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Shannon Entropic Entanglement Criterion in the Simple Harmonic Oscillator

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Abstract: We apply Walborn's Shannon Entropic Entanglement Criterion (SEEC) [S.P. Walborn, B.G. Taketani, A. Salles, F. Toscano, R.L. de Matos Filho, Physical Review Letters **103**, 160505 (2009)], to simple harmonic oscillators. In particular, we investigate the entanglement of a system of coupled harmonic oscillators. A simple form of the entanglement criterion in terms of their interaction is found. It is shown that a pair of interacting ground state coupled oscillators are more likely to be entangled for weaker coupling strengths while coupled oscillators in excited states entangle at higher coupling strengths. Interacting oscillators are more likely to be entangled as their interaction increases.

Keywords: Shannon entropy, entanglement, harmonic oscillator

1 Introduction

In quantum mechanics, entanglement can be thought as a correlation between different states with more than one degree of freedom or between particles [2]. It is a field of active research both theoretically and experimentally. The concept of entanglement is usually introduced via the spin entanglement of fundamental particles [2]. However, entanglement in terms of continuous variables x (or q) and p. [1,3–8] have also been studied.

Various methods of quantifying entanglement [9] have been used such as the positive partial transpose criterion [3], inseparability criterion using variance form of local uncertainty principle [6], and using the entropic functions such as the Shannon, Renyi and von Neumann entropies [1, 4, 10, 11]. The present work will use the entropic uncertainty relation (EUR) statement of entanglement. We specifically choose the Shannon EUR because of its fundamental nature and due to the vast applications of the Shannon entropy [12, 13].

In this paper, we apply the Shannon entropic uncertainty criterion to a system of coupled harmonic oscillators. The entanglement in a system of coupled harmonic oscillators has been studied thoroughly in the literature. For instance, the von Neumann and Renyi entanglement entropies and Schmidt modes have been used to quantify the degree of entanglement for different energies/temperatures [14–16]. Entanglement dynamics (time-dependent interaction strength, sudden death, revival, etc.) have been examined under varying conditions such as initial temperature, damping factors and squeezing of the system and/or its surrounding environment using coupled harmonic oscillators [17–20]. Other applications of the coupled harmonic oscillator among many others include the following. A mathematical formalism that unifies quantum mechanics and special relativity for Lorentz-covariant states was developed through the group symmetries of coupled harmonic oscillators to show that the quark model and Feynman's parton picture can work together to explain the properties of hadrons in high-energy laboratories [21–25]. In [26], the authors used a system of quantum mechanical coupled oscillators to study the effects on a system of interest which can be measured (first oscillator) when one sums over variables of the external system whose variables are not measured (second oscillator which then is "Feynman's rest of the universe") by calculating the entropy using the density matrix formalism.

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In a recent paper, the Shannon entropy has been calculated for a single D-dimensional simple harmonic oscillator [13]. The present work calculates the Shannon entropy for a bipartite system of coupled harmonic oscillators to quantify the entanglement of the ground and excited states of this system.

The paper is organized as follows. In section 2, we review the coordinate space and momentum space solutions of a system of coupled oscillators. The new results using the Shannon entanglement criterion are derived in section 3. We give our conclusions in section 4.

2 Coupled Simple Harmonic Oscillator

In this paper, we consider two masses whose interaction is given by the simple harmonic oscillator (SHO) Hamiltonian

$$\mathscr{H}=\frac{P_1^2}{2m_1}+\frac{P_2^2}{2m_2}+\frac{A}{2}X_1^2+\frac{B}{2}X_2^2+\frac{C}{2}X_1X_2 \tag{1}$$

Equation (1), which describes a coupled simple harmonic oscillator, has a wide variety of applications in physics [18, 26, 27, 33–43].

