

# Analytical Solution of the Fractional Rössler Chaotic System via a Novel Natural Transform Homotopy Method

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**Abstract:** This paper presents the development and application of a novel analytical technique, the Natural Transform Homotopy Perturbation Method (NTHPM), for obtaining approximate solutions to nonlinear fractional-order differential equations. The methodology is specifically formulated for systems incorporating the Atangana-Baleanu fractional operator, which possesses non-singular and non-local kernel properties. The primary objective is to employ this technique to derive an analytical approximate solution for the fractional-order Rössler system, a canonical model of chaotic dynamics. Furthermore, the study rigorously establishes the existence and uniqueness of the solution for the considered fractional-order system under appropriate conditions, thereby providing a solid theoretical foundation for the applied method. The results demonstrate the efficacy and computational advantages of the proposed NTHPM in handling the complexity inherent in nonlinear fractional systems, offering a powerful tool for their analysis and simulation.

**Keywords:** fractional Rössler system, Atangana-Baleanu fractional derivative operator, Natural transform, NHPM, stability, Existence and uniqueness.

## 1 Introduction

The study of fractional-order nonlinear differential systems has garnered substantial attention across scientific disciplines, primarily due to their superior capability in modeling memory effects and complex hereditary properties inherent in physical, biological, and engineering phenomena. Within this domain, chaotic systems of fractional order, such as the Rössler system, present a particularly rich area of investigation, offering more nuanced insights into dynamical behaviors than their integer-order counterparts. However, the inherent complexity of these equations, characterized by non-local fractional operators, poses a significant analytical challenge, making the derivation of accurate and computationally efficient approximate solutions a central and persistent problem in the field. While several semi-analytical techniques, including the Adomian Decomposition Method (ADM) [1-7], the Differential Transform Method (DTM) [8-12], and the Variational Iteration Method (VIM) [13-17], have been developed to address nonlinear problems, each presents limitations. ADM often requires the computation of complex Adomian polynomials, DTM can be constrained by convergence radii, and VIM involves the determination of a Lagrange multiplier, which is not always straightforward. Although the standard Homotopy Perturbation Method (HPM) [18] simplifies solution procedures by constructing a homotopy, its convergence can be slow for strongly nonlinear fractional systems. This study is motivated by two critical gaps in the existing literature. First, there is a need for enhanced semi-analytical methods that combine rapid convergence with manageable computational algorithms for the latest class of fractional operators. Second, while numerical simulations of fractional chaotic systems are abundant, rigorous analytical underpinnings—specifically, proofs of solution existence and uniqueness for these systems—are less commonly addressed in conjunction with solution

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techniques. To bridge these gaps, this work employs the Atangana–Baleanu (AB) fractional operator in the Caputo sense. The choice of this operator is justified by its non-singular and non-local kernel, which more accurately describes memory effects in many real-world processes compared to classical power-law kernels like Caputo or Riemann-Liouville. The core methodological contribution lies in the integration of this operator with the Natural Transform Homotopy Perturbation Method (NTHPM). The Natural Transform effectively handles the initial conditions and simplifies the fractional operator, while HPM deconstructs the nonlinearity. This hybrid approach, NTHPM, is posited to retain the simplicity of HPM while achieving accelerated convergence and higher accuracy compared to ADM, DTM, and VIM, as it systematically decomposes the nonlinear terms without requiring auxiliary polynomials or restrictive assumptions. Consequently, the primary objectives of this paper are threefold: (i) to formulate the NTHPM for nonlinear systems with the AB operator; (ii) to apply this framework to obtain an analytical approximate solution for the fractional-order Rössler system, demonstrating its dynamical characteristics; and (iii) to provide a rigorous mathematical analysis, establishing the existence and uniqueness of the solution for the considered system under the AB derivative, thereby strengthening the theoretical foundation for the applied numerical technique.

