2,\ldots$.

By (3.2), we have

$$\begin{aligned}
&\int_0^{G(ha_{3n},ha_{3n+1},ha_{3n+2})}\varphi(t)dt\\
&= \int_0^{G(H_3(a_{3n-1},b_{3n-1}),H_2(a_{3n},b_{3n}),H_2(a_{3n+1},b_{3n+1}))}\varphi(t)dt\\
&\leq \alpha_1\int_0^{G(ha_{3n},ha_{3n+1},ha_{3n-1})}\varphi(t)dt\\
&+ \alpha_2\int_0^{G(hb_{3n},hb_{3n+1},hb_{3n-1})}\varphi(t)dt\\
&+ \alpha_3\int_0^{G(ha_{3n},ha_{3n+1},ha_{3n+1})}\varphi(t)dt\\
&+ \alpha_4\int_0^{G(hb_{3n},hb_{3n+1},hb_{3n+1})}\varphi(t)dt\\
&+ \alpha_5\int_0^{G(ha_{3n+1},ha_{3n-1},ha_{3n-1})}\varphi(t)dt\\
&+ \alpha_6\int_0^{G(hb_{3n+1},hb_{3n-1},hb_{3n-1})}\varphi(t)dt\\
&+ \alpha_7\int_0^{G(ha_{3n-1},ha_{3n},ha_{3n})}\varphi(t)dt\\
&+ \alpha_8\int_0^{G(hb_{3n-1},hb_{3n},hb_{3n})}\varphi(t)dt\\
&\leq (\alpha_1+\alpha_3+\alpha_5+\alpha_7)\int_0^{G(ha_{3n-1},ha_{3n},ha_{3n+1})}\varphi(t)dt\\
&+ (\alpha_2+\alpha_4+\alpha_6+\alpha_8)\int_0^{G(hb_{3n-1},hb_{3n},hb_{3n+1})}\varphi(t)dt.
\end{aligned}$$

Which implies that

$$\begin{aligned}
&\int_0^{G(ha_{3n},ha_{3n+1},ha_{3n+2})}\\
&\varphi(t)dt \leq (\alpha_1+\alpha_3+\alpha_5+\alpha_7)\int_0^{G(ha_{3n-1},ha_{3n},ha_{3n+1})}\varphi(t)dt\\
&+(\alpha_2+\alpha_4+\alpha_6+\alpha_8)\int_0^{G(hb_{3n-1},hb_{3n},hb_{3n+1})}\varphi(t)dt. \qquad (3.3)
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
&\int_0^{G(hb_{3n},hb_{3n+1},hb_{3n+2})}\varphi(t)dt\\
&\leq (\alpha_1+\alpha_3+\alpha_5+\alpha_7)\int_0^{G(hb_{3n-1},hb_{3n},hb_{3n+1})}\varphi(t)dt\\
&+ (\alpha_2+\alpha_4+\alpha_6+\alpha_8)\int_0^{G(ha_{3n-1},ha_{3n},ha_{3n+1})}\varphi(t)dt. \qquad (3.4)
\end{aligned}$$

Combining (3.3) and (3.4), we get

$$\begin{aligned}
&\int_0^{G(ha_{3n},ha_{3n+1},ha_{3n+2})}\varphi(t)dt\\
&+ \int_0^{G(hb_{3n},hb_{3n+1},hb_{3n+2})}\varphi(t)dt \qquad (3.5)\\
&\leq (\sum_{i=1}^{8} a_i)\int_0^{[G(ha_{3n-1},ha_{3n},ha_{3n+1})+G(hb_{3n-1},hb_{3n},hb_{3n+1})]}\varphi(t)dt.
\end{aligned}$$

