

A New Method for Solving Singularly Perturbed Boundary Value Problems

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Abstract: In this paper, a new initial value method for solving a class of nonlinear singularly perturbed boundary value problems with a boundary layer at one end is proposed. The method is designed for the practicing engineer or applied mathematician who needs a practical tool for these problems (easy to use, modest problem preparation and ready computer implementation). Using singular perturbation analysis the method is distinguished by the following fact: the original problem is replaced by a pair of first order initial value problems; namely, a reduced problem and a boundary layer correction problem. These initial value problems are solved using classical fourth order Runge–Kutta method. Numerical examples are given to illustrate the method. It is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems, Two-point boundary-value problems, Boundary layer, Initial-value methods.

1 Introduction

Singularly perturbed boundary value problems (SPBVPs) are common in applied sciences and engineering. They often occur in, for example, fluid dynamics, quantum mechanics, chemical reactions, electrical networks, etc. A well known fact is that the solution of such problems has a multiscale character, i.e. there are thin transition layers where the solution varies very rapidly, while away from the layers the solution behaves regularly and varies slowly. For a detailed discussion on the analytical and numerical treatment of such problems one may refer to the books of O'Malley [1]; Doolan et al. [2]; Roos et al. [3]; and Miller et al. [4]. Numerically, the presence of the perturbation parameter leads to difficulties when classical numerical techniques are used to solve such problems, this is due to the presence of the boundary layers in these problems; see for example O'Malley [1,5]. Even in the case when only the approximate solution is required, finite difference schemes and finite element methods produced unsatisfactory results; see Samarski [6]. It was shown in [7,8] that the results of using classical methods are also unsatisfactory even when a very fine grid is used. Therefore, the numerical treatment of singular

perturbation problems presents some major computational difficulties. In fact, some numerical techniques employed for solving SPBVPs are based on the idea of replacing a two-point boundary-value problem by two suitable initial-value problems. For example, Kadalbajoo and Reddy [9] considered a class of nonlinear SPBVP which was replaced by an asymptotically equivalent first-order problem and was solved as an initial-value problem. Gasparo and Macconi [10] considered a semilinear SPBVP which was integrated to obtain a first-order IVP, and considered both the inner and outer solutions. A similar matching idea combining the reduced problem and a WKB approximation for the full problem has also been employed by Gasparo and Macconi [11,12] for linear, semilinear and quasilinear problems. These matching ideas are based on the method of asymptotic expansions and on the work of Roberts [13] who considered the matching between inner and outer solutions at an unknown location which was determined iteratively, and referred to his method as a boundary value technique. Robert's idea has been extended by Valanarasu and Ramanujam [14] for boundary-value problems of singularly-perturbed systems of ODEs and used

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exponentially-fitted methods for solving the singularly-perturbed initial-value problem.

A similar approach was followed by Reddy and Chakravarthy [15] who considered the full problem in the inner and outer regions, albeit they determined the boundary condition at the matching point from the solution of the reduced problem. Reddy and Chakravarthy [16] presented a method of reduction of order for solving linear and a class of nonlinear SPBVPs. Then Reddy and Chakravarthy [17] presented three initial-value problems instead of the linear second-order SPBVP. Habib and El-Zahar [18] considered a class of nonlinear SPBVP which was replaced by an asymptotically equivalent first order IVP and was solved using locally exact integration. Wang [19] and Attili [33] presented a boundary value method for a class of nonlinear SPBVPs and used series method and Pade' approximation to obtain a series solution.

In this paper, an initial value method which is simple to use and easy to implement is introduced, for solving a class of nonlinear SPBVPs with a boundary layer at one end. Using singular perturbation analysis the method is distinguished by the following fact: the original problem is replaced by a pair of first order initial value problems; namely, a reduced problem and a boundary layer correction problem. These initial value problems are solved using classical fourth order Runge–Kutta method. Numerical examples are given to illustrate the method. Also, the well-known third order Blasius' viscous flow problem is considered for large suction/injection case and an approximate analytic solution is obtained. It is observed that the present method approximates the exact solution very well.

2 The linear problem

Consider the two point singularly perturbed boundary value problem

$$\varepsilon \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0, \quad x \in [a, b], \quad (1)$$

with the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta, \quad (2)$$

where ε is a small positive parameter $0 < \varepsilon \ll 1$ α and β are given constants, $p(x)$ and $q(x)$ are assumed to be sufficiently continuously differentiable functions, and $p(x) \geq M > 0$ for every $x \in [a, b]$ where M is some positive constant. Under these assumptions, (1) has a solution which, in general, displays a boundary layer of width $O(\varepsilon)$ at $x = a$. Equation (1) can be written as

$$\varepsilon \frac{d^2 y}{dx^2} + \frac{d}{dx}(p(x), y) = F(x, y), \quad x \in [a, b], \quad (3)$$

where

$$F(x, y) = p'(x)y - q(x)y.$$

Now, let $u(x)$ be the solution of the reduced problem

$$p(x) \frac{du}{dx} + q(x)u = 0 \quad \text{with} \quad u(b) = \beta. \quad (4)$$

Then an asymptotically approximation to the given Eq. (3) as follows:

$$\varepsilon \frac{d^2 y}{dx^2} + \frac{d}{dx}(p(x)y) = F(x, u) + O(\varepsilon), \quad x \in [a, b], \quad (5)$$

with the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta. \quad (6)$$

By integrating Eq. 5, we obtain

$$\varepsilon \frac{dy}{dx} + p(x)y = \int F(x, u)dx + O(\varepsilon), \quad (7)$$

where

$$\int F(x, u)dx = \int (p'(x)u - q(x)u) dx.$$

Using Eq. (4), we get

$$\int F(x, u)dx = \int (p'(x)u + p(x)u') dx = p(x)u + k.$$

Then Eq. (7) will be

$$\varepsilon \frac{dy}{dx} + p(x)y = p(x)u + k + O(\varepsilon), \quad (8)$$

where k is an integration constant. In order to determine k , we introduce the condition that the reduced equation of (8) should satisfy the boundary condition at $x = b$. Thus we get $k = 0$.