Let us work out carefully how to transform the Hamiltonian of the SHO, to involve dimensionless variables to enable us to calculate the Shannon entropies. Consider the Hamiltonian $\mathscr{H}$ of one oscillator given by

$$\mathscr{H}=\frac{P^2}{2m}+\frac{1}{2}kX^2=\frac{P^2}{2m}+\frac{1}{2}m\omega^2X^2, \tag{2}$$

where $\omega=\sqrt{\frac{k}{m}}$ with energies $E_n=(n+\frac{1}{2})\hbar\omega$, $n=0,1,2...$ in which $[X,P]=i\hbar$. Dividing the Hamiltonian by $\hbar\omega$, we get

$$H=\frac{1}{2}(p)^2+\frac{1}{2}(y)^2, \tag{3}$$

where we have the dimensionless variables $H\equiv\frac{\mathscr{H}}{\hbar\omega}$, $p\equiv\frac{P}{(m\hbar^2k)^{\frac{1}{4}}}=\frac{P}{\sqrt{m\hbar\omega}}$, and $y\equiv\frac{X}{\left(\frac{\hbar^2}{mk}\right)^{\frac{1}{4}}}=\frac{X}{\sqrt{\frac{\hbar}{m\omega}}}$. Clearly, $p=-i\frac{d}{dy}$ and one can easily show that

$$[y,p]=i. \tag{4}$$

In the literature, the above results are often equivalently obtained by setting $\hbar$, m, k and ω to 1.

Let us reconsider equation (1), the Hamiltonian of two masses interacting as coupled harmonic oscillators, with $P_j=-i\hbar\frac{d}{dX_j}$. One can diagonalize this to give [14, 26]

$$\begin{aligned}\mathscr{H}=&\frac{1}{2M}\left(\left[P_1'\right]^2+\left[P_2'\right]^2\right)\\&+\frac{1}{2}K\left(e^{2\eta}\left[X_1'\right]^2+e^{-2\eta}\left[X_2'\right]^2\right),\end{aligned} \tag{5}$$

where $M=\sqrt{m_1m_2}$, $P_j'=-i\hbar\frac{d}{dX_j'}$, $j=1,2$, $K=\sqrt{AB-\frac{C^2}{4}}$,

$$e^{2\eta}=\frac{A+B+\frac{A-B}{|A-B|}\sqrt{(A-B)^2+C^2}}{\sqrt{4AB-C^2}}, \tag{6}$$

and

$$\begin{aligned}X_1'=&X_1cos\alpha-X_2sin\alpha,\\X_2'=&X_1sin\alpha+X_2cos\alpha\end{aligned} \tag{7}$$

with

$$tan(2\alpha)=\frac{C}{B-A}. \tag{8}$$

We carefully changed our notation with the primed coordinates to explicitly exhibit the transformations to diagonalize the original Hamiltonian $\mathscr{H}$. The details can be found in [26].

Similar to equation (3), which came from equation (2), we can rewrite equation (5) in terms of dimensionless variables p_1, p_2, y_1, and y_2.

$$H=\frac{1}{2}p_1^2+\frac{1}{2}p_2^2+\frac{1}{2}e^{2\eta}y_1^2+\frac{1}{2}e^{-2\eta}y_2^2 \tag{9}$$

with $[y_j,p_k]=i\delta_{jk}$, $H=\frac{\mathscr{H}}{\hbar\omega}$, $p_j\equiv\frac{P_j'}{(M\hbar^2K)^{\frac{1}{4}}}$, $y_j\equiv\frac{X_j'}{\left(\frac{\hbar^2}{MK}\right)^{\frac{1}{4}}}$, and $\omega=\sqrt{\frac{K}{M}}$, or equivalently as mentioned before, let $\hbar$, M, K, and ω equal 1. Defining the dimensionless variable $x_j=\frac{X_j}{\left(\frac{\hbar^2}{MK}\right)^{\frac{1}{4}}}$, equation (7) becomes

$$\begin{aligned}y_1=&x_1cos\alpha-x_2sin\alpha,\\y_2=&x_1sin\alpha+x_2cos\alpha.\end{aligned} \tag{10}$$

We can infer from reference [14] the solution of the Schrodinger equation with the Hamiltonian given by equation (9).

$$\begin{aligned}\Psi_{nm}=&\psi_n\psi_m\\=&c_n^{(1)}c_m^{(2)}e^{\left(-\frac{e^{\eta}}{2}y_1^2\right)}e^{\left(-\frac{e^{-\eta}}{2}y_2^2\right)}H_n\left(e^{\frac{\eta}{2}}y_1\right)H_m\left(e^{\frac{-\eta}{2}}y_2\right),\end{aligned} \tag{11}$$

where

$$c_n^{(1)}=\frac{1}{\sqrt{\sqrt{\pi}n!2^n}}\quad\text{and}\quad c_m^{(2)}=\frac{1}{\sqrt{\sqrt{\pi}m!2^m}}. \tag{12}$$