Next, we can show that

$$\begin{aligned}
&\int_0^{G(ha_{3n-1},ha_{3n},ha_{3n+1})}\varphi(t)dt\\
&+ \int_0^{G(hb_{3n-1},hb_{3n},hb_{3n+1})}\varphi(t)dt\\
&\leq (\sum_{i=1}^{8} a_i)\int_0^{[G(ha_{3n-2},ha_{3n-1},ha_{3n})+G(hb_{3n-2},hb_{3n-1},hb_{3n})]}\varphi(t)dt. (3.6)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{G(ha_{3n-2},ha_{3n-1},ha_{3n})}\varphi(t)dt\\
&+ \int_0^{G(hb_{3n-2},hb_{3n-1},hb_{3n})}\varphi(t)dt \qquad (3.7)\\
&\leq (\sum_{i=1}^{8} a_i)\int_0^{[G(ha_{3n-3},ha_{3n-2},ha_{3n-1})+G(hb_{3n-3},hb_{3n-2},hb_{3n-1})]}\varphi(t)dt.
\end{aligned}$$

It follows from (3.5), (3.6) and (3.7) that for all $n \in N$, we have

$$\begin{aligned}
&\int_0^{G(ha_n,ha_{n+1},ha_{n+2})}\varphi(t)dt + \int_0^{G(hb_n,hb_{n+1},hb_{n+2})}\varphi(t)dt\\
&\leq (\sum_{i=1}^{8} a_i)\int_0^{[G(ha_{n-1},ha_n,ha_{n+1})+G(hb_{n-1},hb_n,hb_{n+1})]}\varphi(t)dt\\
&= k\int_0^{[G(ha_{n-1},ha_n,ha_{n+1})+G(hb_{n-1},hb_n,hb_{n+1})]}\varphi(t)dt\\
&\leq k^2\int_0^{[G(ha_{n-2},ha_{n-1},ha_n)+G(hb_{n-2},hb_{n-1},hb_n)]}\varphi(t)dt\\
&\vdots\\
&\leq k^n\int_0^{[G(ha_0,ha_1,ha_2)+G(hb_0,hb_1,hb_2)]}\varphi(t)dt, \qquad (3.8)
\end{aligned}$$

where $k = \sum_{i=1}^{8} a_i \in [0,1)$. From (G3), we have $G(ha_n,ha_{n+1},ha_{n+2}) \leq G(ha_n,ha_{n+1},ha_{n+2})$ and $G(hb_n,hb_{n+1},hb_{n+2}) \leq G(hb_n,hb_{n+1},hb_{n+2})$. Hence, from (G3) and (3.8), we get

$$\begin{aligned}
&\int_0^{G(ha_n,ha_{n+1},ha_{n+1})}\varphi(t)dt + \int_0^{G(hb_n,hb_{n+1},hb_{n+1})}\varphi(t)dt\\
&\leq \int_0^{G(ha_n,ha_{n+1},ha_{n+2})}\varphi(t)dt + \int_0^{G(hb_n,hb_{n+1},hb_{n+2})}\varphi(t)dt\\
&\leq k^n\int_0^{[G(ha_0,ha_1,ha_2)+G(hb_0,hb_1,hb_2)]}\varphi(t)dt. \qquad (3.9)
\end{aligned}$$

Therefore, for all $n,m \in N, n < m$, by (G5) and (3.9), we have

$$\int_0^{G(ha_n,ha_m,ha_m)} \varphi(t)dt + \int_0^{G(hb_n,hb_m,hb_m)} \varphi(t)dt$$
$$\leq \int_0^{[G(ha_n,ha_{n+1},ha_{n+1})+G(hb_n,hb_{n+1},hb_{n+1})]} \varphi(t)dt$$
$$+\cdots+ \int_0^{[G(ha_{m-1},ha_m,ha_m)+G(hb_{m-1},hb_m,hb_m)]} \varphi(t)dt$$
$$\leq (k^n + k^{n+1} + \cdots + k^{m-1}) \int_0^{[G(ha_0,ha_1,ha_2)+G(hb_0,hb_1,hb_2)]} \varphi(t)dt$$
$$\leq \frac{k^n}{1-k} \int_0^{[G(ha_0,ha_1,ha_2)+G(hb_0,hb_1,hb_2)]} \varphi(t)dt. \quad (3.10)$$

