Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/amisl/140101

Optimized Four-Step Block Hybrid Scheme with One Off-Grid Point for Direct Integration of First-Order Ordinary Differential Equations

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Received: 8 Aug. 2025, Revised: 20 Oct. 2025, Accepted: 21 Nov. 2025
Published online: 1 Jan. 2026

Abstract: This study proposes a four-step numerical scheme with one off-grid point to improve the order of accuracy by incorporating optimized points into the algorithm's formulation. This study explores a four-step hybrid block method designed to solve first-order ordinary differential equations with a constant step size. The method employs interpolation techniques capable of evaluating terms that have not been defined at the grid points. It is shown to be consistent, zero-stable, and convergent. The numerical results obtained are competing with existing methods and performed better.

Keywords: Hybrid block method, Initial value problems, Off-grid point, Zero stability.

1 Introduction

Block hybrid methods are numerical techniques that integrate multistep linear methods with power series through the use of interpolation. Originally introduced by [1] and later refined by [2], these methods include an additional point at each step of the formula, leading to more accurate approximations of the solutions to differential equations and improved convergence rates. Since the groundbreaking work of [3], block methods have gained significant attention in the literature for their effectiveness in solving both initial value problems (IVP) and boundary value problems (BVPs), due to their flexibility, high accuracy, and computational efficiency in handling complex systems. [4] proposed a hybrid overlapping grid block method that combines equally spaced and optimally selected grid points to solve both linear and nonlinear first-order initial value problems (IVP). His results demonstrated that this approach of the overlapping grid significantly reduces the local truncation error, outperforming traditional non-overlapping grid methods. Similarly, [5] applied a block hybrid method with equally spaced grid points to solve first-order linear and non-linear IVPs. He reported that equally spaced grid points yielded high rates of convergence, surpassing those of the fourth-order Runge–Kutta method. [6] developed an implicit block hybrid method for solving first-, second- and third-order initial value problems (IVP). He explored the convergence rates, accuracy, and robustness of these implicit block hybrid algorithms, and further examined their performance when various countable off-points were introduced between grid points during the derivation process. [7] introduced spline functions with four collocation points to address second-order initial value problems (IVP). Their work demonstrated that the spline collocation scheme converges with an order of seven, provided certain conditions on the collocation point parameters are met. The analysis included a detailed investigation of the method's stability properties and the identification of regions of absolute stability, which depend on the values of the parameters. In addition, various iterative techniques were explored for solving both initial value and boundary value problems in ordinary and partial differential equations, where solutions or approximations are obtained through successive iterations. For IVPs, these iterative methods

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can be formulated in either integral or differential form. They linearize the non-linear governing equations around the previous iteration, resulting in linear differential equations at each step. However, since the coefficients of these equations may vary with the independent variable, numerical methods are required to approximate the solutions.

2 Methodology

Numerical analysis involves the study of algorithms that use numerical approximations to solve mathematical problems encountered in various fields such as engineering, physical sciences, life sciences, social sciences, medicine, and business. Many of these problems are dynamic in nature and involve time, space, and other physical quantities, which are often modeled by ordinary differential equations (ODEs). Examine the following equation:

$$\sum_{j}^{k} \alpha_j y_{n+j} = \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{1}$$

For driving continuous systems, [8] method is employed. A k-step linear multi-step method (LMM) obtained by:

$$y(x) = \sum_{j=0}^{T-1} \alpha_j(x) y(x_{I,n+1}) \tag{2}$$

Note $\alpha_j(x)$ and $\beta_j(x)$ make up the continuous coefficients of the method, where $\alpha_j(x)$ and $\beta_j(x)$ are defined as