Hence, a first order initial value problem which is asymptotically equivalent to the second order boundary value problem (1) was obtained.

$$\varepsilon \frac{dw}{dx} + p(x)w = p(x)u, \quad (9)$$

with initial condition

$$w(a) = \alpha.$$

Over most of the interval, the solution of Eq. (4) behaves like the solution of (9) but at the end $x = a$, there is a region in which the solution varies greatly from the solution of (9). To portray the solution over this region, we will use the substitution $x - a = \varepsilon t$, the stretching transformation which means $dx = \varepsilon dt$. This transforms (9) into

$$\frac{dw}{dt} + p(a + \varepsilon t)w = p(a + \varepsilon t)u(a + \varepsilon t). \quad (10)$$

Taking $\varepsilon = 0$ in (10) leads to

$$\frac{dw}{dt} + p(a)w = p(a)u(a). \tag{11}$$

If we require the solution to (11) to compensate for the fact that the solution of the reduce problem (4) does not satisfy the boundary condition at $x = a$, and further that this solution goes to zero as $t \rightarrow \infty$, then we obtain the boundary layer correction problem

$$\frac{dW}{dt} + p(a)W = 0 \quad \text{with} \quad W(0) = \alpha - u(a). \tag{12}$$

Then, from standard singular perturbation theory it follows that the solution of (9) admits the representation in terms of the solutions of the reduced and boundary layer correction problems; that is,

$$y(x) = u(x) + W\left(\frac{x-a}{\varepsilon}\right) + O(\varepsilon). \tag{13}$$

Remark 1. The reduced problem (4) and the resulting problem (12) are easily solvable since they are separable and thus (13) results in an asymptotic analytical solution to the original problem (1) given by

$$y(x) = \beta e^{\int_b^x \frac{-q(s)}{p(s)} ds} + (\alpha - u(a))e^{\frac{-p(a)(x-a)}{\varepsilon}} + O(\varepsilon). \tag{14}$$

Note that both the Eq. (4) and Eq. (12) are independent of ε and this means if the problem is nonlinear, as we will discuss in the next section, we can easily get the numerical solutions by method for one-order initial value differential equation such as the Runge-Kutta method.

3 The nonlinear case

Consider the nonlinear singularly perturbed boundary value problem

$$\varepsilon \frac{d^2y}{dx^2} + p(x,y) \frac{dy}{dx} + q(x,y) = 0, \quad x \in [a, b], \tag{15}$$

with boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta. \tag{16}$$

where $0 < \varepsilon \ll 1$, α and β are given constants, $p(x,y)$ and $q(x,y)$ are assumed to be sufficiently continuously differentiable functions, and $p(x,y) \geq M > 0$ for every $x \in [a, b]$.

Again if we set $\varepsilon = 0$ we obtain the reduced problem

$$p(x,u) \frac{du}{dx} + q(x,u) = 0 \quad \text{with} \quad u(b) = \beta. \tag{17}$$

The solution of this problem satisfies (15) on most of the interval $[a, b]$ and away from $x = a$. If this problem is

separable then it can be integrated easily and if not any initial value solver like fourth order Runge–Kutta method will be used to approximate the solution.

Equation (15) can be written as

$$\varepsilon \frac{d^2y}{dx^2} + \frac{df}{dx}(x,y) = G(x,y), \tag{18}$$

where

$$\begin{aligned} \frac{df}{dx}(x,y) &= \frac{\partial f}{\partial x}(x,y) + p(x,y) \frac{dy}{dx}, \\ G(x,y) &= \frac{\partial f}{\partial x}(x,y) - q(x,y). \end{aligned} \tag{19}$$

Also, Eq. (17) can be written as

$$\frac{df}{dx}(x,u) = G(x,u) \quad \text{with} \quad u(b) = \beta, \tag{20}$$

where

$$\begin{aligned} \frac{df}{dx}(x,u) &= \frac{\partial f}{\partial x}(x,u) + p(x,u) \frac{du}{dx}, \\ G(x,u) &= \frac{\partial f}{\partial x}(x,u) - q(x,u). \end{aligned} \tag{21}$$

Subtracting Eq.(20) from Eq.(18) and integrating the resulting equation, we get

$$\begin{aligned} \int_b^x \left(\varepsilon \frac{d^2y(s)}{ds^2} + \frac{df}{ds}(s,y(s)) \right) ds = \\ \int_b^x \left(\frac{df}{ds}(s,u(s)) \right) ds + E(x), \end{aligned} \tag{22}$$

where

$$E = \int_b^x (G(s,y(s)) - G(s,u(s))) ds, \quad y(a) = u(a) = \beta. \tag{23}$$

Thus

$$\varepsilon \frac{dy}{dx} + f(x,y) = f(x,u) + E(x) + K, \quad y(a) = \alpha, \tag{24}$$

where

$$K = \left(\varepsilon \frac{dy}{ds}(s) + f(s,y(s)) - f(s,u(s)) \right)_{s=b} = \varepsilon y'(b).$$

In what follows we construct approximate solution of Eq. (24).

Lemma 3.1. *Let $y(x)$ and $u(x)$ be respectively the solutions of the BVP (15) and the reduced problem (17). Then,*

$$|y(x) - u(x)| \leq C \left(\varepsilon + e^{-M(x-a)/\varepsilon} \right), \quad x \in [a, b]$$

Proof. See (Lorenz [20, Theorem 3]).