Thus

$$\int_0^{G(ha_n,ha_m,ha_m)} \varphi(t)dt \to 0 \text{ as } n,m \to \infty.$$

Which implies that

$$G(ha_n,ha_m,ha_m) \to 0 \text{ and } G(hb_n,hb_m,hb_m) \to 0 \text{ as } n,m \to \infty.$$

Thus, $\{ha_n\}$ and $\{hb_n\}$ are all G-Cauchy in hY. Since hY is G-complete, we get $\{ha_n\}$ and $\{hb_n\}$ are converges to some $a \in hY$ and $b \in hY$, respectively. Since h is G-continuous, we have $\{hha_n\}$ is G-convergent to ha and $\{hhb_n\}$ is G-convergent to hb. i.e.,

$$hha_n \to ha \quad and \quad hhb_n \to hb \quad as \quad n \to \infty. \quad (3.11)$$

Also, as h commutes with H_1, H_2 and H_3, we have

$$hha_{3n} = hH_3(a_{3n-1}, b_{3n-1}) = H_3(ha_{3n-1}, hb_{3n-1}),$$

$$hhb_{3n} = hH_3(b_{3n-1}, a_{3n-1}) = H_3(hb_{3n-1}, ha_{3n-1}),$$

$$hha_{3n+1} = hH_1(a_{3n}, b_{3n}) = H_1(ha_{3n}, hb_{3n}),$$

$$hhb_{3n+1} = hH_1(b_{3n}, a_{3n}) = H_1(hb_{3n}, ha_{3n}),$$

$$hha_{3n+2} = hH_2(a_{3n+1}, b_{3n+1}) = H_2(ha_{3n+1}, hb_{3n+1}),$$

and

$$hhb_{3n+2} = hH_2(b_{3n+1}, a_{3n+1}) = H_2(hb_{3n+1}, ha_{3n+1}).$$

Thus, from (3.2), we have

$$\int_0^{G(hha_{3n},hha_{3n+1},H_2(a,b))} \varphi(t)dt$$
$$= \int_0^{G(H_1(ha_{3n},hb_{3n}),H_2(a,b),H_3(ha_{3n-1},hb_{3n-1}))} \varphi(t)dt$$
$$\leq \alpha_1 \int_0^{G(hha_{3n},ha,hha_{3n-1})} \varphi(t)dt$$
$$+ \alpha_2 \int_0^{G(hhb_{3n},hb,hhb_{3n-1})} \varphi(t)dt$$
$$+ \alpha_3 \int_0^{G(hha_{3n},ha,ha)} \varphi(t)dt$$
$$+ \alpha_4 \int_0^{G(hhb_{3n},hb,hb)} \varphi(t)dt$$
$$+ \alpha_5 \int_0^{G(ha,hha_{3n-1},hha_{3n-1})} \varphi(t)dt$$
$$+ \alpha_6 \int_0^{G(hb,hhb_{3n-1},hhb_{3n-1})} \varphi(t)dt$$
$$+ \alpha_7 \int_0^{G(hha_{3n-1},hha_{3n},hha_{3n})} \varphi(t)dt$$
$$+ \alpha_8 \int_0^{G(hhb_{3n-1},hhb_{3n},hhb_{3n})} \varphi(t)dt.$$

Letting $n \to \infty$, and using (3.11), also G is continuous, we get
$G(ha,ha,H_2(a,b)) = 0$.
Hence, $ha = H_2(a,b)$. By the same way, we can show that $hb = H_2(b,a)$. Also we may show that $ha = H_1(a,b), hb = H_1(b,a), ha = H_3(a,b)$ and $hb = H_3(b,a)$. Therefore, (a,b) is a common coupled coincidence point of the pair $(H_1,h),(H_2,h)$ and (H_3,h). By Lemma 1, we get

$$ha = H_1(a,b) = H_2(a,b) = H_3(a,b)$$
$$= H_1(b,a) = H_2(b,a) = H_3(b,a) = hb. \quad (3.12)$$