$$\alpha_j(x) = \sum_{j=0}^{T+m-1} \alpha_{j,i+1} x_I^i, \tag{3}$$

$$\beta_j(x) = \sum_{j=0}^{T+m-1} \beta_{j,i+1} x_I^i, \tag{4}$$

The points The points

$$x_{I,n+j} \quad \text{for } j = 0,1,2,\ldots,T-1 \quad (0 \le T \le k)$$

in equation (4) above are arbitrarily chosen T-interpolation points taken from

$$\{x_{I,n}, x_{I,n+1}, \ldots, x_{I,n+k}\},$$

and the points

$$x_{C,j} \quad \text{for } j = 0,1,2,\ldots,m-1$$

are the m collocation points that belong to the set

$$\{x_{C,n}, x_{C,n+1}, \ldots, x_{C,n+k}\}.$$

As an example, for $T = 1$, where $x_{I,0} = x_n$, we have

$$x_{I,n+1} \quad \text{for } j = 0,1,2,\ldots,T-1.$$

$$\begin{aligned} m &= 6, \\ x_{c,0} &= x_n, \\ x_{c,1} &= x_{n+1}, \\ x_{c,2} &= x_{n+2}, \\ x_{c,3} &= x_{n+3}, \\ x_{c,4} &= x_{n+4}, \\ x_{c,5} &= x_{n+5} \end{aligned} \tag{5}$$

[8] established a matrix equation of the form;

$$DC = I \tag{6}$$

where D and C are matrices defined as I is a matrix of the same dimension (T+m)×(T+m). Consider the following equations:

$$\begin{aligned} y(x) = \sum_{j=0}^{T-1} y(x_{I,n+j}) \left(\sum_{i=0}^{T+m-1} \alpha_{j,i+1} x_I^i \right) \\ + \sum_{j=0}^{m-1} f(x_{c,j} y(x_{c,j})) \left(\sum_{i=0}^{T+m-1} \beta_{j,i+1} x_I^i \right) \end{aligned} \tag{7}$$

$$D = \begin{vmatrix} 1 & x_{I,0} & x_{I,0}^2 & \cdots & x_{I,0}^{T+m-1} \\ 1 & x_{I,1} & x_{I,0}^2 & \cdots & x_{I,1}^{T+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{I,n+T-1} & x_{I,n+T-1}^2 & \cdots & x_{I,n+T-1}^{T+m-1} \\ 0 & 1 & 2x_{c,0} & \cdots & (T+m-1)x_{c,0}^{T+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{c,m-1} & \cdots & (T+m-1)x_{c,m-1}^{T+m-1} \end{vmatrix} \tag{8}$$

$$C = \begin{vmatrix} \alpha_{0,1} & \alpha_{1,1} & \alpha_{T-1,1} & \beta_{0,1} & \cdots & \beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \alpha_{T-1,2} & \beta_{0,2} & \cdots & \beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{0,T+m} & \alpha_{1,T+m} & \alpha_{T-1,T+m} & \beta_{0,T+m} & \cdots & \beta_{m-1,T+m} \end{vmatrix} \tag{9}$$

To approximate the generic equation (4), power series are employed, where T and m represent the number of interpolation points and the corresponding collocation points, respectively. The functions $\alpha_j(x)$ and $\beta_j(x)$ are parameters to be determined. The matrix $C = D^{-1}$ has columns that yield the continuous coefficients. Both $\alpha_j(x)$ and $\beta_j(x)$, $j = 1,2,\ldots,k-1$, are combined.

In this section, our objective is to develop a new block hybrid method using the idea of multistep collocation from [8]. From equation (3), we develop a continuous scheme for the four-step block hybrid method, incorporating one off-grid point, $x_{n+1/3}$.

We shall apply the above technique to derive block hybrid methods, specifically a four-step block hybrid method with one off-point within the interval between x_n and x_{n+4}.

We obtain our matrices D and $C = D^{-1}$ from equations (8) and (9), respectively, as follows:

$$D = \begin{vmatrix} 1 & x_{n+v-1} & x^2_{n+v-1} & x^3_{n+v-1} & x^4_{n+v-1} & \cdots & x^m_{n+v-1} \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & \cdots & mx_0^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+v} & 3x^2_{n+v} & 4x^3_{n+v} & \cdots & mx^{m-1}_{n+v} \end{vmatrix} \quad (10)$$

and

$$C = \begin{vmatrix} \alpha_{01} & \alpha_{11} & \cdots & \alpha_{x-1,1} & h\beta_{01} & \cdots & h\beta_{v+1,1} \\ \alpha_{02} & \alpha_{12} & \cdots & \alpha_{x-1,2} & h\beta_{02} & \cdots & h\beta_{v+1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,v+1} & \alpha_{1,v+1} & \cdots & \alpha_{x-1,v+1} & h\beta_{0,v+1} & \cdots & h\beta_{x+1,v+1} \end{vmatrix} \quad (11)$$

