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Generalized Fractional Spline Method with Stability for Fractional Differential equations

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Abstract: In this paper, Fractional initial value problems $y^{(3\alpha)}(x) = f(x,y)$is solved base on the proposed fractional spline interpolation for the case $\alpha,\ 0 < \alpha < 1$, relied on class C^q-splines as a way to approximate the exact solution of such problems. In addition, this fractional spline interpolation contains β parameter, where $\beta \in (0,1]$ and β is taken to be equal to one to test the stability analysis of the method.

Keywords: Fractional spline function, Initial value Problem , Caputo derivative, Stability Analysis.

1 Introduction

Fractional calculus is an important subject that can be described as (Complex order or calculus of integral and derivatives of any arbitrary real). Fractional calculus has achieved too importance and considerable popularity in three past decades. This is referred entirely to have showed applications in numerous seemingly divers and widespread fields of engineering and science. Indeed, differential and integral equations, and other different problems to be solved, it requires to provide many potentially useful tool that involves generalizations in one and more variables and special functions of mathematical physics as well as their extensions.[3,19]
Spline functions are a great tool that can be used for the numerical approximation of functions on the one hand. On other hand, they can recommend new, rewarding problems on the other and challenging. Lacunary interpolation by spline become visible whenever observation points out irregular or scattered in formation that is related to a function and it is derivatives.[2,13]
Create a new fractional spline interpolation based on a new class of C^q-spline interpolation and use it to find a numerical solution of FIVP :

$$\begin{aligned} &If\ 3\alpha > 1\\ &\quad y^{(3\alpha)}(x) = f(x,y),\ y(x_0) = y_0,\ y'(x_0) = y'_0,\ x \in [0,1].\\ &And\ if\ 3\alpha \le 1\\ &\quad y^{(3\alpha)}(x) = f(x,y),\ y(x_0) = y_0,\ x \in [0,1]. \end{aligned} \tag{1}$$

2 Construction of The General Fractional Spline Function

The main aim of this section is Construct general fractional spline function $s(x) \in C^q[0,1]$ interpolating to a function defined on [a, b],such as Abass, Faraidun and Rostam, Sallam and Karaballi, and Sallam and Anwar , [1,12,14] respectively, satisfies (1) at the knots $x_i = ih,\ i = 0,1,...,N$ and $h = \frac{1}{N}$
C^m- spline function is defined as follows
$S^q_{n,7\alpha} = \left[s(x); s \in C^q[0,1], s \in P_{7\alpha}(x), x \in I = [x_i, x_{i+1}]\right]$
where $P_{7\alpha}(x)$ is the set of all fractional polynomials of degree at most 7α

Definition 1.[9,8] *The Caputo fractional derivative of order $\alpha > 0$ is defined by*

$$D^{\alpha}_a f(x) = \begin{bmatrix} \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(s)}{(x-s)^{(\alpha+1-n)}}ds & for n-1 < \alpha < n, n \in N \\ \frac{d^{(n)}}{dx^{(n)}} f(x) & for \alpha = n, n \in N \end{bmatrix} \tag{2}$$

3 Existences and Uniqueness

Theorem 1.*Given the real numbers, $s^{(3\alpha)}_i, i = 0,1,...,N, s_0, s^{(\alpha)}_0 and s^{(5\alpha)}_0$ then there exist a*

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unique cubic spline $s \in S^q_{n,7\alpha}$ such that

$$\left.\begin{array}{l} s_0 = f_0 \\ s_0^{(\alpha)} = f_0^{(\alpha)} \\ s_i^{(3\alpha)} = f_i^{(3\alpha)} \\ and \\ s_0^{(5\alpha)} = f_0^{(5\alpha)} \end{array}\right\} for\ i = 0,1,...,N \tag{3}$$

