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Schrodinger's Equation and the Infinite Potential Well Problem in Curved Spacetime

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Abstract: In this work, I will simply derive a general curved spacetime version of the ordinary quantum mechanical Schroding-er's equation and consider the eigenfunction/eigenvalue problem of the familiar one dimensional infinite potential well problem in a curved background.
Keywords: Schrodinger's equation, Curved spacetime, Infinite potential well

1 Introduction

It is well known to the physics community that the ordinary quantum mechanical Schrodinger's equation has a wide range of applicability ranging from that of the atomic, and subatomic domains up to the condensed state of matter, though, is restricted only to the non-relativistic region, and perhaps also, suffering from an oversimplified inclusion of the potential energy terms considered in an interaction particularly when the gravitational interactions are formally involved. For this reason, the need for a generalized version of the Schrodinger's equation in a general curved background that formally includes the gravitational effects with the quantum domain can really be considered as an important and valuable contribution to the ordinary theory, and perhaps stimulate a whole new formulation of the subject of quantum mechanics in curved spacetimes that reconsiders all the well-known applications and problems of the ordinary theory, but with the extra advantage of joining both effects together, which may further give rise to a one semi-grand (or probably some master) equation that formally deals with such a unification scheme, and hence, the idea of the present paper at hand.

2 Schrodinger's Equation in Curved Space time

Starting with the ordinary quantum mechanical Schrodinger's equation in the usual flat 3+1 dimensional background

$$\frac{\hbar^2}{2m}\partial_i\partial^i\psi + V\psi = i\hbar\,\partial_0\psi \tag{1}$$

(where summation over repeated indices is implied) we can , and by the use of the minimal coupling prescription [1], promote this equation to formally include the gravitational effects by simply replacing the partial derivatives with the corresponding covariant ones via the substitution

$$\frac{\hbar^2}{2m}\nabla_i\nabla^i\psi + V_{NG}\psi = i\hbar\nabla_0\psi \tag{2}$$

and using the formal definition of the generalized Laplacian (or more precisely the Laplace-Beltrami) operator for an n-dimensional space [2] we simply get for the corresponding time-dependent Schrodinger's equation in a generally curved background the formally looking equation

$$\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{|\tilde{g}|}}\partial_i\left(\sqrt{|\tilde{g}|}\tilde{g}^{ij}\partial_j\psi\right)\right) + V_{NG}\psi = i\hbar\,\partial_0\psi \tag{3}$$

where $\tilde{g}_{ij}$ here represents the spatial sector of the spacetime metric, and V_{NG} corresponds to all the non-gravitational potential energy terms involved in the interaction, and since we are assumed to be acting on a scalar wave function ψ the last term has been simply switched from a covariant derivative to the usual partial derivative.

This is by far all what can be said about the simply derived Schrodinger's-Gravitational equation and will come now to consider one of its applications in a 1+1 dimensional spacetime example.

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3 The Infinite Potential Well Problem in Curved Spacetime

Taking a closer look at eq.(3) in a 1+1 dimensional space time, and due to that fact that any two-dimensional Riemannian manifold is necessarily conformally flat [3], the metric tensor used in this case has to be of the form

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \tag{4}$$

where $\Omega(x)$ is some continuous, non-vanishing, finite, smooth, and strictly positive function, and $\eta_{\mu\nu}$ is the two dimensional flat metric tensor [4, 5].
We therefore have

$$\tilde{g}_{ij} = -\Omega^2 \tag{5}$$

and plugging this into eq.(3) (with the choice of a *static* metric) gives for the corresponding stationary state equation

$$\frac{\hbar^2}{2m}\left(-\frac{1}{\Omega^2}\partial_x^2 + \frac{\Omega'}{\Omega^3}\partial_x\right)\varphi + V_{NG}\varphi = E\varphi \tag{6}$$

which although not in a self-ajoint form plays the role of the corresponding time independent Schrodinger's equation appropriate for any 1+1 dimensional quantum system in a statically curved background.

As a simple illustrative example, I will consider the infinite potential well problem for a particle confined in the region between $x = 0$, and $x = L$ with the boundary conditions $\varphi(0) = \varphi(L) = 0$ with the reasonable choice of $\Omega = e^{\alpha x}$ (for a certain parameter α) and aim to find the corresponding eigenvalues and eigenfunctions.