From equations (10) and (11) above, $DC = I$, we get $C = D^{-1}$, and from equation (3) above, we get:

$$y(x) = \alpha_0(x)y_n + h\big[\beta_0(x)f_n + \beta_{1/3}(x)f_{n+1/3} + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}\big] \quad (12)$$

From

$$D = \begin{vmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} \\ 0 & 1 & 2x_{n+2} & 3x^2_{n+2} & 4x^3_{n+2} & 5x^4_{n+2} & 6x^5_{n+2} \\ 0 & 1 & 2x_{n+3} & 3x^2_{n+3} & 4x^3_{n+3} & 5x^4_{n+3} & 6x^5_{n+3} \\ 0 & 1 & 2x_{n+4} & 3x^2_{n+4} & 4x^3_{n+4} & 5x^4_{n+4} & 6x^5_{n+4} \end{vmatrix} \quad (13)$$

$$C = \begin{vmatrix} \alpha_0 & \alpha_1 & h\beta_0 & h\beta_1 & h\beta_2 & h\beta_3 & h\beta_4 \\ \alpha_0 & \alpha_1 & h\beta_0 & h\beta_1 & h\beta_2 & h\beta_3 & h\beta_4 \\ \alpha_0 & \alpha_1 & h\beta_0 & h\beta_1 & h\beta_2 & h\beta_3 & h\beta_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_0 & \alpha_1 & h\beta_0 & h\beta_1 & h\beta_2 & h\beta_3 & h\beta_4 \end{vmatrix} \quad (14)$$

Using the element $C = D^{-1}$, we have

$$\alpha_0(x) = (x - x_n) \quad (15)$$

$$\beta_0(x) = \frac{1}{1440h^5}\Big[-2(x-x_n)^6 + 36(x-x_n)^5h - 255(x-x_n)^4h^2 + 900(x-x_n)^3h^3 - 1644(x-x_n)^2h^4 + 1440(x-x_n)h^5\Big] \quad (16)$$

$$\beta_1(x) = \frac{1}{1440h^5}\Big[10(x-x_n)^6 - 168(x-x_n)^5h + 1065(x-x_n)^4h^2 - 3080(x-x_n)^3h^3 + 3600(x-x_n)^2h^4\Big] \quad (17)$$

$$\beta_2(x) = \frac{1}{720h^5}\begin{bmatrix} -10(x-x_n)^6 + 156(x-x_n)^5 \\ h - 885(x-x_n)^4h^2 \\ +2140(x-x_n)^3h^3 - 1800(x-x_n)^2h^4 \end{bmatrix} \quad (18)$$

$$\beta_3(x) = \frac{1}{720h^5}\begin{bmatrix} 10(x-x_n)^6 - 144(x-x_n)^5 \\ h + 735(x-x_n)^4h^2 \\ -1560(x-x_n)^3h^3 + 1200(x-x_n)^2h^4 \end{bmatrix} \quad (19)$$

$$\beta_4(x) = \frac{1}{1440h^5}\begin{bmatrix} -10(x-x_n)^6 + 132(x-x_n)^5 \\ h - 615(x-x_n)^4h^2 \\ +1200(x-x_n)^3h^3 - 900(x-x_n)^2h^4 \end{bmatrix} \quad (20)$$

Substituting $\alpha_0, \beta_0, \beta_1, \beta_2, \beta_3,$ and β_4 into;.

$$y(x) = \alpha_0(x)y_n + h\big[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}\big] \quad (21)$$

The continuous scheme is obtained as follows.