Lemma 3.2. Let $y(x)$ be the solution of the BVP (15). Then,

$$|y^{(m)}(x)| \leq C \left(1 + \varepsilon^{-m} e^{-M(x-a)/\varepsilon}\right),$$

$x \in [a, b]$, $m = 0, 1, \dots$

Proof. See (Vulanovic [21, Theorem 1]).

With the help of these lemmas we can prove the following theorem.

Theorem 3.1. Let $y(x)$, $u(x)$ and $w(x)$ be respectively the solutions of the BVP (15), the reduced problem (17) and the following initial-value problem

$$\varepsilon \frac{dw}{dx} + f(x, w) = f(x, u) \quad \text{with} \quad w(a) = \alpha. \quad (25)$$

Then,

$$|y(x) - w(x)| \leq C\varepsilon.$$

Proof. From Eq. (23) and by using Lemma 3.1 we get the following bounded error

$$\begin{aligned} E &= \int_b^x (G(s, y(s)) - G(s, u(s))) ds \\ &\leq \int_b^x \left(\left| \frac{\partial G}{\partial y}(s, \xi) \right| |y(s) - u(s)| \right) ds \\ &\leq C\varepsilon, \end{aligned} \quad (26)$$

where, ξ lies between $y(x)$ and $u(x)$. Let $m = 1$ in Lemma 3.2, we get the following bounded error

$$|K| = |\varepsilon y'(b)| \leq C\varepsilon. \quad (27)$$

Therefore, Eq. (24) becomes

$$\varepsilon \frac{dy}{dx} + f(x, y) = f(x, u) + O(\varepsilon), \quad y(a) = \alpha, \quad (28)$$

To estimate the error involved in the solution $w(x)$ of Eq. (25) we proceed as follows:

Let $z(x) = y(x) - w(x)$. Then $z(x)$ satisfies the following IVP

$$\varepsilon \frac{dz}{dx} + \frac{\partial f}{\partial y}(x, \zeta)z = O(\varepsilon), \quad z(a) = 0, \quad (29)$$

where, ζ lies between $y(x)$ and $w(x)$.

By integrating Eq. (29) it can be shown that

$$|z(x)| = |y(x) - w(x)| \leq C\varepsilon.$$

The proof of Theorem 3.1 is completed.

Thus, by Theorem 3.1 the solution of the two-point boundary value problem (15) is approximated by that of the first order initial value problem (25).

Close to the boundary layer problem, we use as before the substitution $x - a = \varepsilon t$, which transforms (25) into

$$\frac{dw}{dt} + f(a + \varepsilon t, w) = f(a + \varepsilon t, u(a + \varepsilon t)). \quad (30)$$

Setting $\varepsilon = 0$ and remembering that we require the solution to (30) to compensate for the fact that the solution of the reduce problem (20) does not satisfy the boundary condition at $x = a$, and further that this solution goes to zero as $t \rightarrow \infty$, then we obtain the boundary layer correction problem

$$\frac{dW}{dt} + f(a, W + u(a)) = f(a, u(a)), \quad W(0) = \alpha - u(a). \quad (31)$$

As a result the solution to the original problem (15) will be a combination of the reduced and the boundary layer correction problems; that is,

$$y(x) = u(x) + W\left(\frac{x-a}{\varepsilon}\right) + O(\varepsilon). \quad (32)$$

4 The error analysis

The numerical error of the present method has two sources: one from the asymptotic approximation and the other from the numerical approximation. Let h_o and h_{in} be the mesh spacing on the non-boundary layer and on the boundary layer domain respectively.

4.1 Error on the non-boundary layer domain

Let y be the exact solution of the original problem, u be the exact solution of the reduced problem, and u_N be the numerical solution of the reduced problem. Assume u_N is obtained from the fourth-order Runge-Kutta method. On the non-boundary layer domain, the error is

$$\|u - u_N\| = \max_{i=1 \dots N-1} \{|u(x_i) - u_N(x_i)|\}.$$

By the triangle inequality, we conclude

$$\|y - u_N\| \leq \|y - u\| + \|u - u_N\| = O(\varepsilon) + O(h_o^4).$$

In more times, the exact solution of the reduced problem can be easily obtained and the second term of the above error inequality is vanished.

4.2 Error on the boundary layer domain

On the boundary layer domain, the asymptotic approximation error is generated from the reduction of order method and the numerical approximation error from the numerical methods.

Let W be the exact solution of the Eq. (31), and W_N be the numerical solution of Eq. (31). Assume W_N is obtained from the fourth-order Runge-Kutta method. On the boundary layer domain, the error is

$$\|y - (W_N + u_N)\| \leq \|y - (W + u)\| + \|(W + u) - (W_N + u_N)\| = O(\epsilon) + O(h_{in}^4).$$

The new method works well for singular perturbation problems since the singular perturbation parameter ϵ is extremely small.

5 Numerical results

To demonstrate the applicability of the method we have applied it on three nonlinear singular perturbation problems. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. Also, the well-known third order Blasius' viscous flow problem is considered for large suction/injection case and an approximate analytic solution is obtained.