Since the sequences $\{ha_{3n-1}\}$, $\{hb_{3n}\}$ and $\{ha_{3n+1}\}$ are the subsequences of the sequence $\{ha_n\}$, so all are G-convergent to a. By the same process, we may show that $\{hb_{3n-1}\}$, $\{hb_{3n}\}$ and $\{ha_{3n+1}\}$ are converges to b. From condition (3.2), we have

$$\int_0^{G(ha_{3n},ha,ha)} \varphi(t)dt = \int_0^{G(H_1(a,b),H_2(a,b),H_3(a_{3n-1},b_{3n-1}))} \varphi(t)dt$$
$$\leq \alpha_1 \int_0^{G(ha,ha,ha_{3n-1})} \varphi(t)dt + \alpha_2 \int_0^{G(hb,hb,hb_{3n-1})} \varphi(t)dt$$
$$+ \alpha_3 \int_0^{G(ha,ha,ha)} \varphi(t)dt + \alpha_4 \int_0^{G(hb,hb,hb)} \varphi(t)dt$$
$$+ \alpha_5 \int_0^{G(ha,ha_{3n-1},ha_{3n-1})} \varphi(t)dt + \alpha_6 \int_0^{G(hb,hb_{3n-1},hb_{3n-1})} \varphi(t)dt$$
$$+ \alpha_7 \int_0^{G(ha_{3n-1},ha,ha)} \varphi(t)dt + \alpha_8 \int_0^{G(hb_{3n-1},hb,hb)} \varphi(t)dt.$$

Letting $n \to \infty$, also as G is continuous, we get

$$\int_0^{G(a,ha,ha)} \varphi(t)dt \le (\alpha_1+\alpha_7)\int_0^{G(ha,ha,a)} \varphi(t)dt$$
$$+(\alpha_2+\alpha_8)\int_0^{G(hb,hb,b)} \varphi(t)dt$$
$$+\ \alpha_5\int_0^{G(ha,a,a)} \varphi(t)dt+\alpha_6\int_0^{G(hb,b,b)} \varphi(t)dt.$$

Further, we can show that

$$\int_0^{G(b,hb,hb)} \varphi(t)dt \le (\alpha_1+\alpha_7)$$
$$\int_0^{G(hb,hb,b)} \varphi(t)dt+(\alpha_2+\alpha_8)\int_0^{G(ha,ha,a)} \varphi(t)dt$$
$$+\ \alpha_5\int_0^{G(hb,b,b)} \varphi(t)dt+\alpha_6\int_0^{G(ha,a,a)} \varphi(t)dt.$$

By using Proposition 5(iii), we have

$$\int_0^{G(a,ha,ha)} \varphi(t)dt$$
$$+\int_0^{G(b,hb,hb)} \varphi(t)dt \le (\alpha_1+\alpha_2+\alpha_7+\alpha_8)$$
$$\int_0^{[G(ha,ha,a)+G(hb,hb,b)]} \varphi(t)dt$$
$$+\ (\alpha_5+\alpha_6)\int_0^{[G(ha,a,a)+G(hb,b,b)]} \varphi(t)dt$$
$$\le (\alpha_1+\alpha_2+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8)$$
$$\int_0^{[G(ha,ha,a)+G(hb,hb,b)]} \varphi(t)dt.$$