$$\begin{aligned} y(x) &= (x-x_n)y_n \\ &+ \frac{1}{1440h^5}\begin{bmatrix} -2(x-x_n)^6 + 36(x-x_n)^5h - \\ 255(x-x_n)^4h^2 + 900(x-x_n)^3h^3 \\ -1644(x-x_n)^2h^4 + 1440(x-x_n)h^5 \end{bmatrix} f_n \\ &+ \frac{1}{1440h^5}\begin{bmatrix} 10(x-x_n)^6 - 168(x-x_n)^5h + \\ 1065(x-x_n)^4h^2 - 3080(x-x_n)^3h^3 \\ +3600(x-x_n)^2h^4 \end{bmatrix} f_{n+1} \\ &+ \frac{1}{720h^5}\begin{bmatrix} -10(x-x_n)^6 + 156(x-x_n)^5h - \\ 885(x-x_n)^4h^2 + 2140(x-x_n)^3h^3 \\ -1800(x-x_n)^2h^4 \end{bmatrix} f_{n+2} \\ &+ \frac{1}{720h^5}\begin{bmatrix} 10(x-x_n)^6 - 144(x-x_n)^5h + \\ 735(x-x_n)^4h^2 - 1560(x-x_n)^3h^3 \\ +1200(x-x_n)^2h^4 \end{bmatrix} f_{n+3} \\ &+ \frac{1}{1440h^5}\begin{bmatrix} -10(x-x_n)^6 + 132(x-x_n)^5h - \\ 615(x-x_n)^4h^2 + 1200(x-x_n)^3h^3 \\ -900(x-x_n)^2h^4 \end{bmatrix} f_{n+4} \end{aligned} \quad (22)$$

Evaluate equation (22) at the nodes to obtain

$$y_{n+\frac{1}{3}} - y_n = h\Big[\frac{13667}{58320}f_n + \frac{5051}{29160}f_{n+1} - \frac{277}{2430}f_{n+2} + \frac{1421}{29160}f_{n+3} - \frac{523}{58320}f_{n+4}\Big] \quad (23)$$

$$y_{n+1}-y_n=h\Big[\frac{251}{720}f_n+\frac{323}{360}f_{n+1}-\frac{11}{30}f_{n+2}+\frac{53}{360}f_{n+3}-\frac{19}{720}f_{n+4}\Big] \quad (24)$$

$$y_{n+2}-y_n=h\Big[\frac{29}{90}f_n+\frac{62}{45}f_{n+1}+\frac{4}{15}f_{n+2}+\frac{2}{45}f_{n+3}-\frac{1}{90}f_{n+4}\Big] \quad (25)$$

$$y_{n+3}-y_n=h\Big[\frac{27}{80}f_n+\frac{51}{40}f_{n+1}+\frac{9}{10}f_{n+2}+\frac{21}{40}f_{n+3}-\frac{3}{80}f_{n+4}\Big] \quad (26)$$

$$y_{n+4}-y_n=h\Big[\frac{14}{45}f_n+\frac{64}{45}f_{n+1}+\frac{8}{15}f_{n+2}+\frac{64}{45}f_{n+3}-\frac{14}{45}f_{n+4}\Big] \quad (27)$$

Equation (23) – (27) can be written as in the form.

$$\begin{cases} y_{n+\frac{1}{3}}-y_n=\frac{h}{58320}\Big(13667f_n+10102f_{n+1}-6648f_{n+2}+2842f_{n+3}-523f_{n+4}\Big) \\ y_{n+1}-y_n=\frac{h}{720}\Big(351f_n+646f_{n+1}-264f_{n+2}+106f_{n+3}-19f_{n+4}\Big) \\ y_{n+2}-y_n=\frac{h}{90}\Big(29f_n+126f_{n+1}+24f_{n+2}+4f_{n+3}-f_{n+4}\Big) \\ y_{n+3}-y_n=\frac{h}{80}\Big(27f_n+102f_{n+1}+72f_{n+2}+42f_{n+3}-3f_{n+4}\Big) \\ y_{n+4}-y_n=\frac{h}{45}\Big(14f_n+64f_{n+1}+24f_{n+2}+64f_{n+3}+14f_{n+4}\Big) \end{cases} \quad (28)$$

3 Order and Error Constant of the Method

The derived finite difference scheme, given in equation (28) are discrete schemes that fall within the class of Linear Multistep Methods (LMMs).

$$\sum_{i=0}^{k}\alpha_i y(x_{n+i})=\sum_{i=0}^{k}\beta_i f(x_{n+i}) \quad (29)$$

$y'=\frac{d}{dx}y=f(x,y), \quad h=\Delta x$
This technique is connected to a linear difference operator.