The H_n and H_m are the Hermite polynomials. The corresponding eigenvalues are

$$E_{nm}=e^{\eta}\left(n+\frac{1}{2}\right)+e^{-\eta}\left(m+\frac{1}{2}\right). \tag{13}$$

Similarly, one can either solve the momentum space wave functions by finding the Fourier transform of equation (11) or simply replacing $e^{\frac{\eta}{2}}y_1$ with $e^{\frac{-\eta}{2}}p_1$ and $e^{\frac{-\eta}{2}}y_2$ with $e^{\frac{\eta}{2}}p_2$ in equation (11) similar to [13].

$$\begin{aligned}\Phi_{nm} =&\phi_n\phi_m\\ =&c_n^{(1)}c_m^{(2)}e^{-\frac{e^{-\eta}}{2}p_1^2}e^{-\frac{e^{\eta}}{2}p_2^2}H_n\left(e^{\frac{-\eta}{2}}p_1\right)H_m\left(e^{\frac{\eta}{2}}p_2\right).\end{aligned} \tag{14}$$

3 Shannon Entropic Entanglement Criterion for the SHO

EURs were introduced as an alternative way to express the Heisenberg uncertainty principle, to address some shortcomings of the variance statement $\Delta x\Delta p \geq \frac{\hbar}{2}$ [12, 28]. One of the advantages of using the EUR is that it is an excellent mathematical framework to quantify uncertainties for correlated systems such as entangled systems [28]. The first measure of uncertainty in terms of entropy given by Hartley (which he called "information") was $U(n) = k\log n$, where U is the uncertainty of n outcomes of a random experiment. For the case in which the probabilities of the outcomes are unequal, this is generalized to $U_i = k\sum_i^n P_i \log\frac{1}{P_i}$, where P_i is the probability of the ith outcome. For $k = 1$, we get the Shannon entropy $S = -\sum_{i=1}^n P(x_i)\ln P(x_i)$ with x as a discrete random variable. It can be shown that the preceding equation can be generalized to

$$S = -\int_{-\infty}^{\infty} p(x)\ln p(x)\,dx \tag{15}$$

for a continuous variable x with $p(x)$ as the probability density function. We will refer to equation (15) as the Shannon entropy. We refer the reader to [12] for more details.

To study the entanglement of coupled harmonic oscillators using the Shannon entropy, consider a bipartite system which we label as system 1 and system 2. In the succeeding discussions, subscripts 1 and 2 correspond to systems 1 and 2 respectively. As in [1, 10], we define the dimensionless variables as

$$x_\pm = x_1 \pm x_2 \quad\text{and}\quad p_\pm = p_1 \pm p_2 \tag{16}$$

with $[x_j, p_k] = i\delta_{jk}$ as in equation (4). A pure state of the system is described by the wavefunctions $\Psi(x_1,x_2) = \psi_1(x_1)\psi_2(x_2)$ in coordinate space and $\mathscr{P}(p_1,p_2) = \phi_1(p_1)\phi_2(p_2)$ in momentum space. A change in variables using equation (16), gives $\Psi(x_+,x_-) = \frac{1}{\sqrt{2}}\psi_1\left(\frac{x_++x_-}{2}\right)\psi_2\left(\frac{x_+-x_-}{2}\right)$ and $\mathscr{P}(p_+,p_-) = \frac{1}{\sqrt{2}}\phi_1\left(\frac{p_++p_-}{2}\right)\phi_2\left(\frac{p_+-p_-}{2}\right)$.