Since $0 \le \alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8 < 1$, so the last inequality happens only if $G(a,ha,ha)=0$ and $G(b,hb,hb)=0$. Hence $a=ha$ and $b=hb$. From (3.12), we have $a=ha=hb=b$, thus, we get
$ha=H_1(a,a)=H_2(a,a)=H_3(a,a)=a$.
For uniqueness, let $p_0 \in Y$ with assumption that $p_0 \ne a$ such that
$p_0=hp_0=H_1(p_0,p_0)=H_2(p_0,p_0)=H_3(p_0,p_0)$.
Once again using the condition (3.2) and Proposition 5(iii), we have

$$\int_0^{G(p_0,p_0,a)} \varphi(t)dt = \int_0^{G(H_1(p_0,p_0),H_2(p_0,p_0),H_3(a,a))} \varphi(t)dt$$
$$\le \alpha_1\int_0^{G(hp_0,hp_0,ha)} \varphi(t)dt+\alpha_2\int_0^{G(hp_0,hp_0,ha)} \varphi(t)dt$$
$$+\ \alpha_3\int_0^{G(hp_0,hp_0,hp_0)} \varphi(t)dt+\alpha_4\int_0^{G(hp_0,hp_0,hp_0)} \varphi(t)dt$$
$$+\ \alpha_5\int_0^{G(hp_0,ha,ha)} \varphi(t)dt+\alpha_6\int_0^{G(hp_0,ha,ha)} \varphi(t)dt$$
$$+\ \alpha_7\int_0^{G(ha,hp_0,hp_0)} \varphi(t)dt+\alpha_8\int_0^{G(ha,hp_0,hp_0)} \varphi(t)dt$$
$$\le (\alpha_1+\alpha_2+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8)$$
$$\int_0^{G(p_0,p_0,a)} \varphi(t)dt.$$

Since $0 \le \alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8 < 1$, we get

$$\int_0^{G(p_0,p_0,a)} \varphi(t)dt < \int_0^{G(p_0,p_0,a)} \varphi(t)dt,$$

which gives a contradiction. Thus, H_1, H_2, H_3 and h have a unique common fixed point.

Corollary 3.2 *Let (Y,G) be a G-metric space. Let $H_1,H_2,H_3 : Y\times Y \to Y$ and $h : Y \to Y$ be mappings such that*

$$\int_0^{G(H_1(a,b),H_2(p,q),H_3(r,c))} \varphi(t)dt$$
$$\le \alpha_1\int_0^{G(ha,hp,hr)} \varphi(t)dt+\alpha_2\int_0^{G(hb,hq,hc)} \varphi(t)dt \quad (3.13)$$

for all $a,b,c,p,q,r \in Y$, where $a_i \ge 0, i=1,2,$ and $\alpha_1+\alpha_2 < 1$. Also $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Assume that H_1,H_2,H_3 and h satisfies the following conditions:
(i)
$H_1(Y\times Y) \subset h(Y), H_2(Y\times Y) \subset h(Y), H_3(Y\times Y) \subset h(Y)$;
(ii) $h(Y)$ is G-complete;
(iii) h is G-continuous and continuous with H_1,H_2,H_3.
Then there exists a unique $a \in Y$ such that $ha=H_1(a,a)=H_2(a,a)=H_3(a,a)=a$.

Example 3.3 *Suppose $Y=[0,1]$. Define $G : Y\times Y\times Y \to R^+$ by*

$$G(a,b,c) = |a-b|+|b-c|+|c-a|$$

for all $a,b,c \in Y$. Then (Y,G) is a complete G-metric space. Define a map
$H_1,H_2,H_3 : Y\times Y \to Y$
by

$$H_1(a,b)=H_2(a,b)=H_3(a,b)=\frac{a+b}{8}$$

for all $a,b \in Y$. Also, define $h : Y \to Y$ by $ha=\frac{a}{2}$ and $\varphi(t)=\frac{t}{2}$ for some $t \in Y$. Then $H(Y\times Y) \subseteq hY$.

Thus the condition of Corollary 3.2 holds, in fact,

$$\int_0^{G(H_1(a,b),H_2(p,q),H_3(r,c))} \varphi(t)dt \le \int_0^{G(\frac{a+b}{8},\frac{p+q}{8},\frac{r+c}{8})} \varphi(t)dt$$
$$= \frac{1}{4}\int_0^{G(ha,hp,hr)} \varphi(t)dt+\frac{1}{4}\int_0^{G(hb,hq,hc)} \varphi(t)dt.$$

By subadditivity, we have

$$= \frac{1}{4}\int_0^{[G(ha,hp,hr)+G(hb,hq,hc)]} \varphi(t)dt.$$

Thus we see that the condition (3.13) of Corollary 3.2 is satisfied with $\alpha_1=\alpha_2=\frac{1}{4}$. So, we may say that H_1,H_2,H_3 and h have a common fixed point. Further, 0 is the unique common fixed point for all maps H_1,H_2,H_3 and h.