$$L[y(x);h]=\sum_{i=0}^{k}\big[\alpha_i(x)\,y(x+ih)+h\beta_i(x)\,y'(x+ih)\big] \quad (30)$$

An arbitrary function that is continuously differentiable on the interval $[a,b]$ is denoted by $y(x)$. The extension of the Taylor series around the point x,

$$L[y(x);h]=c_0\,y(x)+c_1\,hy'(x)+c_2\,hy''(x)+\cdots+c_q\,h^q\,y^{(q)}(x) \quad (31)$$

The linear multistep method given in equation (3) above is said to have order p if, for

$c_0=c_1=c_2=\cdots=c_p=0$ and
$c_{p+1}\neq 0$ is the error constant,

$$c_0=\alpha_0+\alpha_1+\alpha_2+\cdots+\alpha_k,$$

$$c_1=(\alpha_0+\alpha_1+2\alpha_2+\cdots+k\alpha_k)-(\beta_0+\beta_1+\beta_2+\cdots+\beta_k)$$

$$c_p=\frac{1}{q!}\left(\alpha_1+2^q\alpha_2+\cdots+k^q\alpha_k\right)-\frac{1}{(q-1)!}\left(\beta_1+2^{q-1}\beta_2+\cdots+k^{q-1}\beta_k\right), \quad q=2,3,\ldots,N \quad (32)$$

Let $h=\Delta x=\omega H=\omega\Delta T$

$$y_{n+\frac{1}{3}}=y\left(x_n+\frac{1}{3}\Delta x\right)=y\left(x_n+\frac{1}{3}\omega\Delta T\right)=y_{n+\frac{1}{3}\omega}$$

$$y\left(x_{n+\frac{1}{3}}\right)=y\left(x_n+\frac{1}{3}\Delta x\right)=y\left(x_n+\frac{1}{3}\omega\Delta T\right)$$

$$\sum_j \alpha_j y_{n+\frac{1}{3}}=\Delta x\sum_j \beta_j f_{n+\frac{1}{3}}$$

$$\Rightarrow \sum_j \alpha_j y_{n+\frac{1}{3}\omega}=\omega\Delta T\sum_j \beta_j f_{n+\frac{1}{3}\omega}$$

Let $J=\omega j$, where j is a fraction, we get:

$$\sum_J \alpha_J^n y_{n+\frac{1}{3}}=\Delta T\sum_J \beta_J^n f_{n+\frac{1}{3}}$$

By transforming $\alpha_J^n=\alpha_J$, $\beta_J^n=\omega\beta_J$, the aforementioned methodology may be used to determine the truncation error of the linear multiscale block method.

$$\frac{1}{3}\Delta x=\Delta T \;\Rightarrow\; \omega=3, \quad h=\Delta x=\omega\Delta T$$

Definition 3.1: According to [9], technique (28) is of order P if the error constant is $C_{p+1}\neq 0$ and $C_0=C_1=C_2=\cdots=C_P=0$.

Using this definition in block method (28), it is confirmed that each of the five difference schemes has error constants and is of order

$$P = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 6 \end{pmatrix}^T$$

$$C_7 = \begin{pmatrix} -1.43 \times 10^{-2} \\ -9.79 \times 10^{-2} \\ -1.29 \times 10^{-2} \\ -8.47 \times 10^{-2} \\ -2.27 \times 10^{-2} \end{pmatrix}^T$$

3.1 The Consistency of the Method

Definition 3.2: An LMM of type (28) is regarded consistent if its order is $p \geq 1$. The discrete schemes generated in (28) are consistent as they are of order $6 \geq 1$ according to Definition (3.1).