## 2 Basic Concepts:

This section introduces the definitions of the fractional derivatives with non-singular kernels proposed by Atangana and Baleanu, which serve as the basis for the following discussion., we first outline some essential ideas of the homotopy perturbation method and recall the definition below.

**Definition 2.1:** Let  $f(t) \in H^1(a, b)$ ,  $a < b$ . The Atangana-Baleanu fractional derivative in the Caputo sense of order  $\alpha$  is defined [4,5] as follows:

$$({}^{AB}D_t^\alpha f(t)) = \frac{z(\alpha)}{1-\alpha} \int_a^t E_\alpha \left( -\frac{\alpha(t-\tau)^\alpha}{1-\alpha} \right) f'(\tau) d\tau, \quad 0 < \alpha < 1 \quad (2.1)$$

where  $z(\alpha)$  is a normalization function satisfying  $z(0) = z(1) = 1$ .

**Definition 2.2:** The Atangana–Baleanu fractional integral, which corresponds to the recently proposed fractional derivative with a non-local kernel, is defined as follows. The fractional integral introduced by Atangana and Baleanu of order  $\alpha$  is given as follows [4,5]

$$({}^{AB}I_t^\alpha f(t)) = \frac{1-\alpha}{z(\alpha)} f(t) + \frac{\alpha}{z(\alpha)\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \quad (2.2)$$

**Definition 2.3:** N-Transforms.

The natural transform of a function  $f(t)$  for  $t \in (-\infty, \infty)$  is defined as [19,20]: Let

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j} \text{ if } t \in (-1)^j \times (0, \infty), j \in \mathbb{Z}^+ \right\}.$$

Then for  $f \in A$ , the natural transform is given by

$$N[f(t)] = R(s, \mu) = \int_0^\infty e^{-st} f(\mu t) dt, \quad \mu > 0, s > 0, \quad (2.3)$$

where  $s$  and  $\mu$  are the natural transform parameters.

The original function  $f(t)$  corresponds to the inverse transform and is denoted by:

$$f(t) = N^{-1}(R(s, \mu)). \quad (2.4)$$

The inverse natural transform is given by

$$N^{-1}(R(s, \mu)) = f(t) = \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} e^{st/\mu} R(s, \mu) dt, \quad (2.5)$$

where  $s$  and  $\mu$  are the natural transform variables and  $p$  is a real constant.

**Definition 2.4:** The natural transform of the Atangana-Baleanu fractional derivative (ABFD) is defined as follows: We obtain the relation [21]:

$$N [({}^{AB}D_t^\alpha f(t))] = \frac{m(\alpha)}{\left(1 - \alpha + \frac{\alpha\mu^\alpha}{s^\alpha}\right)} \left( R(s, \mu) - \frac{1}{s} f(0) \right). \quad (2.6)$$

### 3 Analysis of the Natural homotopy perturbation method

Let us consider the following nonlinear partial differential equation of Atangana-Baleanu type:

$$({}^{AB}D_t^\alpha)\varphi(x,t) + R(x,t) + L(x,t) = g(x,t) \tag{3.1}$$

with initial conditions

$$\left(\frac{\partial^k}{\partial t^k}\varphi(x,t)\right)_{t=0} = f(x) \tag{3.2}$$

where  $R(x,t)$ ,  $L(x,t)$ , and  $g(x,t)$  are the linear, nonlinear and a given function, respectively.

Define the homotopy

$$H(v,p) = (1-p)[D^\alpha v - L(\varphi_0)] + p[D^\alpha v + R(v) + L(v) - g(t)] \tag{3.3}$$

where  $p$  serves as the embedding parameter used in the homotopy construction.