Example 5.1. Consider the second-order nonlinear equation

$$\epsilon \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + e^y = 0; x \in [0, 1], \tag{33}$$

where $y(0) = y(1) = 0$. The reduced problem is

$$2 \frac{du}{dx} + e^u = 0,$$

with $u(1) = 0$, which has the solution $u(x) = -\ln\left(\frac{x+1}{2}\right)$. Hence, the corresponding initial value problem is given by

$$\epsilon \frac{dw}{dx} + 2w = -2\ln\left(\frac{x+1}{2}\right); \quad w(0) = 0.$$

And the boundary layer correction problem is given by

$$\frac{dW}{dt} + 2(W + \ln 2) = 2\ln 2; W(0) = \ln\left(\frac{1}{2}\right),$$

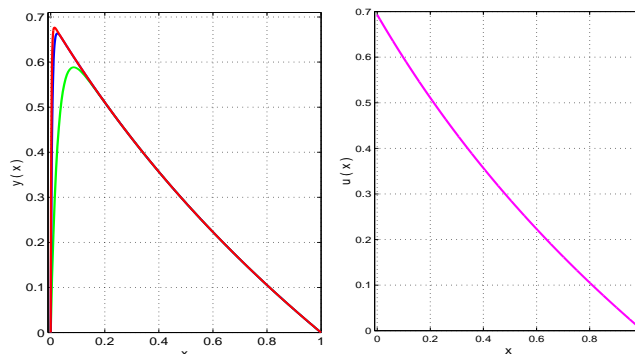
or simply

$$\frac{dW}{dt} + 2W = 0; W(0) = \ln\left(\frac{1}{2}\right). \tag{34}$$

The solution is $W(t) = \ln\left(\frac{1}{2}\right) e^{-2t}$ with $t = \frac{x}{\epsilon}$. Hence the solution of (33) is approximated by

$$y(x) = -\ln\left(\frac{x+1}{2}\right) + \ln\left(\frac{1}{2}\right) e^{-\frac{2x}{\epsilon}}, \tag{35}$$

which is the asymptotic approximate solution obtained by Bender and Orszag [22]. As shown from the previous example, the present method offers a relatively simple and easy tool for obtaining asymptotic approximate analytical solution for singular perturbation problems.



From left to right $\epsilon = 0.005, 0.01, 0.05$ The reduced solution

Fig. 5.1: Graphs of the solutions of Example 5.1 and its reduced problem.

Example 5.2. See Kevorkian and Cole [23], and Roberts [24]. The quasilinear problem

$$\epsilon \frac{d^2y}{dx^2} + y \frac{dy}{dx} = y, \tag{36}$$

where $y(0) = A, y(1) = B, 0 \leq |A| < B - 1$. The problem has a uniformly valid approximation for comparison [23]

$$y = x + (B - 1) \tanh\left(\frac{(B - 1)(x/\epsilon)}{2} + (1/2) \log\left(\frac{(B - 1 + A)}{(B - 1 - A)}\right)\right).$$

The reduced problem is $u \frac{du}{dx} - u = 0$ with $u(1) = B$ which has the solution $u(x) = x + \frac{B}{2} - 1$. Hence, the corresponding initial value problem is given by

$$\epsilon \frac{dw}{dx} + \frac{1}{2} w^2 = \frac{1}{2} (x + B - 1)^2, \quad w(0) = A.$$

And the boundary layer correction problem is given by

$$\frac{dW}{dt} + \frac{1}{2} (W + B - 1)^2 = \frac{1}{2} (B - 1)^2, \tag{37}$$

where

$$w(0) = A - B + 1.$$

The results for $A = -1$ and $B = 3.9995$ are presented in Fig. 5.2 for $\epsilon = 0.01$ and in Fig. 5.3 for $\epsilon = 0.0001$ using classical fourth order Rung-Kutta method at $h_{in} = 0.1$ over the boundary region $[0, t_p]$, where $|W(t_p)| < \delta$, and δ is a user specified tolerance. In our computations we take $\delta = 10^{-4}$.

We compare the numerical results of the present method with different values of the step size h_{in} (different number of grid points, N) as shown in Table 5.1. The numerical results indicate that as the step size decreases, the error decreases. Moreover, with a constant number of mesh points, the numerical error is maintained at the same level for a family of singular perturbation problems.

Table 5.1: Maximal error comparison with different values of the step size h_{in} for Example 5.2.

$A = -1, B = 3.9995$		Maximal error			
		$\epsilon = 10^{-3}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-8}$	$\epsilon = 10^{-10}$
$h_{in} = 0.1$	$N = 40$	2.7457e-005	2.7457e-005	2.7457e-005	2.7457e-005
$h_{in} = 0.2$	$N = 21$	5.0144e-004	5.0144e-004	5.0144e-004	5.0144e-004
$h_{in} = 0.3$	$N = 15$	3.0000e-003	3.0000e-003	3.0000e-003	3.0000e-003
$h_{in} = 0.4$	$N = 12$	1.1300e-002	1.1300e-002	1.1300e-002	1.1300e-002
$h_{in} = 0.5$	$N = 10$	2.8800e-002	2.8800e-002	2.8800e-002	2.8800e-002

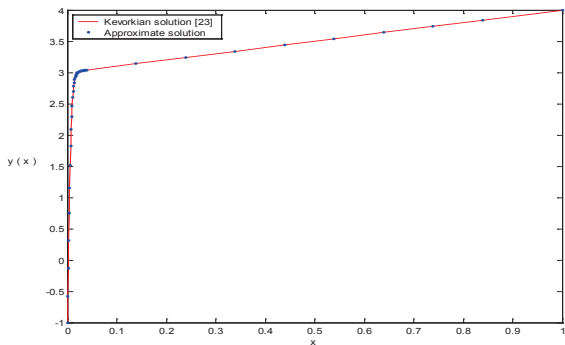


Fig. 5.2: Keivorkian solution [23] and the obtained approximate solution of Example 5.2 at $\epsilon = 0.01$.

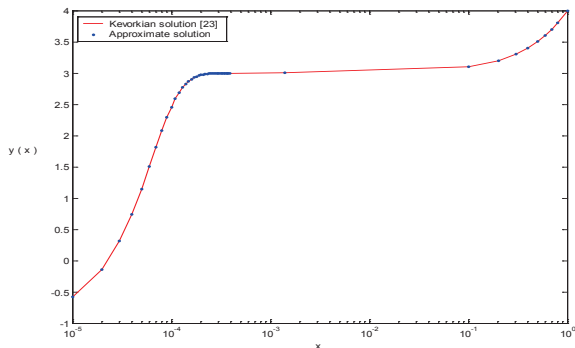


Fig. 5.3: Keivorkian solution [23] and the obtained approximate solution of Example 5.2 at $\epsilon = 0.0001$.