*Proof.*The unique spline function $s(x) \in S^q_{n,7\alpha}$ in $[x_i, x_{i+1}]$ where $\beta \in (0,1)$ will be

$$s(x) = A(x)s_i + B(x)h^{\alpha}s_i^{(\alpha)} + h^{3\alpha}[C(x)s_i^{(3\alpha)} + D(x)s_{i+\beta}^{(3\alpha)}] + E(x)h^{5\alpha}s_i^{(5\alpha)} \tag{4}$$

where

$$A(x) = 1,$$
$$B(x) = \frac{1}{\Gamma(\alpha+1)}x^{(\alpha)},$$
$$C(x) = \frac{1}{\Gamma(3\alpha+1)}x^{(3\alpha)} - \frac{\Gamma(4\alpha+1)}{\beta^{4\alpha}\Gamma(7\alpha+1)}x^{(7\alpha)},$$
$$D(x) = \frac{\Gamma(4\alpha+1)}{\beta^{4\alpha}\Gamma(7\alpha+1)}x^{(7\alpha)},$$
and
$$E(x) = \frac{1}{\Gamma(5\alpha+1)}x^{(5\alpha)} - \frac{\Gamma(4\alpha+1)}{\beta^{2\alpha}\Gamma(2\alpha+1)\Gamma(7\alpha+1)}x^{(7\alpha)}. \tag{5}$$

We can express any $p(t)$ in $[0,1]$ in the following form:

$$p(t) = p_0A_0(t) + p_0^{(\alpha)}B(t) + p_0^{(3\alpha)}C(t) + p_\beta^{(3\alpha)}D(t) + p_0^{(5\alpha)}E(t),$$

to determine A, B, C, D and E we write the above equality for
$p(t) = 1, t^{\alpha}, t^{3\alpha}, t^{5\alpha}, t^{7\alpha}$ we get

$$\begin{array}{l} A = 1, \\ B = \frac{1}{\Gamma(\alpha+1)}t^{\alpha}, \\ \Gamma(3\alpha+1)C + \Gamma(3\alpha+1)D = t^{3\alpha}, \\ \frac{\Gamma(5\alpha+1)\beta^{2\alpha}}{\Gamma(2\alpha+1)}D + \Gamma(5\alpha+1)E = t^{5\alpha}, \\ \frac{\Gamma(7\alpha+1)\beta^{4\alpha}}{\Gamma(4\alpha+1)}D = t^{7\alpha} \end{array},$$

solving these we obtained the equation (5), and $x = x_i + t\beta h, 0 \le t \le 1$ with a similar expression for $s(x) in [x_{i-1}, x_i]$.
Since $s \in C^q[0,1]$, and $S(x_i^+) = S(x_i^-)$ to $S^{(5\alpha)}(x_i^+) = S^{(5\alpha)}(x_i^-)$ respectively, for $i = 0,1,...,N$, leads to the following linear system of equations:

$$s_i = s_{i-1} + \frac{1}{\Gamma(\alpha+1)}\beta^{\alpha}h^{\alpha}s_{i-1}^{(\alpha)} + h^{3\alpha}\beta^{3\alpha}[\frac{[\Gamma(7\alpha+1) - \Gamma(3\alpha+1)\Gamma(4\alpha+1)]}{\Gamma(3\alpha+1)\Gamma(7\alpha+1)}s_{i-1}^{(3\alpha)} + \frac{\Gamma(4\alpha+1)}{\Gamma(7\alpha+1)}s_{i-1+\beta}^{(3\alpha)}] + \frac{\Gamma(2\alpha+1)\Gamma(7\alpha+1)}{\Gamma(4\alpha+1)\Gamma(5\alpha+1)}\beta^{5\alpha}h^{5\alpha}s_{i-1}^{(5\alpha)}. \tag{6}$$

$$h^{\alpha}s_i^{(\alpha)} = h^{\alpha}s_{i-1}^{(\alpha)} + \left[\frac{\Gamma(6\alpha+1) - \Gamma(2\alpha+1)\Gamma(5\alpha+1)}{\Gamma(2\alpha+1)\Gamma(6\alpha+1)}\right] \beta^{2\alpha}h^{3\alpha}s_{i-1}^{(3\alpha)} + \frac{\Gamma(4\alpha+1)}{\Gamma(7\alpha+1)}\beta^{2\alpha}h^{3\alpha}s_{i-1+\beta}^{(3\alpha)} + \frac{1}{\Gamma(2\alpha+1)\prod_{k=4}^{6}\Gamma(k\alpha+1)}\left[\Gamma(2\alpha+1)\prod_{k=1}^{2}\Gamma(6\alpha+1) - \Gamma(5\alpha+1)\prod_{k=1}^{2}\Gamma(4\alpha+1)\right]\beta^{4\alpha}h^{5\alpha}s_{i-1}^{(5\alpha)} \tag{7}$$