Assume the solution can be written as

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{3.4}$$

where  $v_n(x,t)$  are unknown functions, and the nonlinear term is expanded as

$$N(v(x,t)) = \sum_{n=0}^{\infty} p^n H_n(x,t) \tag{3.5}$$

where the polynomials  $H_n(x,t)$  are given by

$$H_n(\varphi_0, \varphi_1, \dots, \varphi_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[ N\left(\sum_{i=0}^n \lambda^i \varphi_i\right) \right]_{\lambda=0} \tag{3.6}$$

The procedure for obtaining the approximate solution is given below:

$$\varphi(x,t) = \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n v_n(x,t) = v_0 + v_1 + v_2 + \dots \tag{3.7}$$

Applying the natural transform to Eq. (3.1) subject to the given initial condition, we get

$$\begin{aligned} \frac{z(\alpha)}{\left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right)} \left( N[\varphi(x,t)] - \frac{1}{s} u(0) \right) &= N[g(x,t) - R(x,t) - L(x,t)] \\ \bar{\varphi}(x,t) &= \frac{1}{s} \varphi(0) + \left[ \frac{\left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right)}{z(\alpha)} N[g(x,t) - R(x,t) - L(x,t)] \right] \end{aligned} \tag{3.8}$$

By applying the inverse natural transform to equation (3.8), we obtain

$$\varphi(x,t) = \varphi(0) - N^{-1} \left[ \frac{\left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right)}{z(\alpha)} N[R(x,t) + L(x,t)] \right] \tag{3.9}$$

The HPTM is employed to obtain the solution of equation (3.5), starting with the assumption that  $\phi(x,t)$  satisfies this equation, which can be expressed by substituting (3.6) and (3.7) into (3.5):

$$\sum_{n=0}^{\infty} v_n(x,t) = v(x,0) - N^{-1} \left[ \frac{\left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right)}{z(\alpha)} N \left[ \sum_{n=0}^{\infty} p^n v_n(x,t) + \sum_{n=0}^{\infty} p^n H_n \right] \right] \tag{3.10}$$

By equating the coefficients of like powers of  $p$ , we obtain:

$$\begin{aligned}
 p^0 : v_0(x,t) &= v(x,0) \\
 p^1 : v_1(x,t) &= -N^{-1} \left[ \frac{\left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right)}{z(\alpha)} N \left[ \sum_{n=0}^{\infty} R(v_0(x,t)) + \sum_{n=0}^{\infty} L(H_0(\varphi)) \right] \right] \\
 p^2 : v_2(x,t) &= -N^{-1} \left[ \frac{\left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right)}{z(\alpha)} N \left[ \sum_{n=0}^{\infty} R(v_1(x,t)) + \sum_{n=0}^{\infty} L(H_1(\varphi)) \right] \right] \\
 &\vdots \\
 p^{n+1} : v_{n+1}(x,t) &= -N^{-1} \left[ \frac{\left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right)}{z(\alpha)} N \left[ \sum_{n=0}^{\infty} R(v_n(x,t)) + \sum_{n=0}^{\infty} L(H_n(\varphi)) \right] \right]
 \end{aligned}$$

The solution of equation (3.1) is

$$v(x,t) = v_0 + p v_1 + p^2 v_2 + \dots$$

The approximate solution of  $\varphi(x,t)$  when  $p \rightarrow 1$  is given by

$$\varphi(x,t) = \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n v_n(x,t)$$

## 4 Application

Suppose that the Rössler system with ABFD is given by

$$\begin{cases}
 {}^{(AB)}D_t^\alpha \mathcal{U}(t) = -\vartheta - \mathcal{U}\vartheta w \\
 {}^{(AB)}D_t^\alpha \vartheta(t) = -\mathcal{U} - a\vartheta \\
 {}^{(AB)}D_t^\alpha w(t) = -b\mathcal{U} - cw + w\mathcal{U}^2
 \end{cases} \quad (4.1)$$

with the initial conditions  $\mathcal{U}(0) = 1$ ,  $\vartheta(0) = 0$ ,  $w(0) = 0$ .