3.2 Zero Stability of the Method

$$A^1 = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} \quad A^0 = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} - \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{14}{3}} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} =$$

$$h \begin{bmatrix} 0 & \frac{39937}{65610} & -\frac{14629}{32805} & \frac{9031}{32805} & -\frac{3269}{32805} \\ 0 & \frac{1427}{1440} & -\frac{133}{240} & \frac{241}{720} & -\frac{173}{1440} \\ 0 & \frac{43}{30} & -\frac{7}{45} & \frac{7}{45} & -\frac{1}{15} \\ 0 & \frac{219}{160} & -\frac{57}{80} & \frac{57}{80} & -\frac{21}{160} \\ 0 & \frac{64}{45} & -\frac{8}{15} & \frac{64}{45} & -\frac{14}{45} \end{bmatrix} \begin{bmatrix} f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} +$$

$$\begin{bmatrix} 0&0&0&0&\frac{10256}{32805} \\ 0&0&0&0&\frac{95}{288} \\ 0&0&0&0&\frac{14}{45} \\ 0&0&0&0&\frac{51}{160} \\ 0&0&0&0&\frac{14}{45} \end{bmatrix} \begin{bmatrix} f_{n-\frac{13}{3}} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

$$A^{(0)} \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} - A^{(1)} \begin{bmatrix} y_{n-\frac{14}{3}} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = hB^{(0)} \begin{bmatrix} f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} +$$

$$hB^{(1)} \begin{bmatrix} f_{n-\frac{13}{3}} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $y_n = r^n$ so that we get

$$Y_n = \begin{bmatrix} r^{n+\frac{1}{3}} \\ r^{n+1} \\ r^{n+2} \\ r^{n+3} \\ r^{n+4} \end{bmatrix}, \quad Y_{n-1} = \begin{bmatrix} r^{n+\frac{14}{3}} \\ r^{n+3} \\ r^{n+2} \\ r^{n+1} \\ r^n \end{bmatrix}$$

$A^{(0)}Y_n - A^{(1)}Y_{n-1} = 0$

$$A^{(0)}r \begin{bmatrix} r^{n+\frac{1}{3}} \\ r^{n+1} \\ r^{n+2} \\ r^{n+3} \\ r^{n+4} \end{bmatrix} - A^{(1)} \begin{bmatrix} r^{n+\frac{14}{3}} \\ r^{n+3} \\ r^{n+2} \\ r^{n+1} \\ r^n \end{bmatrix} = 0$$

$\Rightarrow A_B Y_{n-1} = 0$

For the nontrivial solution, we require

$|A_B| = \left|A^{(0)}r - A^{(1)}\right| = 0$

$$\mathbf{A}^{(0)} = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}, \quad \mathbf{A}^{(1)} = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \end{bmatrix}$$

and

$$B^{(0)} = \begin{bmatrix} 0 & \frac{39937}{65610} & -\frac{14629}{32805} & \frac{9031}{32805} & -\frac{3269}{32805} \\ 0 & \frac{1427}{1440} & -\frac{133}{240} & \frac{241}{720} & -\frac{173}{1440} \\ 0 & \frac{43}{30} & -\frac{7}{45} & \frac{7}{45} & -\frac{1}{15} \\ 0 & \frac{219}{160} & -\frac{57}{80} & \frac{57}{80} & -\frac{21}{160} \\ 0 & \frac{64}{45} & -\frac{8}{15} & \frac{64}{45} & -\frac{14}{45} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} 0&0&0&0&\frac{10256}{32805} \\ 0&0&0&0&\frac{95}{288} \\ 0&0&0&0&\frac{14}{45} \\ 0&0&0&0&\frac{51}{160} \\ 0&0&0&0&\frac{14}{45} \end{bmatrix}$$

where $A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}$ are 5×5 matrices. $\rho(r) = \det\left(rA^{(0)} - A^{(1)}\right)$

Substituting $A^{(0)}$ and $A^{(1)}$ into the characteristic polynomial, it gives

$$\rho(r) = \left|rA^{(0)} - A^{(1)}\right| \tag{34}$$

Therefore, the method in equation (28) is zero-stable if the zeros of the characteristic polynomial's initial roots $\rho(r)$ satisfy $|r| \leq 1$, and it is consistent since it is of order $\rho > 1$ for $r_i = [0,0,0,0,1]$ where $i = 1,2,\ldots,k+1$. Therefore, according to [9] and [10], the technique in equation (28) is convergent, since it is both zero-stable and consistent.