$$s_i^{(3\alpha)} = f(x_i, s_i), s_i^{(3\alpha)} = s_{i-1}^{(3\alpha)} \tag{8}$$

and

$$h^{5\alpha}s_i^{(5\alpha)} = \frac{-\Gamma(4\alpha+1)}{\beta^{2\alpha}\Gamma(2\alpha+1)}h^{3\alpha}s_{i-1}^{(3\alpha)} + \frac{-\Gamma(4\alpha+1)}{\beta^{2\alpha}\Gamma(2\alpha+1)}h^{3\alpha}s_{i-1+\beta}^{(3\alpha)} + \left[\frac{\prod_{k=1}^{2} - \Gamma(4\alpha+1)}{\prod_{k=1}^{2}\Gamma(2\alpha+1)}\right]^{5\alpha} s_{i-1}^{(5\alpha)},\ i = 1,2,...,N \tag{9}$$

and hence $s(x)$ is uniquely determined in $[0,1]$.

*Remark.*If $\beta = 1$ in the theorem 1 then the linear system (6)-(9) will be written as:

$$s_i = s_{i-1} + \frac{1}{\Gamma(\alpha+1)}h^{\alpha}s_{i-1}^{(\alpha)} + h^{3\alpha}[\frac{[\Gamma(7\alpha+1) - \Gamma(3\alpha+1)\Gamma(4\alpha+1)]}{\Gamma(3\alpha+1)\Gamma(7\alpha+1)}s_{i-1}^{(3\alpha)} + \frac{\Gamma(4\alpha+1)}{\Gamma(7\alpha+1)}s_i^{(3\alpha)}] + \frac{\Gamma(2\alpha+1)\Gamma(7\alpha+1)}{\Gamma(4\alpha+1)\Gamma(5\alpha+1)}h^{5\alpha}s_{i-1}^{(5\alpha)}, \tag{10}$$

$$h^{\alpha}s_i^{(\alpha)} = h^{\alpha}s_{i-1}^{(\alpha)} + \left[\frac{\Gamma(6\alpha+1) - \Gamma(2\alpha+1)\Gamma(5\alpha+1)}{\Gamma(2\alpha+1)\Gamma(6\alpha+1)}\right]h^{3\alpha}s_{i-1}^{(3\alpha)} + \frac{\Gamma(4\alpha+1)}{\Gamma(7\alpha+1)}h^{3\alpha}s_i^{(3\alpha)} + \left[\frac{\Gamma(2\alpha+1)\prod_{k=1}^{2}\Gamma(6\alpha+1) - \Gamma(5\alpha+1)\prod_{k=1}^{2}\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\prod_{k=4}^{6}\Gamma(k\alpha+1)}\right] h^{5\alpha}s_{i-1}^{(5\alpha)}, \tag{11}$$

$$s_i^{(3\alpha)} = f(x_i, s_i), s_i^{(3\alpha)} = s_{i-}^{(3\alpha)}, \tag{12}$$

and

$$h^{5\alpha}s_i^{(5\alpha)} = \frac{-\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)}h^{3\alpha}s_{i-1}^{(3\alpha)} + \frac{-\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)}h^{3\alpha}s_i^{(3\alpha)} + \Big[\frac{\prod_{k=1}^{2} -\Gamma(4\alpha+1)}{\prod_{k=1}^{2}\Gamma(2\alpha+1)}\Big]h^{5\alpha}s_{i-1}^{(5\alpha)}, i=1,...,N. \tag{13}$$

Theorem 2.*[4,7,10,?] Let $g \in C^{2m}[0,h]$ be given. let P_{2m-1} be the unique Hermite interpolation polynomial of degree $2m-1$ that matches g and its first $m-1$ derivatives $g^{(r)}$ at 0 and h. then*

$$\mid e^{(r)}(x) \mid \leq \frac{h^r\Big[x(h-x)\Big]^{m-r}G}{r!(2m-1)!},\ r=0(1)m,\ 0\leq x\leq h \tag{14}$$

where

$$\begin{aligned} &\mid e^{(r)}(x) \mid = \mid g^{(r)}(x) - P_{2m-1}^{(r)}(x) \mid \text{ and} \\ &G = \max_{0\leq x\leq h} \mid g^{(2m)}(x) \mid \end{aligned} \tag{15}$$

the bounds in (14) are best possible for $r=0$ only.