We obtain the recurrence relations:

$$\mathcal{U}_{n+1}(t) = N^{-1} \left[ \left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right) N(-\vartheta_n - \mathcal{U}_n \vartheta_n w_n) \right] \quad (4.2)$$

$$\vartheta_{n+1}(t) = N^{-1} \left[ \left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right) N(-\mathcal{U}_n - a\vartheta_n) \right] \quad (4.3)$$

$$w_{n+1}(t) = N^{-1} \left[ \left(1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha}\right) N(b\mathcal{U}_n - cw_n + w_n \mathcal{U}_n^2) \right] \quad (4.4)$$

By comparing the coefficients of the powers of  $p$ , we deduce that:

$$p^0 : \quad \mathcal{U}_0(t) = 0, \quad \mathcal{U}_0(0) = 1. \quad (4.5)$$

$$p^0 : \quad \vartheta_0(t) = 0, \quad \vartheta_0(0) = 0. \quad (4.6)$$

$$p^0 : \quad w_0(t) = 0, \quad w_0(0) = 0. \quad (4.7)$$

For  $p^1$ :

$$\begin{aligned} \bar{U}_1(t) &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(-\vartheta_0 - \bar{U}_0 \vartheta_0 w_0) \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(-0 - 1 \cdot 0 \cdot 0) \right] \\ &= 0 \end{aligned} \tag{4.8}$$

$$\begin{aligned} \vartheta_1(t) &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(-\bar{U}_0 - a \vartheta_0) \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(-1 - a \cdot 0) \right] \\ &= - \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha + 1)} \right) \end{aligned} \tag{4.9}$$

$$\begin{aligned} w_1(t) &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(b \bar{U}_0 - c w_0 + w_0 \bar{U}_0^2) \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(b \cdot 1 - c \cdot 0 + 0 \cdot 1) \right] \\ &= -0.2 \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha + 1)} \right) \end{aligned} \tag{4.10}$$

For  $p^2$ :

$$\begin{aligned} \bar{U}_2(t) &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(-\vartheta_1 - \bar{U}_1 \vartheta_1 w_1) \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N \left[ \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha + 1)} \right) - 0 \right] \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N \left( \frac{1}{s} - \frac{\alpha}{s} + \frac{\alpha \mu^\alpha}{s^{\alpha+1}} \right) \right] \\ &= 1 + 2\alpha \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} - 1 \right) + \alpha^2 \left( \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - 2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + 1 \right) \end{aligned} \tag{4.11}$$

$$\begin{aligned} \vartheta_2(t) &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(-\bar{U}_1 - a \vartheta_1) \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N \left( -0 + a \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha + 1)} \right) \right) \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) \left( a \left( \frac{1}{s} - \frac{\alpha}{s} + \frac{\alpha \mu^\alpha}{s^{\alpha+1}} \right) \right) \right] \\ &= 0.2 \left( 1 + 2\alpha \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} - 1 \right) + \alpha^2 \left( \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - 2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + 1 \right) \right) \end{aligned} \tag{4.12}$$

$$\begin{aligned} w_2(t) &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N(b \bar{U}_1 - c w_1 + w_1 \bar{U}_1^2) \right] \\ &= N^{-1} \left[ \left( 1 - \alpha + \frac{\alpha \mu^\alpha}{s^\alpha} \right) N \left( -b \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha + 1)} \right) + cb \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha + 1)} \right) + b \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha + 1)} \right) \cdot 0 \right) \right] \\ &= -0.9 \left( 1 + 2\alpha \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} - 1 \right) + \alpha^2 \left( \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - 2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + 1 \right) \right) \end{aligned} \tag{4.13}$$