The Shannon entropic entanglement criterion is given by

$$\begin{aligned}H[w_\pm]+H[v_\mp] &< \ln(2\pi e) \quad\text{or}\\ H[w_\pm]+H[v_\mp]-\ln(2\pi e) &< 0,\end{aligned} \tag{17}$$

where the Shannon entropies are given by

$$\begin{aligned}H[w_\pm] &= -\int_{-\infty}^{\infty} dx_\pm w_\pm(x_\pm)\ln(w_\pm(x_\pm)) \quad\text{and}\\ H[v_\pm] &= -\int_{-\infty}^{\infty} dp_\pm v_\pm(p_\pm)\ln(v_\pm(p_\pm))\end{aligned} \tag{18}$$

with

$$\begin{aligned}w_\pm(x_\pm) &= \int_{-\infty}^{\infty} dx_\mp|\Psi(x_+,x_-)|^2 = \frac{1}{2}\int_{-\infty}^{\infty} dx_\mp|\psi_1|^2|\psi_2|^2,\\ v_\pm(p_\pm) &= \int_{-\infty}^{\infty} dp_\mp|\mathscr{P}(p_+,p_-)|^2 = \frac{1}{2}\int_{-\infty}^{\infty} dp_\mp|\phi_1|^2|\phi_2|^2.\end{aligned} \tag{19}$$

Strictly speaking in equation (6), the case $A = B$ is not defined. However, to facilitate a simpler calculation, we will look at the case in which $A \approx B$. Let us look at the two cases in which $A \to B$ for values $A > B$ and for values $A < B$. For the case $A > B$, equation (6) becomes $e^{2\eta} = \frac{A+B+\sqrt{(A-B)^2+C^2}}{\sqrt{4AB-C^2}}$, and $A \to B$ yields

$$e^{2\eta} \approx \sqrt{\frac{2A+|C|}{2A-|C|}} > 1 \tag{20}$$

for non-zero C. This implies that $\eta > 0$. For the case $A < B$, equation (6) becomes $e^{2\eta} = \frac{A+B-\sqrt{(A-B)^2+C^2}}{\sqrt{4AB-C^2}}$, and $A \to B$ yields $e^{2\eta} \approx \sqrt{\frac{2A-|C|}{2A+|C|}} > 1$ for nonzero C. This implies $\eta < 0$. As will later be seen in this section, entanglement occurs only for $\eta > 0$. Let us consider the case when $A > B$. Since the value of η in section 2 above is independent of the sign of C, let us assume for the moment that $C < 0$[1]. Equation (8) gives $\alpha = 45^\circ$ and, from equation (7), $y_1 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}$, $y_2 = \frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}$. With equation (16), we get

$$y_1 = \frac{x_-}{\sqrt{2}} \quad\text{and}\quad y_2 = \frac{x_+}{\sqrt{2}} \tag{21}$$

and similarly

$$p_1 = \frac{p_-}{\sqrt{2}} \quad\text{and}\quad p_2 = \frac{p_+}{\sqrt{2}}. \tag{22}$$

[1] The case $C > 0$ results in $\alpha = -45^\circ$ with the same qualitative results. In providing the details, we chose to use $C < 0$ to get the $\alpha = 45^\circ$ instead which has been discussed in many of the applications mentioned in [26] and which makes the calculations more straightforward without bothering about the extra negative sign introduced by $\alpha = -45^\circ$.

From equations (21) and (22), we can rewrite the solutions in equations (11) and (14) as

$$\Psi_{nm} = c_n^{(1)} c_m^{(2)} e^{\left(-\frac{e^{\eta}}{4} x_-^2\right)} e^{\left(-\frac{e^{-\eta}}{4} x_+^2\right)} H_n\left(\frac{e^{\frac{\eta}{2}}}{\sqrt{2}} x_-\right) H_m\left(\frac{e^{\frac{-\eta}{2}}}{\sqrt{2}} x_+\right) \tag{23}$$

and

$$\Phi_{nm} = c_n^{(1)} c_m^{(2)} e^{\left(-\frac{e^{-\eta}}{4} p_-^2\right)} e^{\left(-\frac{e^{\eta}}{4} p_+^2\right)} H_n\left(\frac{e^{\frac{-\eta}{2}}}{\sqrt{2}} p_-\right) H_m\left(\frac{e^{\frac{\eta}{2}}}{\sqrt{2}} p_+\right). \tag{24}$$