Theorem 3.*suppose that s(x) be the fractional spline defined in theorem 1 ,$f^{\alpha}$and$f^{3\alpha} \in C^q[0,1]$and that $f^p(0)=0, p=1,2$ then for any $x\in[0,1]$ we have*

$$\mid s(x) - f(x) \mid \leq \frac{h^2\Gamma(\alpha+1)}{4}f^{(2+\alpha)} \tag{16}$$

*Proof.*because $s^{\alpha}(x)$ is hermite interpolation polynomial of degree 3 matching $f^{\alpha}(x)$, $atx = x_i, x_{i+1}$ so for any $x \in [x_i, x_{i+1}]$ we have using (14) with $m=1, g=f^{(\alpha)}, andp = s^{(\alpha)}$

$$\mid s^{(\alpha)} - f^{(\alpha)} \mid \leq \frac{h^2}{(1)!(4)}D^2D^{(\alpha)}f$$

also if we put $g = f^{(3\alpha)}, andp = s^{(3\alpha)}$ we get

$$\mid s^{(3\alpha)} - f^{(3\alpha)} \mid \leq \frac{h^2}{(1)!(4)}D^2D^{(3\alpha)}f$$

Then we can get

$$\mid I_{0|x}^{\alpha}\Big[s^{(\alpha)} - f^{(\alpha)}\Big] \mid \leq I_{0|x}^{\alpha}\Big[\frac{h^2}{(1)!(4)}D^2D^{(\alpha)}f\Big]$$

$$\mid s(x) - s(0) - f(x) + f(0) \mid \leq \Gamma(\alpha+1)x^{\alpha}\Big[\frac{h^2}{4}D^2D^{(\alpha)}f\Big]$$

since $s(0)=f(0)$ and $x\in[0,1]$ then the last equation becomes

$$\mid s(x) - f(x) \mid \leq \frac{\Gamma(\alpha+1)h^2}{(4)}D^2D^{(\alpha)}f$$

and since $f^p(0)=0, p=1,2,$ following [10] we have $D^2D^{(\alpha)}f = D^{(2+\alpha)}f = f^{(2+\alpha)}$
which direct to

$$\mid s(x) - f(x) \mid \leq \frac{\Gamma(\alpha+1)h^2}{(4)}f^{(2+\alpha)}.$$

4 Stability Analysis

The presented method (10),(11) and (13) is under consideration for it's stability analysis and executing the method to the test equation

$$y^{(3\alpha)}(x) = -\lambda^{(3\alpha)}y(x), \lambda\in R \tag{17}$$

with the initial condition of (1) setting $\lambda h = Z$, using (17) to obtain

$$\begin{aligned} s_i =& \frac{1}{\Gamma(3\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}]}\Big[\Gamma(3\alpha+1) \\ &\Gamma(7\alpha+1)+[\Gamma(7\alpha+1)-\Gamma(3\alpha+1)\Gamma(4\alpha+1)]\lambda^{3\alpha}h^{3\alpha}\Big]s_{i-1} \\ &+\frac{\Gamma(7\alpha+1)}{\Gamma(\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}}h^{\alpha}s_{i-1}^{\alpha} \\ &+\Big[\frac{\Gamma(2\alpha+1)\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\Gamma(5\alpha+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}}\Big]h^{5\alpha}s_{i-1}^{5\alpha}. \end{aligned}$$

$$\begin{aligned} h^{\alpha}s_i^{\alpha} =& \frac{1}{\prod_{k=2}^{3}\Gamma(k\alpha+1)\Gamma(6\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}]} \\ &\Big[\Gamma(3\alpha+1)\prod_{k=6}^{7}\Gamma(k\alpha+1)\lambda^{3\alpha}h^{3\alpha}+[\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(7\alpha+1) \\ &-\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)]\lambda^{6\alpha}h^{6\alpha}\Big]s_{i-1} \\ &+\frac{1}{\Gamma(\alpha+1)\Gamma(6\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}]}\Big[\Gamma(\alpha+1) \\ &\prod_{k=6}^{7}\Gamma(k\alpha+1)+[\Gamma(4\alpha+1)\Gamma(7\alpha+1)-\Gamma(\alpha+1)\Gamma(4\alpha+1) \\ &\Gamma(6\alpha+1)]\lambda^{3\alpha}h^{3\alpha}\Big]h^{\alpha}s_{i-1}^{\alpha} \\ &+\Big[\frac{1}{\Gamma(2\alpha+1)\prod_{k=4}^{6}\Gamma(k\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}}\Big] \\ &\Big[\Gamma(2\alpha+1)\prod_{k=5}^{7}\Gamma(k\alpha+1)-(\prod_{k=1}^{2}\Gamma(4\alpha+1))\Gamma(5\alpha+1)\Gamma(7\alpha+1)] \\ &+[\Gamma(2\alpha+1)(\prod_{k=1}^{2}\Gamma(4\alpha+1))\Gamma(7\alpha+1)-\Gamma(2\alpha+1)\prod_{k=4}^{6}\Gamma(k\alpha+1)] \\ &\lambda^{3\alpha}h^{3\alpha}\Big]h^{5\alpha}s_{i-1}^{5\alpha}. \end{aligned}$$