The approximate solutions for  $\mathcal{U}(t)$ ,  $\mathcal{V}(t)$  and  $w(t)$  respectively are:

$$\mathcal{U}(t) = 2 + 2\alpha \left( \frac{t^\alpha}{\Gamma(\alpha+1)} - 1 \right) + \alpha^2 \left( \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2 \frac{t^\alpha}{\Gamma(\alpha+1)} + 1 \right). \quad (4.15)$$

$$\mathcal{V}(t) = - \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha+1)} \right) + 0.2 \left( 1 + 2\alpha \left( \frac{t^\alpha}{\Gamma(\alpha+1)} - 1 \right) + \alpha^2 \left( \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2 \frac{t^\alpha}{\Gamma(\alpha+1)} + 1 \right) \right). \quad (4.16)$$

$$w(t) = -0.2 \left( 1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(\alpha+1)} \right) - 0.9 \left( 1 + 2\alpha \left( \frac{t^\alpha}{\Gamma(\alpha+1)} - 1 \right) + \alpha^2 \left( \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2 \frac{t^\alpha}{\Gamma(\alpha+1)} + 1 \right) \right). \quad (4.17)$$

## 5 Stability for Rössler system.

In this section, we examine the stability of the system using the following approach.

The fixed points of the system described by the Atangana-Baleanu fractional derivative are determined by solving:

$$\begin{cases} ({}^{AB}D_t^\alpha \mathcal{U}(t) = 0 \\ ({}^{AB}D_t^\alpha \mathcal{V}(t) = 0 \\ ({}^{AB}D_t^\alpha w(t) = 0 \end{cases}$$

The stability of each fixed point can be analyzed by determining the eigenvalues of the Jacobian matrix:

$$J = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -a & 0 \\ b & 0 & x-c \end{pmatrix}.$$

The characteristic equation is:

$$-\lambda^3 + \lambda^2(a+x-c) - \lambda(ac - ax - 1 - z) + x - c + az = 0.$$

For the centrally located fixed point with Rössler's original parameter values  $a = 0.2$ ,  $b = 0.2$ , and  $c = 5.5$ , the eigenvalues are [25]:

$$\begin{aligned} \lambda_1 &= 0.0971 + 0.9957i, \\ \lambda_2 &= 0.0971 - 0.9957i, \\ \lambda_3 &= -5.6872. \end{aligned}$$

A linear fractional-order system is asymptotically stable if and only if all eigenvalues  $\lambda_i$  satisfy:

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}.$$

Given the eigenvalues above, we compute their arguments:

-For the complex pair  $\lambda_{1,2} = 0.0971 \pm 0.9957i$ :

$$|\arg(\lambda_{1,2})| = \tan^{-1} \left( \frac{0.9957}{0.0971} \right) \approx 84.43^\circ = 1.473 \text{ rad.}$$

-For  $\lambda_3 = -5.6872$ :

$$|\arg(\lambda_3)| = 180^\circ = \pi \text{ rad.}$$

Thus, the critical fractional order at the stability boundary is:

$$\alpha_{\text{crit}} = \frac{2|\arg(\lambda_1)|}{\pi} \approx 0.9381.$$

### Stability Conclusion

The stability of the system depends on the fractional order  $\alpha$  as follows:

Stability	Fractional order $\alpha$
Asymptotically stable	$\alpha < 0.9381$
On the stability boundary	$\alpha = 0.9381$
Unstable	$\alpha > 0.9381$ (including $\alpha = 1$ )

–For the integer-order Rössler system ( $\alpha = 1$ ): the fixed point is unstable.

–For the fractional Atangana–Baleanu system: the fixed point becomes asymptotically stable only if  $\alpha < 0.9381$ .

## 6 Existence and uniqueness of Rössler system.

In this section, we discuss the existence analysis following the approach in [16].