The numerical results of the present method and the initial value method in [12] are compared for different boundary condition values. As shown in Table 5.2, the present method gives more accurate results.

Example 5.3. Consider the second-order nonlinear example from O'Malley [5]

$$\epsilon \frac{d^2y}{dx^2} + e^y \frac{dy}{dx} - \frac{\pi}{2} \sin\left(\frac{\pi x}{2}\right) e^{2y} = 0; \quad (38)$$

$$y(0) = y(1) = 0.$$

The problem has a uniformly valid approximation for comparison [5]

$$y = -\ln\left[(1 + \cos(\pi x/2))(1 - 0.5e^{-x/2\epsilon})\right].$$

The reduced problem is $\frac{du}{dx} - \frac{\pi}{2} \sin\left(\frac{\pi x}{2}\right) e^u = 0$, with $u(1) = 0$, which has the solution $u(x) = -\ln(1 + \cos(\pi x/2))$. Hence, the corresponding initial value problem is given by

$$\epsilon \frac{dw}{dx} + e^w = \left(\frac{1}{1 + \cos(\pi x/2)}\right); w(0) = 0.$$

And the boundary layer correction problem is given by

$$\frac{dW}{dt} + \frac{1}{2}e^W = \frac{1}{2}; \quad W(0) = \ln 2. \quad (39)$$

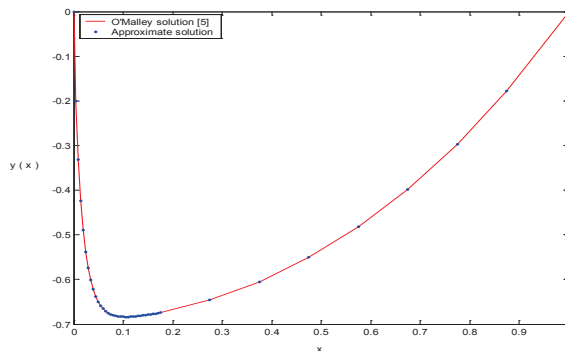


Fig. 5.4: O'Malley solution [5] and the obtained approximate solution of Example 5.3 at $\epsilon = 0.01$.

Table 5.2: Maximal error comparison when $\epsilon = 10^{-12}$ for Example 5.2.

Method	Boundary conditions		
	$A = 2, B = 6$	$A = -2, B = 6$	$A = -1, B = 3.9995$
Gasparo and Maconi [12]	8.e-001	1.e-001	3.e-001
The present method	4.e-004	4.e-004	3.e-005

Table 5.3: Maximal error comparison with different values of the step size h_{in} for Example 5.3.

		Maximal error			
		$\epsilon = 10^{-3}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-8}$	$\epsilon = 10^{-10}$
$h_{in} = 0.1$	$N = 172$	3.8347e-008	3.8347e-008	3.8347e-008	3.8347e-008
$h_{in} = 0.2$	$N = 87$	6.3420e-007	6.3420e-007	6.3420e-007	6.3420e-007
$h_{in} = 0.3$	$N = 58$	3.2899e-006	3.2899e-006	3.2899e-006	3.2899e-006
$h_{in} = 0.4$	$N = 44$	1.0700e-005	1.0700e-005	1.0700e-005	1.0700e-005
$h_{in} = 0.5$	$N = 36$	2.6786e-005	2.6786e-005	2.6786e-005	2.6786e-005

Table 5.4: Maximal error comparison when $h_{in} = 0.1$ for Example 5.3.

Method	$\epsilon = 10^{-3}$	$\epsilon = 10^{-5}$
Iterative method by Gasparo and Maconi [11]	5.962e-002	5.962e-002
Iterative method by Jayakumar and Ramanujam [25]	4.869e-003	7.584e-004
The present method with third order Rung kutta method	6.504e-006	6.504e-006
The present method with fourth order Rung kutta method	3.834e-008	3.834e-008

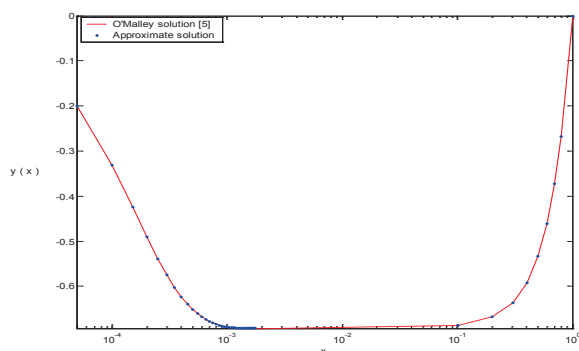


Fig. 5.5: O'Malley solution [5] and the obtained approximate solution of Example 5.3 at $\epsilon = 0.0001$.

We compare the present method with the iterative methods using Newton's method of quasilinearization from [11,25]. In [11] the approximate solution of the resultant linear problems is obtained by solving two initial-value problems using the variable step size initial-value integrator LSODA with local error tolerances 10^{-12} . In [25] the resultant linear problems are solved by the numerical method suggested in [26] which is a combination of an exponentially-fitted finite difference method and a classical numerical method.

As shown in Table 5.4, the present method gives more accurate results compared to the methods in [11,25]. Table 5.3 and Table 5.4 show the advantage of the present method in using high order methods for one-order initial value differential equation such as the Runge-Kutta

methods to obtain more accurate results without any new restriction on the step size.