$$\begin{aligned} h^{5\alpha}s_i^{5\alpha} =& \frac{\Gamma(4\alpha+1)\Gamma(7\alpha+1)\lambda^{6\alpha}h^{6\alpha}}{\prod_{k=2}^{3}\Gamma(k\alpha+1)\Gamma(6\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}}s_{i-1} \\ &+\frac{\Gamma(4\alpha+1)\Gamma(7\alpha+1)}{\prod_{k=1}^{2}\Gamma(k\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}]}h^{\alpha}s_{i-1}^{\alpha} \\ &+[\frac{1}{\prod_{k=1}^{2}\Gamma(k\alpha+1)\Gamma(5\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}h^{3\alpha}]} \\ &[[\prod_{k=1}^{2}\Gamma(k\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)-(\prod_{k=4}^{5}\Gamma(k\alpha+1))\Gamma(7\alpha+1)] \\ &+[\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(7\alpha+1)-\prod_{k=1}^{2}\Gamma(\alpha+1)\Gamma(4\alpha+1) \\ &\Gamma(5\alpha+1)]\lambda^{3\alpha}h^{3\alpha}\Big]h^{5\alpha}s_{i-1}^{5\alpha}. \end{aligned}$$

or in matrix notation $S_i = BS_{i-1};\ i=1,...,N$

$$S_i = \begin{bmatrix} s_i \\ s_i^{\alpha} \\ s_i^{5\alpha} \end{bmatrix},\quad S_{i-1} = \begin{bmatrix} s_{i-1} \\ s_{i-1}^{\alpha} \\ s_{i-1}^{5\alpha} \end{bmatrix}\quad and\quad M = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where

$$b_{11}=\frac{1}{\Gamma(3\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}]}\Big[\Gamma(3\alpha+1)\Gamma(7\alpha+1)+[\Gamma(7\alpha+1)-\Gamma(3\alpha+1)\Gamma(4\alpha+1)]Z^{3\alpha}\Big]$$

$$b_{12}=\frac{\Gamma(7\alpha+1)}{\Gamma(\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}}h^{\alpha}$$

$$b_{13}=\Big[\frac{1}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}}\Big]\Big[\Gamma(2\alpha+1)\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\Gamma(5\alpha+1)\Big]h^{5\alpha}$$

$$b_{21}=\frac{1}{\prod_{k=2}^{3}\Gamma(k\alpha+1)\Gamma(6\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}]}\Big[\Gamma(3\alpha+1)\prod_{k=6}^{7}\Gamma(k\alpha+1)Z^{3\alpha}+[\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(7\alpha+1)-\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)]Z^{6\alpha}]\Big]$$

$$b_{22}=\frac{1}{\Gamma(\alpha+1)\Gamma(6\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}]}\Big[\Gamma(\alpha+1)\prod_{k=6}^{7}\Gamma(k\alpha+1)+[\Gamma(4\alpha+1)\Gamma(7\alpha+1)-\Gamma(\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)]Z^{3\alpha}\Big]h^{\alpha}$$

$$b_{23}=\frac{1}{\Gamma(2\alpha+1)\prod_{k=4}^{6}\Gamma(k\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}}\Big[\Gamma(2\alpha+1)\prod_{k=5}^{7}\Gamma(k\alpha+1)-(\prod_{k=1}^{2}\Gamma(4\alpha+1))\Gamma(5\alpha+1)\Gamma(7\alpha+1)]+[\Gamma(2\alpha+1)(\prod_{k=1}^{2}\Gamma(4\alpha+1))\Gamma(7\alpha+1)-\Gamma(2\alpha+1)\prod_{k=4}^{6}\Gamma(k\alpha+1)]Z^{3\alpha}\Big]h^{5\alpha}$$