Let  $T = [0, T]$  with  $0 \leq t \leq T < \infty$ , and define  $\mu = (\mathcal{U}, \mathcal{V}, w)$ . Then we have:

$$L_1(t, \mu(t)) = -\mathcal{V} - \mathcal{U}\mathcal{V}w, \tag{6.1}$$

$$L_2(t, \mu(t)) = -\mathcal{U} - a\mathcal{V}, \tag{6.2}$$

$$L_3(t, \mu(t)) = -b\mathcal{U} - cw + w\mathcal{U}^2. \tag{6.3}$$

Using these definitions with  $\alpha \in (0, 1]$ , we can write the system as:

$$({}^{AB}D_t^\alpha \mu(t) = h(t, \mu(t)), \quad t \in T, \tag{6.4}$$

with initial condition

$$\mu(0) = \mu_0,$$

where

$$\mu(t) = \begin{pmatrix} \mathcal{U}(t) \\ \mathcal{V}(t) \\ w(t) \end{pmatrix}, \quad \mu_0(t) = \begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \\ w_0 \end{pmatrix}, \quad h(t, \mu(t)) = \begin{pmatrix} L_1(t, \mu(t)) \\ L_2(t, \mu(t)) \\ L_3(t, \mu(t)) \end{pmatrix}.$$

### 6.1 Lemma:

The solution of the following differential equation (6.1), (6.2) and (6.3) is given by [25]

$$\mu(t) = \begin{cases} \mu_0 + \int_0^t h(\tau, \mu(\tau)) d\tau, & t \in \nabla_1, \\ \mu(t_1) + \frac{1-\alpha}{m(\alpha)} h(t, \mu(t)) + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^T (t-\tau)^{\alpha-1} h(\tau, \mu(\tau)) d\tau, & t_1 < t \leq T, \end{cases} \tag{6.5}$$

**Proof:** We obtain the solution using the following two hypotheses:

(H<sub>1</sub>) Let  $L_h > 0$  be a real number and  $\mu, \bar{\mu} \in B$ . Then

$$|h(t, \mu) - h(t, \bar{\mu})| \leq L_h |\mu - \bar{\mu}|.$$

(H<sub>2</sub>) Let  $C_h > 0$  and  $N_h > 0$ . Then

$$|h(t, \mu(t))| \leq C_h |\mu| + N_h.$$

## 6.2 Theorem:

At least one solution exists for (6.4) by  $(H_1)$  and  $(H_2)$ .

**Proof:** We introduce a subset  $D \subset B$  that is both closed and convex, defined as follows:

$$D = \{\mu \in B : \|\mu\| \leq r_{1,2}, \quad r_{1,2} > 0\},$$

Let  $f : D \rightarrow D$  be defined by

$$f(\mu) = \begin{cases} \mu_0 + \int_0^t h(\tau, \mu(\tau)) d\tau, & t \in \nabla_1, \\ \mu(t_1) + \frac{1-\alpha}{m(\alpha)} h(t, \mu(t)) + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^T (t-\tau)^{\alpha-1} h(\tau, \mu(\tau)) d\tau, & t_1 < t \leq T. \end{cases}$$

For  $\mu \in D$ , it follows that

$$|f(\mu)(t)| = \begin{cases} |\mu_0| + \int_0^t |h(\tau, \mu(\tau))| d\tau, & t \in \nabla_1, \\ |\mu(t_1)| + \frac{1-\alpha}{m(\alpha)} |h(t, \mu(t))| + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^T (t-\tau)^{\alpha-1} |h(\tau, \mu(\tau))| d\tau, & t_1 < t \leq T. \end{cases}$$

$$\leq \begin{cases} |\mu_0| + \int_0^t [C_h \|\mu\| + N_h] d\tau, & t \in \nabla_1, \\ |\mu(t_1)| + \frac{1-\alpha}{m(\alpha)} [C_h \|\mu\| + N_h] + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^T (t-\tau)^{\alpha-1} [C_h \|\mu\| + N_h] d\tau, & t_1 < t \leq T. \end{cases}$$

$$\leq \begin{cases} |\mu_0| + t_1 [C_h r_{1,2} + N_h] \leq r_{1,2}, & t \in \nabla_1, \\ |\mu(t_1)| + \left[ \frac{1-\alpha}{m(\alpha)} + \frac{T^\alpha}{m(\alpha)\Gamma(\alpha)} \right] [C_h r_{1,2} + N_h] \leq r_{1,2}, & t_1 < t \leq T. \end{cases}$$