Note that, the boundary layer correction problem (39) has a solution $W(t) = -\ln(-0.5 + e^{t/2})$ with $t = \frac{x}{\epsilon}$ and hence an approximate analytical solution can be obtained and given by

$$y(x) = -\ln(1 + \cos(\pi x/2)) - \ln(-0.5 + e^{x/2\epsilon}).$$

Example 5.4. Consider the well-known third order Blasius Equation in the following general form

$$\begin{aligned} f'''(\eta) + f(\eta)f''(\eta) &= 0; \\ \eta &\in [0, \infty), \\ f(0) = -\alpha, f'(0) = -\beta, \lim_{\eta \rightarrow \infty} f'(\eta) &= \gamma, \end{aligned} \tag{40}$$

where α, β and γ are constants. According to Guedda [27], in the event of $f(0) = -\alpha$, α represents a suction/injection parameter where $-\alpha > 0$ represents suction and $-\alpha < 0$ corresponds to injection of the fluid. The initial condition, $f'(0) = -\beta$, indicates the slip condition at the wall [28]. The case where $\beta = 0$ represents no-slip. In practice we can find numerical solutions only on a finite interval. For this reason, we introduce a one-parameter family of problems related to Blasius problem on the finite interval $(0, L)$ where the length L of the interval is taken as the parameter of the family. The typical problem in this family is defined for each value of L in the range $1 \leq L \leq \infty$ by

$$\begin{aligned} f_L'''(\eta) + f_L(\eta)f_L''(\eta) &= 0; \\ \eta &\in [0, L), \\ f_L(0) = -\alpha, f_L'(0) = -\beta, f_L'(L) &= \gamma. \end{aligned} \tag{41}$$

When we reformulate (41) as a singularly perturbed problem, it transpires that $\frac{1}{L}$ is the singular perturbation parameter and so it is appropriate to introduce the temporary notation $\delta = \frac{1}{L}$. Then the problem (41) can be written in the form

$$\begin{aligned} f_\delta'''(\eta) + f_\delta(\eta)f_\delta''(\eta) &= 0; \\ f_\delta(0) &= -\alpha, f_\delta(1) = \gamma, \end{aligned} \tag{42}$$

Putting $g(\eta) = f_\delta'(\eta)$, this problem becomes

$$g''(\eta) + f_\delta(\eta)g'(\eta) = 0; \quad g(0) = -\beta, g\left(\frac{1}{\delta}\right) = \gamma, \tag{43}$$

Changing variables from η to $x = \delta\eta$, and writing $y(x) = g(\eta)$ and $h(x) = f_\delta(\eta)$, we obtain

$$\varepsilon y''(x) + p(x)y'(x) = 0; y(0) = -\beta, y(1) = \gamma, \tag{44}$$

where $\varepsilon = -\delta/\alpha$, $p(x) = -h(x)/\alpha$ and $p(0) = 1$. For moderate-to-large values of suction/injection parameter α , Eq. (44) is a singularly perturbed problem for $y(x)$ with a boundary layer of width $O(\varepsilon)$ at $x = 0$ or $x = 1$.

-Solution for large suction case

The reduced problem of (44) is given by $p(x)u'(x) = 0$, with $u(1) = \gamma$ which has the solution $u(x) = \gamma$. Hence, the corresponding initial value problem is given by

$$\varepsilon \frac{dw}{dx} + p(x)w = p(x); w(0) = -\beta.$$

And the boundary layer correction problem is given by

$$\frac{dW}{dt} + p(0)(W + \gamma) = p(0)\gamma; W(0) = -\beta - \gamma,$$

or simply

$$\frac{dW}{dt} + W = 0; W(0) = -\beta - \gamma. \tag{45}$$

The solution is $W(t) = -(\beta + \gamma)e^{-t}$ with $t = \frac{x}{\varepsilon}$. Hence the solution of (44) is

$$y(x) = f_\delta'(x) = \gamma - (\beta + \gamma)e^{-\alpha x}, \quad f_\delta(0) = -\alpha, \tag{46}$$

which results in

$$f_\delta(\eta) = -\alpha + \gamma\eta + \frac{\beta + \gamma}{\alpha}(1 - e^{\alpha\eta}), \quad \alpha \neq 0, \tag{47}$$

$$f_\delta''(\eta) = -\alpha(\beta + \gamma)e^{\alpha\eta}. \tag{48}$$

Thus we have obtained an asymptotic approximate solution for Blasius' viscous flow problem for large suction case.

In the following figures, we compare the obtained numerical solutions of the problem (40), in MATLAB environment using the function `bvp4c`

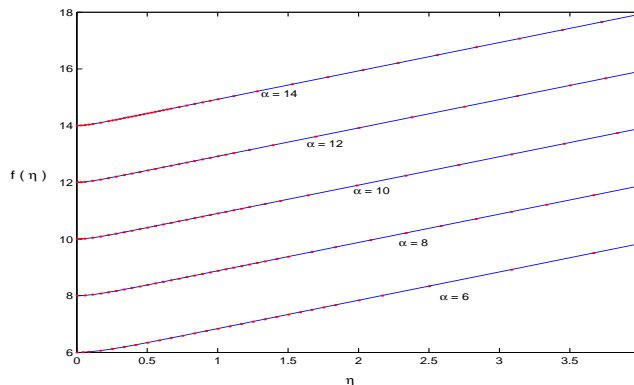


Fig. 5.6: Solution comparison, numerical solution (dotted lines) and Eq. (47) solution (solid lines) at $\beta = 0, \gamma = 1$.