3.3 Absolute Stability Region

If, for a given $\bar{\lambda}$, all roots r_s of the stability polynomial $\pi(\bar{\lambda})$ of equation (28) satisfy $|r_s| \leq 1$, $s = 0, 1, \ldots, k$, then the LMM (28) is stable; if not, it is unstable for that $\bar{\lambda}$.

$$y' = \frac{d}{dx}y(x) = f(x,y) = \lambda y_n \tag{35}$$

$$\Rightarrow A^{(0)} \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} - A^{(1)} \begin{bmatrix} y_{n-\frac{14}{3}} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} =$$

$$\lambda h B^{(0)} \begin{bmatrix} f_{n+\frac{1}{3}} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} + \lambda h B^{(1)} \begin{bmatrix} f_{n-\frac{14}{3}} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(A^{(0)} - \lambda h B^{(0)}\right) \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} -$$

$$\left(A^{(1)} + \lambda h B^{(1)}\right) \begin{bmatrix} y_{n-\frac{14}{3}} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Leaving $y_n = r^n$, we get

$$\left[\left(A^{(0)} - \lambda h B^{(0)}\right) r - \left(A^{(1)} + \lambda h B^{(1)}\right)\right] \begin{bmatrix} r^{n-\frac{14}{3}} \\ r^{n-3} \\ r^{n-2} \\ r^{n-1} \\ r^n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution, we set the characteristic equation to zero:

$$\det\left[\left(A^{(0)} - \lambda h B^{(0)}\right) r - \left(A^{(1)} + \lambda h B^{(1)}\right)\right] = 0.$$

Let $\bar{\lambda} = \lambda h$, then we get:

$$\det\left[\left(A^{(0)} - \bar{\lambda} B^{(0)}\right) r - \left(A^{(1)} + \bar{\lambda} B^{(1)}\right)\right] = 0.$$

$$r = p(\bar{\lambda}), \quad |r| = |p(\bar{\lambda})| < 1 \tag{36}$$

As shown in Figure 1 below, the region of absolute stability of the new block technique is created after substituting the values from equation (29) into the stability matrix. The computation is carried out using the MATLAB environment. Note that the stability region is only plotted in the domain $x \in (-4, 0)$, $y \in (-2, 2)$.

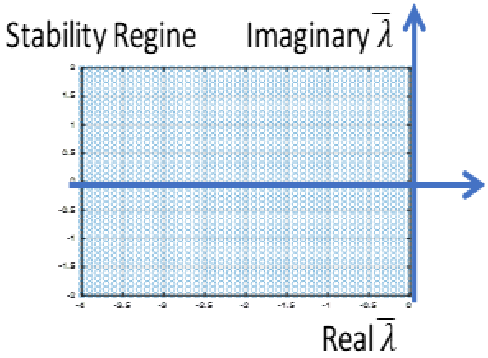

Fig. 1: Absolute stability region

4 Result and Discussion

The performance of our unique block approach is evaluated through a series of numerical tests presented in this section. The following test problems are considered. This technique, known as the multistep collocation method (MC), is based on the continuous finite difference approximation approach with the collocation criterion. We used the block technique to verify that the approach is more effective than the existing methods. All calculations were performed with MATLAB, and the results were both precise and accurate. In the numerical experiments, we used a step size of $h = 0.01$.

4.1 Numerical Experiment

4.1.1 Problem 1

$$y' = F(x,y) = 2x, \quad h = 0.01, \quad 0 \leq x \leq 1 \tag{37}$$

The exact solution is given as:

$$y(x) = x^2 \tag{38}$$

[*Source* : [11]&[12]]

4.1.2 Problem 2

$$y' = F(x,y) = xy, \quad h = 0.01, \quad 0 \le x \le 1 \tag{39}$$

The exact solution is as follows.

$$y(x) = e^{\frac{x^2}{2}} \tag{40}$$

[*Source* : [13]]

4.1.3 Problem 3

Numerical example (SIR model) [13]
The SIR model is a fundamental epidemiological framework that is used to estimate the theoretical number of individuals infected with a contagious disease within a closed population over time. Its name, **SIR**, derives from the three key compartments it tracks: the number of susceptible individuals $S(t)$, infected individuals $I(t)$, and recovered individuals $R(t)$, respectively. These compartments are governed by a set of coupled differential equations. Despite its simplicity, the SIR model effectively describes the spread of many infectious diseases, such as measles, mumps, and rubella. The model is represented by the following system of three coupled equations.