In this case, the previously given time independent Schrodinger's equation reduces to the form

$$\frac{d^2\varphi}{dx^2} - \alpha\frac{d\varphi}{dx} + \frac{2mE}{\hbar^2}e^{2\alpha x}\varphi = 0 \tag{7}$$

which with the given boundary conditions represent a Sturm-Liouville system provided we use $I = e^{-\alpha x}$ as an integrating factor and $w(x) = e^{\alpha x}$ as the corresponding weight function [6].
The general solution of eq.(7) (resorting to Mathematica) is

$$\varphi = C_1 \cos\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}e^{\alpha x}\right) + C_2 \sin\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}e^{\alpha x}\right) \tag{8}$$

and imposing now the given boundary conditions gives us the following system of equations for the general coefficients C_1, and C_2

$$\begin{pmatrix} \cos\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}\right) & \sin\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}\right) \\ \cos\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}e^{\alpha L}\right) & \sin\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}e^{\alpha L}\right) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

and to avoid trivial solutions we must set

$$\begin{vmatrix} \cos\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}\right) & \sin\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}\right) \\ \cos\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}e^{\alpha L}\right) & \sin\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}e^{\alpha L}\right) \end{vmatrix}$$

$$= \sin\left(\sqrt{\frac{2mE}{\hbar^2\alpha^2}}(e^{\alpha L} - 1)\right) = 0 \tag{10}$$

from which we get

$$\sqrt{\frac{2mE}{\hbar^2\alpha^2}}(e^{\alpha L} - 1) = n\pi \tag{11}$$

or

$$E_n = \frac{n^2\pi^2\hbar^2\alpha^2}{2m(e^{\alpha L} - 1)^2} \tag{12}$$

(for $n = 1,2, \ldots$) representing the energy spectrum of the infinite well problem in a conformally flat background.
Therefore, the general solution of the given problem (with the given set of eigenvalues) can be written as

$$\varphi_n = C_1 \cos\left(\frac{n\pi e^{\alpha x}}{e^{\alpha L} - 1}\right) + C_2 \sin\left(\frac{n\pi e^{\alpha x}}{e^{\alpha L} - 1}\right) \tag{13}$$

If we further now choose

$$C_1 = N \sin\left(\frac{n\pi e^{\alpha L}}{e^{\alpha L} - 1}\right) \tag{14}$$

and

$$C_2 = -N \cos\left(\frac{n\pi e^{\alpha L}}{e^{\alpha L} - 1}\right) \tag{15}$$

(for some normalization constant N) our solution becomes

$$\varphi_n = N \sin\left(\frac{n\pi(e^{\alpha x} - e^{\alpha L})}{1 - e^{\alpha L}}\right) \tag{16}$$

which clearly satisfies the given boundary conditions and can be normalized to unity according to the orthonormality condition

$$N^2 \int_0^L e^{\alpha x}\left(\sin\left(\frac{n\pi(e^{\alpha x} - e^{\alpha L})}{1 - e^{\alpha L}}\right)\right)^2 dx = 1 \tag{17}$$

from which we get

$$N = \sqrt{\frac{2\alpha}{e^{\alpha L} - 1}} \tag{18}$$

Therefore, the complete set of orthonormalized solutions of the infinite potential well problem together with the given set of energy eigenvalues in a statically and conformally flat background can finally be written as

$$\varphi_n = \sqrt{\frac{2\alpha}{e^{\alpha L} - 1}}\sin\left(\frac{n\pi(e^{\alpha x} - e^{\alpha L})}{1 - e^{\alpha L}}\right) \tag{19}$$

It's worth mentioning at this stage that in the $\alpha \longrightarrow 0$ limit we get the corresponding flat spacetime quantities, i.e

$$\lim_{\alpha\to 0} E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{20}$$

and

$$\lim_{\alpha\to 0} \varphi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \tag{21}$$

which exactly match the well-known energy spectrum and eigenfunctions of the corresponding infinite potential well quantities in the absence of gravity (as can be easily found in any reference book on ordinary quantum mechanics [7]).

4 Conclusions and Further Remarks

As we can see from the previous analysis, the Schrodinger's equation with the gravitational effect has been simply derived via the minimal coupling prescription, and a consideration of the special case of a two-dimensional quantum mechanical problem has been investigated by the use of the infinite potential well example that has been considered for the case of a static and conformally flat metric where a calculation of the eigenfunctions and corresponding eigenvalues were found and perfectly matched the flat background results in the no gravity limit. And as a final remark, I would like to add that a further consideration of other ordinary quantum mechanical problems such as the 1D harmonic oscillator (or perhaps the Hydrogen atom in a 3D problem) or any of the well-known problems of the ordinary theory may be similarly motivated from the previous analysis in some future work.

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