Let us now outline the calculation of the Shannon entropic entanglement criterion by calculating the function

$$f(\eta) = H[w_-] + H[v_+] - \ln(2\pi e). \tag{25}$$

Using instead $H[w_+] + H[v_-] - \ln(2\pi e)$ yields the same results. We start with $H[w_-]$. Letting equation (26) equal

$$\begin{cases} a) t \equiv \frac{e^{\frac{\eta}{2}}}{\sqrt{2}} \\ b) z_1 \equiv t x_- = \left(\frac{e^{\frac{\eta}{2}}}{\sqrt{2}}\right) x_- \\ c) z_2 \equiv \frac{x_+}{2t} = \left(\frac{e^{\frac{-\eta}{2}}}{\sqrt{2}}\right) x_+ \, , \end{cases} \tag{26}$$

equation (23) becomes (note equation (11) too)

$$\Psi_{nm} = \psi_n \psi_m = c_n^{(1)} c_m^{(2)} e^{-\frac{\left(z_1^2\right)}{2}} e^{-\frac{\left(z_2^2\right)}{2}} H_n(z_1) H_m(z_2). \tag{27}$$

From equation (19), we get $w_- = \frac{1}{2}\int_{-\infty}^{\infty} dx_+ |\psi_n|^2 |\psi_m|^2$ and with the change in variables in equation (26), we get using equation (27)

$$w_- = q_{nm} e^{-z_1^2} H_n^2(z_1) \quad \text{with} \quad q_{nm} = \frac{t I_0}{\pi n! m! 2^n 2^m} \tag{28}$$

where the integral

$$I_0 \equiv \int_{-\infty}^{\infty} e^{-z_2^2} H_m^2(z_2)\, dz_2. \tag{29}$$

From equation (18), $H[w_-] = -\int_{-\infty}^{\infty} dx_- w_- \ln w_-$. Using equation (28) and expanding gives

$$H[w_-] = -\frac{1}{t} q_{nm} \{(\ln q_{nm}) I_1 + I_2 + I_3\} \tag{30}$$

where

$$\begin{cases} a) I_1 \equiv \int_{-\infty}^{\infty} e^{-z_1^2} H_n^2(z_1) dz_1 \\ b) I_2 \equiv -\int_{-\infty}^{\infty} z_1^2 e^{-z_1^2} H_n^2(z_1) dz_1 \\ c) I_3 \equiv \int_{-\infty}^{\infty} e^{-z_1^2} H_n^2(z_1) \ln\left(H_n^2(z_1)\right) dz_1 . \end{cases} \tag{31}$$

From [13] we get analytic expressions for the integrals in equations (29) and (31).

$$\begin{cases} a) I_0 = 2^m m! \sqrt{\pi} \\ b) I_1 = 2^n n! \sqrt{\pi} \\ c) I_2 = -2^n n! \sqrt{\pi}\left(n + \frac{1}{2}\right) \\ d) I_3 = 2^n n! \sqrt{\pi} \ln 2^{2n} - 2\sum_{k=1}^{n} V_n\left(x_{n,k}\right), \end{cases} \tag{32}$$

where $x_{n,k}$ are the roots of $H_n(z_1)$ and V_n, called the logarithmic potential of the Hermite polynomial H_n, is given by

$$V_n(x_{n,k}) = 2^n n! \sqrt{\pi}\left[\ln 2 + \frac{\gamma}{2} - x_{n,k}^2 \; {}_2F_2\left(1,1;\frac{3}{2},2;-x_{n,k}^2\right) + \frac{1}{2}\sum_{i=1}^{n} \binom{n}{k} \frac{(-1)^k 2^k}{k} \; {}_1F_1\left(1;\frac{1}{2};-x_{n,k}^2\right)\right]$$

with $\gamma \approx 0.577$ as the Euler constant, and ${}_1F_1$ and ${}_2F_2$ are the hypergeometric functions.