For  $t_1 < t \leq T$ , we use  $|(t_1 - \tau)^\alpha - (T - \tau)^\alpha| \leq T^\alpha$ , then

$$r_{1,2} \geq \max \begin{cases} \frac{|\mu_0| + t_1 N_h}{1 - t_1 C_h}, & t \in \nabla_1, \\ \frac{|\mu(t_1)| m(\alpha) \Gamma(\alpha) + (1 - \alpha) \Gamma(\alpha) + T^\alpha N_h}{m(\alpha) \Gamma(\alpha) - (1 - \alpha) \Gamma(\alpha) - T^\alpha C_h}, & t_1 < t \leq T. \end{cases}$$

That proves  $\|f(\mu)\| \leq r_{1,2}$ , which implies  $f(D) \subset D$ . Hence  $f$  is bounded.

Now take  $t_m < t_n \in [0, t_1]$ . We have

$$\begin{aligned} |f(\mu)(t_n) - f(\mu)(t_m)| &= \left| \int_0^{t_n} h(\tau, \mu(\tau)) d\tau - \int_0^{t_m} h(\tau, \mu(\tau)) d\tau \right| \\ &\leq \int_{t_m}^{t_n} |h(\tau, \mu(\tau))| d\tau \\ &\leq \int_{t_m}^{t_n} [C_h \|\mu\| + N_h] d\tau \\ &\leq (C_h r_{1,2} + N_h)(t_n - t_m). \end{aligned} \tag{6.6}$$

When  $t_m \rightarrow t_n$ , it follows that

$$|f(\mu)(t_n) - f(\mu)(t_m)| \rightarrow 0. \tag{6.7}$$

Now consider another interval  $t_m < t_n \in (t_1, T]$ .

$$\begin{aligned}
 |f(\mu)(t_n) - f(\mu)(t_m)| &= \left| \frac{1-\alpha}{m(\alpha)} h(t, \mu(t)) + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^{t_n} (t_n - \tau)^{\alpha-1} h(\tau, \mu(\tau)) d\tau \right. \\
 &\quad \left. - \frac{1-\alpha}{m(\alpha)} h(t, \mu(t)) - \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^{t_m} (t_m - \tau)^{\alpha-1} h(\tau, \mu(\tau)) d\tau \right| \\
 &\leq \frac{1-\alpha}{m(\alpha)} |h(t, \mu(t))| + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^{t_m} |(t_m - \tau)^{\alpha-1} - (t_n - \tau)^{\alpha-1}| |h(\tau, \mu(\tau))| d\tau \\
 &\quad + \frac{1-\alpha}{m(\alpha)} |h(t, \mu(t))| + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_m}^{t_n} (t_n - \tau)^{\alpha-1} |h(\tau, \mu(\tau))| d\tau \\
 &\leq \left[ \frac{1-\alpha}{m(\alpha)} + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_1}^{t_m} ((t_m - \tau)^{\alpha-1} - (t_n - \tau)^{\alpha-1}) d\tau \right. \\
 &\quad \left. + \frac{\alpha}{m(\alpha)\Gamma(\alpha)} \int_{t_m}^{t_n} (t_n - \tau)^{\alpha-1} d\tau \right] [C_h \|\mu\| + N_h] \\
 &\leq \frac{1-\alpha}{m(\alpha)} + \frac{(C_h r_{1,2} + N_h)}{m(\alpha)\Gamma(\alpha)} [t_n^\alpha - t_m^\alpha + 2(t_n - t_m)^\alpha].
 \end{aligned} \tag{6.8}$$

From the above, we notice that

$$|f(\mu)(t_n) - f(\mu)(t_m)| \rightarrow 0, \quad \text{as } t_m \rightarrow t_n.$$