$$b_{31}=\frac{\Gamma(4\alpha+1)\Gamma(7\alpha+1)Z^{6\alpha}}{\prod_{k=2}^{3}\Gamma(k\alpha+1)\Gamma(6\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)\lambda^{3\alpha}}$$

$$b_{32}=\frac{\Gamma(4\alpha+1)\Gamma(7\alpha+1)}{\prod_{k=1}^{2}\Gamma(k\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}]}h^{\alpha}$$

$$b_{33}=[\frac{1}{\prod_{k=1}^{2}\Gamma(k\alpha+1)\Gamma(5\alpha+1)[\Gamma(7\alpha+1)-\Gamma(4\alpha+1)Z^{3\alpha}]}\Big[[\prod_{k=1}^{2}\Gamma(k\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)-(\prod_{k=4}^{5}\Gamma(k\alpha+1))\Gamma(7\alpha+1)]+[\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(7\alpha+1)-\prod_{k=1}^{2}\Gamma(\alpha+1)\Gamma(4\alpha+1)\Gamma(5\alpha+1)]Z^{3\alpha}\Big]h^{5\alpha}$$

The characteristic equation can be expressed as below:

$$r^3-(trB)r^2+(B_1+B_2+B_3)r-det(B)=0 \qquad (18)$$

The cofactors of the diagonal elements can be denoted by B_1,B_2,B_3 respectively where r is the eigenvalue ,In addition, (10),(11) and (13) can define the cubic spline approximation method that have interval of periodicity $(0,Z^{3\alpha})$, where the eigenvalues $r_{1,2}$ of the matrix M are complex conjugate and $\mid r_3 \mid\leq 1$. If all complex eigenvalues have negative real parts, the characteristic equation will be stable as the characteristic polynomial (18) says.[1,5,17]

5 Algorithm

1. $$s(x)=A(x)s_i+B(x)h^{\alpha}s_i^{(\alpha)}+h^{3\alpha}[C(x)s_i^{(3\alpha)}+D(x)s_{i+\beta}^{(3\alpha)}]+E(x)h^{5\alpha}s_i^{(5\alpha)}$$
   was constructed and derived in $C^q[a,b]$, $0<\alpha<1$ and $\beta\in(0,1]$.

2. $\beta=1$ was put in equation (6), (7), and (9) in theorem (1) and equation (10), (11) and (13) were achieved.

3. The error bounded in theorem (3) was discussed.

4. The stability analysis of equation type $y^{(3\alpha)}(x)=-\lambda^{(3\alpha)}y(x),\lambda\in R,$ was demonstrated.

6 Numerical Examples

In this section, some numerical results are shown to illustrate the presented method and comparison between them depending on the value of h and α.

Example 1.[18] Consider the following nonlinear fractional differential equation

$$D^{\alpha}y(x)+xy^2(x)=x^6+\frac{\Gamma(3.5)}{\Gamma(3.5-\alpha)}x^{2.5-\alpha},$$
$$0<\alpha<1 \textit{ with initial condition } y(0)=0,$$
$$\textit{and the exact solution is } y(x)=x^{5/2}.$$

The absolute errors $|e^{m\alpha}| = |D^{(m\alpha)}s(x) - D^{(m\alpha)}y(x)|$ where m=0,1,...,6.

Table 1: Absolute Error of Example(1) for $\alpha = \frac{1}{10}$

h	$\lvert e \rvert$	$\lvert e^{\frac{1}{2}} \rvert$	$\lvert e^{\frac{3}{2}} \rvert$	Exact solution	Approximation solution
0.4	0.0749	0.0787	0.1049	0.0179	0.0608
0.6	0.2223	0.2337	0.2712	0.1012	0.1325
0.8	0.4648	0.4885	0.5229	0.2789	0.2096

Table 2: Absolute Error of Example(1) for $\alpha = \frac{1}{6}$

h	$\lvert e \rvert$	$\lvert e^{\frac{1}{2}} \rvert$	$\lvert e^{\frac{3}{2}} \rvert$	Exact solution	Approximation solution
0.4	0.0589	0.0634	0.1329	0.0179	0.0455
0.6	0.1668	0.1798	0.2991	0.1012	0.0786
0.8	0.3395	0.3660	0.5317	0.2789	0.0871