Next we calculate $H[v_+]$. Similar to equation (26), we let equation (33) equal

$$\begin{cases} a) t \equiv \frac{e^{\frac{\eta}{2}}}{\sqrt{2}} \\ b) \wp_1 \equiv \frac{p_-}{2t} = \left(\frac{e^{\frac{-\eta}{2}}}{\sqrt{2}}\right) p_- \\ c) \wp_2 \equiv t p_+ = \left(\frac{e^{\frac{\eta}{2}}}{\sqrt{2}}\right) p_+ . \end{cases} \tag{33}$$

Equation (24) becomes (note equation (14), too)

$$\begin{aligned} \Phi_{nm} &= \phi_n \phi_m \\ &= c_n^{(1)} c_m^{(2)} e^{\left(-\frac{\wp_1^2}{2}\right)} e^{\left(-\frac{\wp_2^2}{2}\right)} H_n(\wp_1) H_m(\wp_2). \end{aligned} \tag{34}$$

From equation (19), we get $v_+ = \frac{1}{2}\int_{-\infty}^{\infty} dp_- |\phi_n|^2 |\phi_m|^2$. With the change in variables in equation (33) and using equation (34), we get

$$v_+ = r_{nm} e^{-\wp_2^2} H_m^2(\wp_2) \quad \text{with} \quad r_{nm} = \frac{t J_0}{\pi n! m! 2^n 2^m}, \tag{35}$$

where the integral $J_0 \equiv \int_{-\infty}^{\infty} e^{-\wp_1^2} H_n^2(\wp_1) d\wp_1$. From equation (18), $H[v_+] = -\int_{-\infty}^{\infty} dp_+ v_+ \ln(v_+)$.

Using equation (35) and expanding, we get

$$H[v_+] = -\frac{1}{t} r_{nm} \{(\ln r_{nm}) J_1 + J_2 + J_3\}, \tag{36}$$

where $J_1 \equiv \int_{-\infty}^{\infty} e^{-\wp_2^2} H_m^2(\wp_2) d\wp_2$, $J_2 \equiv -\int_{-\infty}^{\infty} \wp_2^2 e^{-\wp_2^2} H_n^2(\wp_2) d\wp_2$ and $J_3 \equiv \int_{-\infty}^{\infty} e^{-\wp_2^2} H_n^2(\wp_2) \ln\left(H_n^2(\wp_2)\right) d\wp_2$. With a proper change in variables, one can easily evaluate J_0, J_1, J_2, and J_3 similar to the evaluation of the integrals in equations (29) and (31), as given by equation (32). The above discussion of the integrals facilitates the Maple calculation of $f(\eta)$ in equation (25) using equations (30) and (36) for different values of n and m. In general, the $f(\eta)$ has the form

$$f(\eta) = \eta_0 - \eta, \tag{37}$$

where η_0 is a constant (threshold value) which is the η-intercept (horizontal axis intercept).

In Figure 1, we plot $f(\eta)$ in equation (25). For instance, $f(\eta) = -\eta$ for $(n = m = 0)$; $f(\eta) \approx 0.541 - \eta$ for $(n = m = 1)$; $f(\eta) \approx 0.852 - \eta$ for $(n = m = 2)$; $f(\eta) \approx 1.07 - \eta$ for $(n = m = 3)$, etc. Similar graphs can be plotted for states with unequal values of n and m. From equation (17), we note that entanglement occurs when $f(\eta) < 0$. The graph shows the strong dependence of entanglement on the quantum numbers (n, m).

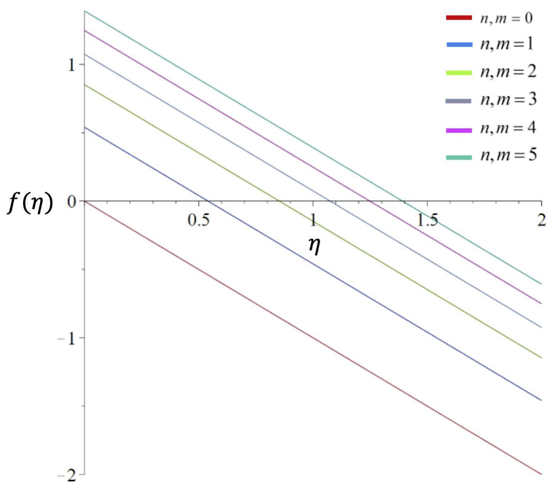

Fig. 1: Plot of $f(\eta) = H[w_-] + H[v_+] - \ln(2\pi e)$ of the ground state and some excited states.