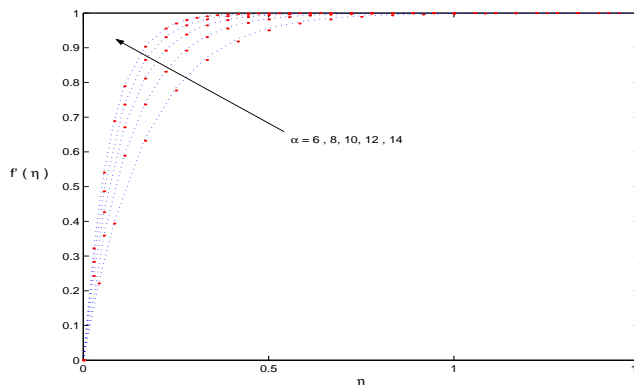


Fig. 5.7: Solution comparison, numerical solution (dotted lines) and Eq. (46) solution (dashed lines) at $\beta = 0, \gamma = 1$.

($atol = 10^{-6}, rtol = 10^{-3}$), with the results from Eqs. (46) to (48) at $\alpha \neq 0, \beta = 0, \gamma = 1$.

According to Weyl [29], “the value $[f''(0)]$ is the essential factor in the formula for the skin friction along the immersed plate”. Due to its importance, we compare the results of the present method and Adomian Decomposition Method (ADM) in [30]. As shown in Table 5.5, the present method gives more accurate results compared to the results in [30] for large suction case. Moreover, as the suction value increases, the numerical error decreases.

Fig 5.9 presents a comparison of the results obtained on the differential analyser, at Manchester University [31] at $\alpha = 2\sigma, \sigma = -5, -10, \beta = 0, \gamma = 2$, and the results obtained by the present method to show the effectiveness of the present method in approximating the solution of Blasius problem for large suction case.

Table 5.5: The present method and ADM values of $f''(0)$ for $\alpha \neq 0, \beta = 0, \gamma = 1$.

α	Numerical (bvp4c)	ADM [30]	Relative error %	present method	Relative error %
-1.5	1.7319	2.50943	44.9	1.5	13.4
-2.0	2.1945	3.00028	36.7	2.0	08.9
-2.5	2.6666	1.49962	43.8	2.5	06.2
-3.0	3.1451	-	-	3	04.6
-6.0	6.0799	-	-	6	01.3
-12	12.0412	-	-	12	00.3

The sign '-' means not available.

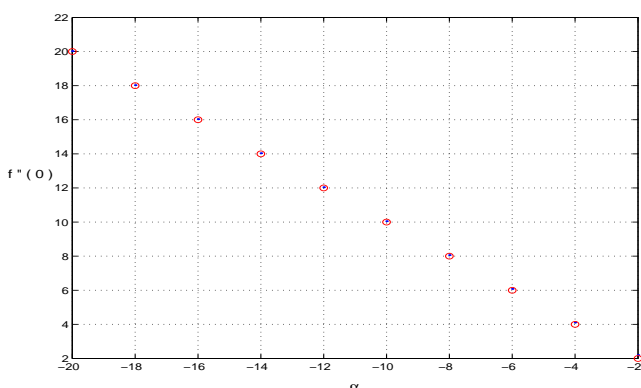


Fig. 5.8: Solution values of $f''(0)$, numerical solution (dotted) and Eq. (48) solution (circles) at $\beta = 0, \gamma = 1$.

6 Extension to higher order problems

The present method can be applied on problems of higher order, for example, on fourth-order singularly perturbed boundary value problems. A model example which can be solved is [32]

$$-\epsilon y^{iv} - a(x)y''' + b(x)y'' - c(x)y = -f(x), \quad (49)$$

$$y(0) = p, y(1) = q, y''(0) = -r, y''(1) = -s. \quad (50)$$

The fourth order problem (49) and (50) will be transformed into a system of weakly coupled system of two second order ODEs, one without the parameter and the other with the parameter ϵ , multiplying the highest derivative, and with their suitable boundary conditions as follows

$$y'' = z, y(0) = p, y(1) = q, \quad (51)$$

$$\epsilon z'' + a(x)z' - b(x)z + c(x)y = f(x), z(0) = -r, z(1) = -s.$$

The reduced system of (51) is given by

$$y_0'' = z_0, y_0(0) = p, y_0(1) = q, \quad (52)$$

$$a(x)z_0' - b(x)z_0 + c(x)y_0 = f(x), z_0(1) = -s.$$

Using Remark 1 (in section 2 for linear problems), we have

$$z = z_0 + \left((-r - z_0(0))e^{-\frac{a(0)x}{\epsilon}} \right) + O(\epsilon). \quad (53)$$

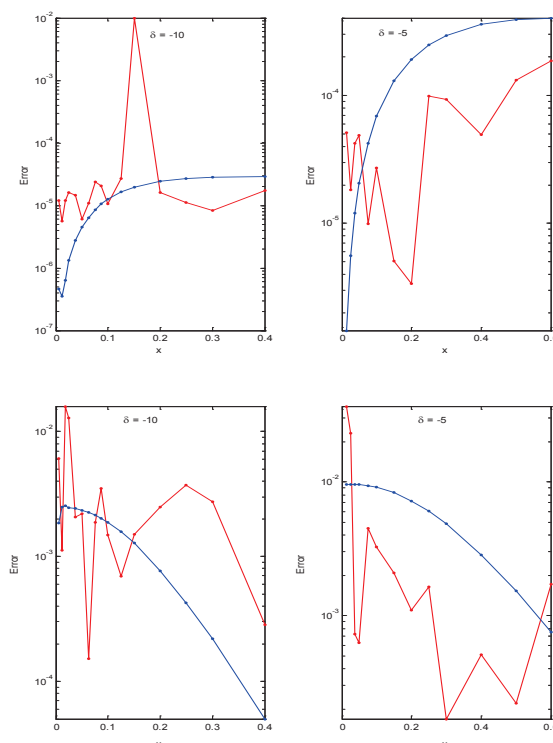


Fig. 5.9: Relative error comparison between, Eq. (46) solution (blue line) and the solution in [31] (red line).