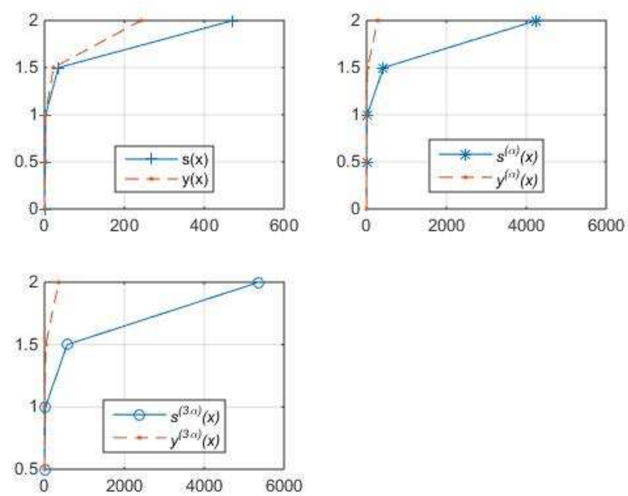

Fig. 1: Exact and Absolute Error of Example 2 when $\alpha = \frac{1}{10}$.

Example 2.[6] Consider the following fractional differential equation

$$D^{(\alpha)}y(x) = \frac{40320}{\Gamma(9-\alpha)}x^{8-\alpha} - 3\frac{\Gamma(5-\frac{\alpha}{2})}{\Gamma(5+\frac{\alpha}{2})}x^{4-\frac{\alpha}{2}} + 2.25\Gamma(\alpha+1)$$
$$+(1.5x^{\frac{\alpha}{2}} - x^4)^3 - (y(x))^{\frac{3}{2}}$$

$0 < \alpha < 1$ with initial condition $y(0) = 0,\ y^{'}(0) = 0$. The exact solution is $y(x) = x^8 - x^{4+\frac{\alpha}{2}} + 2.25x^{\alpha}$.

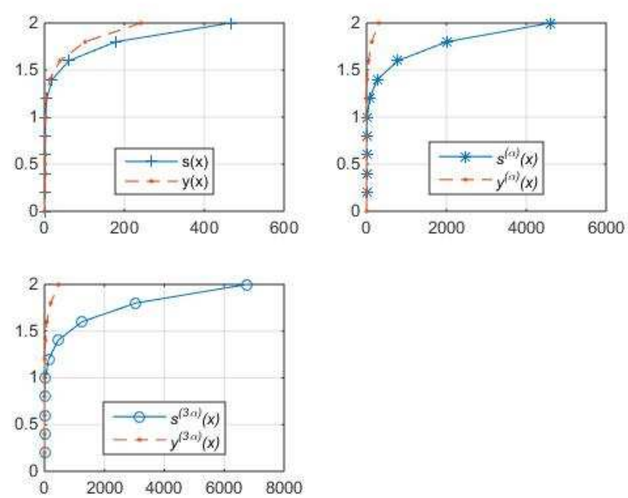

Fig. 2: Exact and Absolute Error of Example 2 when $\alpha = \frac{1}{6}$.

Table 3: Absolute Error of Example (2) for $\alpha = \frac{1}{10}$

h	$\lvert e \rvert$	$\lvert e^{\alpha} \rvert$	$\lvert e^{3\alpha} \rvert$	Exact solution	Approximation solution
0.4	0.0222	0.233	0.5707	1.914	1.681
0.6	0.1683	0.177	0.0541	2.0292	1.8522
0.8	0.2572	0.2703	0.0126	2.0284	1.7581

Table 4: Absolute Error of Example (2) for $\alpha = \frac{1}{6}$

h	$\lvert e \rvert$	$\lvert e^{\alpha} \rvert$	$\lvert e^{3\alpha} \rvert$	Exact solution	Approximation solution
0.4	0.0247	0.0266	1.112	1.7192	1.6926
0.6	0.1813	0.1954	0.1978	1.9083	1.7129
0.8	0.243	0.2619	0.3364	1.959	1.6971

7 Conclusion

The fractional spline function has been proposed to be applicable for the case $0 < \alpha < 1$ and to be used to solve fractional initial value problems on other hand , error bounded and stability analysis was discussed and examples was used to clarify this presented method based on step size h and the value of α.

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