For the case we are considering in equation (20), for no interaction, $C = 0$, implies $\eta = 0$. As mentioned earlier, indeed entanglement occurs for $\eta > \eta_0 \geq 0$. As expected when there is no interaction $(\eta = 0)$, there is no entanglement. A higher value of $|C|$ (and thus η) implies a stronger coupling. For a particular state (given n and m), stronger interactions (higher η values) result in higher degrees of entanglement (more negative $f(\eta)$). Ground state $(n = m = 0)$ interacting oscillators are entangled for $\eta > \eta_0 = 0$. For a given η (given couplings A and C) however, oscillators in excited states ($n > 0$ or $m > 0$) are entangled at a nonzero threshold value of η_0 in equation (37). For example, as given above, for the state $(n = m = 3)$, the threshold value is $\eta_0 = 1.07$. The next higher energy state from the ground state, $n = 1$, $m = 0$ or $n = 0$, $m = 1$ give a threshold of $\eta_0 = 0.270$. Excited states generally have a higher threshold than the ground state because as shown in [29] excited states have higher entropy than the ground state. In addition, excited states will need stronger interactions (corresponding to higher η_0) to lead to entanglement. By using different values of n and m one can check that the value of the threshold η_0 is the same for ψ_{nm} and ψ_{mn}, although these are different states with different eigenvalues as can be seen from equations (11) and (13). In other words, the SEEC is symmetric in the quantum numbers n and m. Figure 2 shows an increase in entropy as shown by an increase in the threshold η_0 and also exhibits the tendency of entanglement at a higher interaction threshold for increasing quantum numbers (n, m).

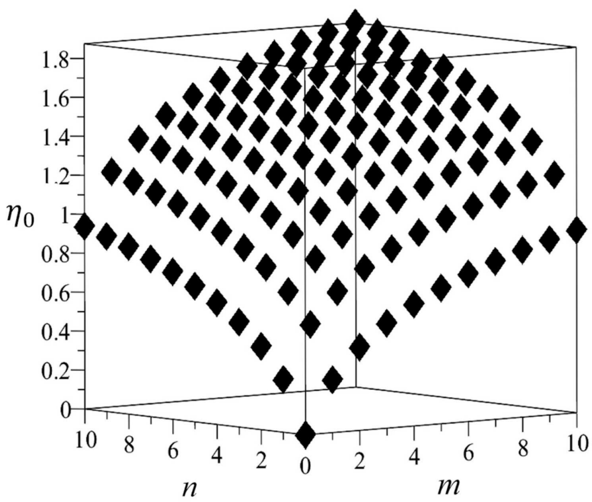

Fig. 2: Plot of the threshold values η_0 given values of n and m.

4 Conclusions

Using the Shannon Entropic Entanglement Criterion (SEEC), we study the entanglement of coupled harmonic oscillators. A simple entanglement criterion is found in terms of the interaction parameter η, given by $f(\eta) = H[w] + H[v] - \ln(2\pi e) = \eta_0 - \eta < 0$ where η_0 is a threshold value depending on the state (n, m). It is found that coupled harmonic oscillators in the ground state have a lower threshold for entanglement than excited states. More interaction is needed for the higher energy states to "share" information and hence entangle due to the increased number of possible states. In addition, the SEEC is shown to be symmetric in the quantum numbers n and m. For a given state, increasing the interaction increases the entanglement as expected since a higher interaction results in more information between the coupled oscillators leading to more entanglement. The SEEC can also be applied to quantum wells and preliminary calculations seem to indicate similar results. However, unlike the SHO, in which the integrals involve well known special functions which can be evaluated and expressed in analytic form, all the integrals involved in quantum wells can only be evaluated numerically [12, 30–32].

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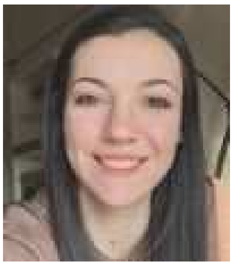

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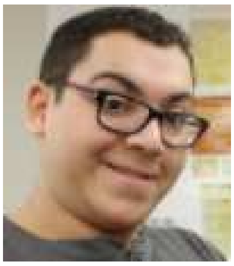

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