Integrating (53) twice with the boundary conditions $y(0) = p, y(1) = q$, we get an approximate solution for y . For example, consider the fourth order singularly perturbed problem [32]

$$-\epsilon \frac{d^4 y}{dx^4} - 4 \frac{d^3 y}{dx^3} = 1, x \in (0, 1), \quad (54)$$

where $y(0) = y(1) = 1, y''(0) = y''(1) = -1$. The problem has an exact solution given by

$$y =$$

$$\left\{ 1 + \frac{\varepsilon^2}{64(1-e^{-4/\varepsilon})} + \left[\frac{1}{24} - \frac{\varepsilon^2}{64} - \frac{(4(1-e^{-4/\varepsilon})+1)}{(8(1-e^{-4/\varepsilon}))} \right] x \right. \\ \left. + \left[\frac{4(1-e^{-4/\varepsilon})+1}{(8(1-e^{-4/\varepsilon}))} \right] x^2 - \frac{\varepsilon^2 e^{-4x/\varepsilon}}{64(1-e^{-4/\varepsilon})} - \frac{x^3}{24} \right\} \\ + \left\{ 1 + \frac{(4(1-e^{-4/\varepsilon})+1)}{(4(1-e^{-4/\varepsilon}))} + \frac{e^{-4x/\varepsilon}}{4(1-e^{-4/\varepsilon})} - \frac{x}{4} \right\}.$$

The equivalent system of problem (54) is given by

$$y'' = z, \quad y(0) = y(1) = 1, \\ \varepsilon z'' + 4z' = -1, \quad z(0) = z(1) = -1. \quad (55)$$

The reduced system of (55) is given by

$$y_0'' = z_0, \quad y_0(0) = y_0(1) = 1, \\ 4z_0 = -1, \quad z_0(1) = -1. \quad (56)$$

which has a solution given by

$$y_0 = -\frac{x^3}{24} - \frac{3x^2}{8} + \frac{5x}{12} + 1, \\ z_0 = -\frac{x}{4} - \frac{3}{4}. \quad (57)$$

Using Remark 1, we have

$$z = -\frac{x}{4} - \frac{3}{4} - \frac{1}{4}e^{-4x/\varepsilon}. \quad (58)$$

And thus, we have

$$y'' = -\frac{x}{4} - \frac{3}{4} - \frac{1}{4}e^{-4x/\varepsilon}, \quad y(0) = y(1) = 1, \quad (59)$$

which results in

$$y = -\frac{x^3}{24} - \frac{3x^2}{8} + \frac{5x}{12} + 1 + \frac{\varepsilon^2}{64}(1 - e^{-4x/\varepsilon} - x(1 - e^{-4/\varepsilon})). \quad (60)$$

The results obtained using Eqs. (58) and (60) compare very well with the exact solutions.

7 Conclusions

In this article, an initial value method is presented for solving quasilinear singularly perturbed boundary value problems with a boundary layer at one end. The method is similar in some respect to the asymptotic expansion methods, but differs in detail. The method differs in how we use the available data. The solution of the given problem is computed numerically by solving two initial value problems easily deduced from the original problem. The two initial value problems, the reduced problem and the boundary layer correction problem, are independent of the perturbation parameter ε and therefore we get easily the numerical solution using classical fourth order Runge–Kutta method. In fact, any standard asymptotic or

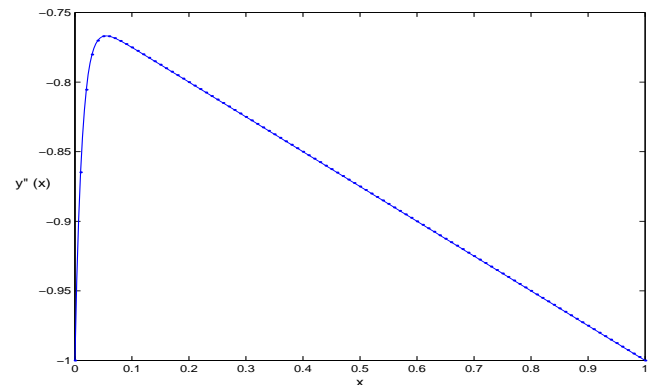


Fig. 6.1: Solution comparison, exact derivative solution (solid line) and Eq.(58) solution (dotted line) at $\varepsilon = 0.05$.

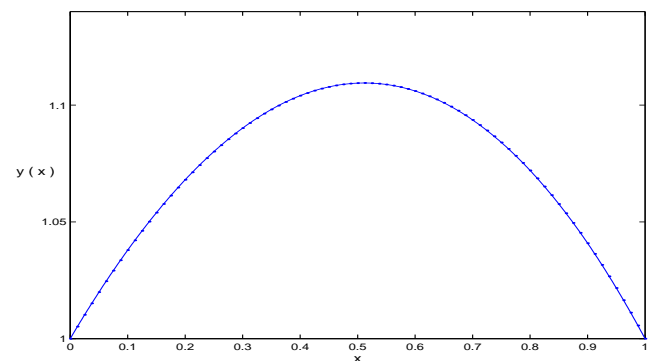


Fig. 6.2: Solution comparison, exact solution (solid line) and Eq. (60) solution (dotted line) at $\varepsilon = 0.05$.

numerical method for first order ODEs can be used. The method is simple to use, very easy to implement on any computer with minimum problem preparation and offers a relatively simple tool for obtaining approximate asymptotic solution for singular perturbation problems. We have implemented it on three nonlinear problems by taking different values of ε and have presented the computational results as well as the results obtained by other methods, in figures and tables. Moreover, we have applied the method on the well-known third order Blasius' viscous flow problem and have obtained approximate asymptotic solutions for large suction case. The method can be applied to higher order singular perturbation problems, possibly easier than other approaches. It can be observed from the tables and figures that the present method approximates the solution of singular perturbation problems very well.

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