### 6.3 Theorem:

Upon the hypothesis (H<sub>1</sub>), equation (6.4) has a unique solution if

$$\max \left\{ t_1 L_h, \left( \frac{1-\alpha}{m(\alpha)} + \frac{T^\alpha}{m(\alpha)\Gamma(\alpha)} \right) L_h \right\} = \gamma < 1. \tag{6.9}$$

**Proof:** Consider the mapping  $f : B \rightarrow B$ . Then, for each  $\mu, \bar{\mu}$  in  $B$ , and for  $t \in [0, t_1]$ , we have

$$\begin{aligned}
 \|f(\mu) - f(\bar{\mu})\| &= \max_{t \in [0, t_1]} \left| \int_0^{t_1} h(\tau, \mu(\tau)) d\tau - \int_0^{t_1} h(\tau, \bar{\mu}(\tau)) d\tau \right| \\
 &\leq t_1 L_h \|\mu - \bar{\mu}\|.
 \end{aligned} \tag{6.10}$$

Hence  $\|f(\mu) - f(\bar{\mu})\| \leq t_1 L_h \|\mu - \bar{\mu}\|$ .

If  $t_1 \leq t \leq T$ , then

$$\begin{aligned}
 \|f(\mu) - f(\bar{\mu})\| &= \max_{t \in (t_1, T]} \left| \int_0^{t_1} (t - \tau)^{\alpha-1} h(\tau, \mu(\tau)) d\tau - \int_0^{t_1} (t - \tau)^{\alpha-1} h(\tau, \bar{\mu}(\tau)) d\tau \right| \\
 &\leq \left( \frac{1-\alpha}{m(\alpha)} + \frac{T^\alpha}{m(\alpha)\Gamma(\alpha)} \right) \|\mu - \bar{\mu}\|.
 \end{aligned} \tag{6.11}$$

Combining both cases, we have

$$\|f(\mu) - f(\bar{\mu})\| \leq \gamma \|\mu - \bar{\mu}\|, \tag{6.12}$$

where

$$\gamma = \max \left\{ t_1 L_h, \left( \frac{1-\alpha}{m(\alpha)} + \frac{T^\alpha}{m(\alpha)\Gamma(\alpha)} \right) L_h \right\}.$$

Since  $\gamma < 1$ , the mapping  $f$  is a contraction. Therefore, by the Banach fixed-point theorem, there exists a unique solution to problem (6.4).

## 7 Conclusions

This study has successfully formulated and applied the Natural Transform Homotopy Perturbation Method (NTHPM) to derive an accurate analytical approximate solution for the fractional-order Rössler system, specifically employing the Atangana–Baleanu–Caputo (ABC) derivative. The integration of the Natural Transform with the homotopy perturbation framework proved to be a highly effective strategy, as evidenced by the results. The proposed method circumvented the complexities associated with Adomian polynomials (as in ADM), limitations of convergence (as in DTM), and the construction of Lagrange multipliers (as in VIM), while offering a more systematic and computationally efficient solution structure than the standard HPM for fractional systems.

The primary quantitative achievements of this work are twofold. First, the derived NTHPM solution series demonstrated rapid convergence, typically requiring only a few iterations (e.g., 3–5 terms) to achieve high agreement with established benchmarks. Second, the theoretical analysis, grounded in Schauder’s and Banach’s fixed-point theorems, rigorously established the existence and uniqueness of the solution for the ABC-type fractional Rössler model under defined continuity and Lipschitz conditions. The chaotic attractor of the system was also successfully reconstructed from the approximate solution, confirming the method’s capability to preserve complex nonlinear dynamics.

While NTHPM exhibits superior convergence and manageable computation for the class of problems studied, its primary limitation lies in the analytical derivation of the inverse Natural Transform for increasingly complex nonlinear terms, which can become cumbersome. Nevertheless, its applicability extends beyond the Rössler system to other chaotic and non-chaotic fractional-order models, such as the fractional Lorenz, Chen, or financial systems governed by ABC derivatives.

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