$$\frac{dS}{dt} = \mu(I - S) - \beta IS \tag{41}$$

$$\frac{dI}{dt} = \mu I - \gamma I + \beta IS \tag{42}$$

$$\frac{dR}{dt} = \mu R - \gamma I \tag{43}$$

where μ, γ, and β are positive parameters. Define y as:

$$Y = S + I + R \tag{44}$$

and adding equations (43)–(45), we obtain the following evolution equation:

$$y' = \mu(1 - y) \tag{45}$$

Taking $\mu = 0.5$, $y(0) = 0.5$, and applying an initial condition, we obtain:

$$y'(x) = 0.5(1 - y), \quad y(0) = 0.5, \quad h = 0.01, \quad 0 \le x \le 1 \tag{46}$$

with exact solution:

$$y(x) = 1 - 0.5e^{-0.5x} \tag{47}$$

4.1.4 Problem 4

Consider the following first-order linear ordinary differential equation:

$$y' = F(x,y) = x + y, \quad y(0) = 1, \quad h = 0.01, \quad 0 \le x \le 1 \tag{48}$$

The exact solution is;

$$y(x) = 2e^x - x - 1 \tag{49}$$

[*Source* : [14]]

4.1.5 Problem 5

Consider the following first-order nonlinear ordinary differential equation;

$$y' = F(x,y) = x^2 + y^2, \quad y(0) = 1, \quad h = 0.01, \quad 0 \le x \le 1 \tag{50}$$

The exact solution is given by;

$$1 + x + x^2 + \frac{4x^3}{3} + \frac{5x^4}{3} + \frac{16x^5}{15} \tag{51}$$

[Source: See [15]]
The tables below compare the absolute errors for Problems 1–5 with the method of [16], respectively.

Table 1: Absolute Error for Problem 1

h	[16] Method	Present Method
0.01	Err1: 2.7105×10^{-20} Err2: 2.7105×10^{-20}	Err1: -1.6941×10^{-21} Err2: -1.6941×10^{-21}

Table 2: Absolute Error for Problem 2

h	[16] Method	Present Method
0.01	Err1: -1.6326×10^{-10} Err2: -1.6326×10^{-10}	Err1: -5.0768×10^{-11} Err2: -5.0768×10^{-11}

Table 3: Absolute Error for Problem 3

h	[16] Method	Present Method
0.01	Err1: 5.5970×10^{-5} Err2: 5.5970×10^{-5}	Err1: 6.9406×10^{-7} Err2: 6.9406×10^{-7}

Table 4: Absolute Error for Problem 4

h	[16] Method	Present Method
0.01	Err1: 2.0178×10^{-5} Err2: 2.0178×10^{-5}	Err1: 2.1524×10^{-6} Err2: 2.1524×10^{-6}

Table 5: Absolute Error for Problem 5

h	[16] Method	Present Method
0.01	Err1: 4.1354×10^{-5} Err2: 4.1354×10^{-5}	Err1: 4.3560×10^{-6} Err2: 4.3560×10^{-6}

5 Conclusion

In this work, a four-step block hybrid algorithm for solving first order ordinary differential equations is derived and presented. The strategy is based on a hybrid block method incorporating one off-grid point, developed for the numerical solution of first-order ordinary differential equations (ODEs). The fundamental properties of the method, including consistency, zero stability, convergence, stability region, and local truncation errors, have been analysed. After computations, the results have been validated which shows the accuracy of the present method.

Authors' contributions

Conceptualization, methodology and Writing the original draft is done by Ibrahim Mohammed Dibal while editing and validation was done by Yeak Su Hoe. All authors have read and approved the final version of the manuscript and consent to its submission for publication.

Conflicts of Interest

The authors declare no conflicts of interest concerning the publication of this research.

Acknowledgement

I extend my deepest gratitude to Almighty God for His continuous guidance, protection, and strength throughout the course of this research. I am sincerely thankful to my supervisor, Assoc. Prof. Yeak Su Hoe, from the Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia for his invaluable support, insightful guidance, and encouragement during every stage of this research. I also gratefully acknowledge the financial support provided by the Tertiary Education Trust Fund (TETFUND), Nigeria.

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