Corollary 3.4 *Let (Y,G) be a G-metric space. Let $H : Y \times Y \to Y$ and $h : Y \to Y$ be four mappings such that*

$$\int_0^{G(H(a,b),H(p,q),H(r,c))} \varphi(t)dt \leq \alpha_1 \int_0^{G(ha,hp,hp)} \varphi(t)dt + \alpha_2 \int_0^{G(hb,hq,hq)} \varphi(t)dt + \alpha_3 \int_0^{G(hp,hr,hr)} \varphi(t)dt + \alpha_4 \int_0^{G(hq,hc,hc)} \varphi(t)dt + \alpha_5 \int_0^{G(hr,ha,ha)} \varphi(t)dt + \alpha_6 \int_0^{G(hc,hb,hb)} \varphi(t)dt \quad (3.14)$$

for all $a,b,c,p,q,r \in Y$, where $\alpha_i \geq 0, i = 1,2,\ldots,6$ and $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 < 1$. Also $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Suppose that H and h satisfy the below conditions:
(i) $H(Y \times Y) \subseteq hY$;
(ii) hY is G-complete;
(iii) h is G-continuous and commutes with H.
Then there exists a unique $a \in Y$ such that $ha = H(a,b) = a$.

Example 3.5 *Suppose $Y = [0,1]$. Define $G : Y \times Y \times Y \to R^+$ by*

$$G(a,b,c) = |a-b| + |b-c| + |c-a|$$

for all $a,b,c \in Y$. Then (Y,G) is a complete G-metric space. Define a mapping $H : Y \times Y \to Y$ by

$$H(a,b) = \frac{ab}{8}$$

for all $a,b \in Y$. Also, define a map $g : Y \to Y$ by $ha = a$ and $\varphi(t) = t$ for $t \in Y$.
Then the condition of Corollary 3.4 holds, in fact,

$$\int_0^{G(H(a,b),H(p,q),H(r,c))} \varphi(t)dt = \int_0^{G(\frac{ab}{8},\frac{pq}{8},\frac{rc}{8})} \varphi(t)dt = \frac{1}{16}\int_0^{G(ha,hp,hp)} \varphi(t)dt + \frac{1}{16}\int_0^{G(hb,hq,hq)} \varphi(t)dt + \frac{1}{16}\int_0^{G(hp,hr,hr)} \varphi(t)dt + \frac{1}{16}\int_0^{G(hq,hc,hc)} \varphi(t)dt + \frac{1}{16}\int_0^{G(hr,ha,ha)} \varphi(t)dt + \frac{1}{16}\int_0^{G(hc,hb,hb)} \varphi(t)dt.$$

Clearly we see that the condition (3.14) of Corollary 3.4 is satisfied with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \frac{1}{16}$. So, H and h have a unique common fixed point. Moreover, 0 is the unique common fixed point for all the mappings H and h.

4 Conclusion

In this paper we use the idea of A. Branciari [6], about integral type contraction to produce new common coupled coincidence fixed point results in generalized metric spaces.

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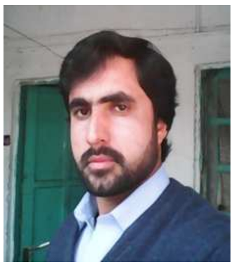

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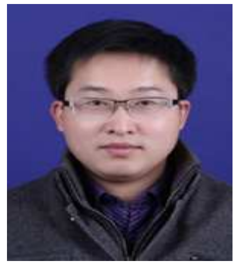

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