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On the Existence Results of a Coupled System of Generalized Katugampola Fractional Differential Equations

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Abstract: This paper is mainly devoted to investigate the existence of solutions to a coupled system of fractional differential equations involving generalized Katugampola derivative with non local initial conditions. The existence results are carried out by using some standard fixed point theorem techniques. A suitable example is also provided to illustrate the applications on our main results.

Keywords: Fractional differential equation, coupled system, generalized Katugampola derivative, nonlocal initial value problem, existence.

1 Introduction

In the last few decades, Fractional calculus remains one of the most important and popular subject in many mathematical research works. Certainly, the fractional order differential equations (FDEs) have been extensively studied by many authors in theoretical form as well as in applications. Fractional differential equations have been proved to be an excellent tool in various fields of Physics, Engineering, Bio-Engineering, Control theory, Material viscoelastic theory, Aerodynamics and other applied sciences, see [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16] and references therein.

Since coupled systems of integer or fractional order differential equations are useful to obtain the mathematical modeling of physical phenomena, many authors have started paying attention towards the study of such problems. For some recent development in the study of coupled system of FDEs, one can see [17,18,19,20,21,22] and references therein.

Being inspired by the aforementioned work, this paper establishes existence results for the solutions of a coupled system of fractional differential equations involving the generalized Katugampola derivative:

$$ {}^{\rho}D_{a^+}^{\mu_1,\nu_1}u(t) = f_1(t,u(t),v(t)), \tag{1}$$

$$ {}^{\rho}D_{a^+}^{\mu_2,\nu_2}v(t) = f_2(t,u(t),v(t)), \tag{2}$$

with the following nonlocal initial conditions

$$ {}^{\rho}I_{a^+}^{1-\beta_1}u(a) = \sum_{i=1}^{m}\lambda_{1i}u(\omega_i), \tag{3}$$

$$ {}^{\rho}I_{a^+}^{1-\beta_2}v(a) = \sum_{i=1}^{m}\lambda_{2i}v(\omega_i), \tag{4}$$

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where $\rho > 0$, $\mu_j \in (0,1), \nu_j \in [0,1]$ with $\mu_j \leq \beta_j = \mu_j + \nu_j - \mu_j\nu_j \leq 1$, $j = 1,2$ and $\beta_1 \leq \beta_2$. $f_j : (a,b] \times E \times E \to E$, $j = 1,2$ are given functions, where $(E, \|\cdot\|_E)$ are real Banach spaces. The operators ${}^{\rho}D_{a^+}^{\mu_j,\nu_j}$, $j = 1,2$ are the so called generalized Katugampola fractional derivatives of order μ_j and type $\nu_j, j = 1,2$ and the operators ${}^{\rho}I_{a^+}^{1-\beta_j}$ are the Katugampola fractional integrals of order $1-\beta_j$, $j = 1,2$ with $a > 0$. $\omega_i \in (a,b]$, $i = 1,2,...,m$ are prefixed points such that $a < \omega_1 \leq \omega_2 \leq ... \leq \omega_m \leq b$. $\lambda_{1i} \geq 0$ and $\lambda_{2i} \geq 0$ for all $i = 1,2,...,m$.

We used Krasnoselskii fixed point theorem to establish our main existence results for the coupled system (1) – (4).

2 Preliminary results

Let $0 < a < b < \infty$ be a finite interval on $\mathbb{R}^+$ and let $C[a,b]$ be the Banach space of all continuous functions $h : [a,b] \to \mathbb{R}$ with associated norm

$$\|h\|_C = \max\{|h(t)| : t \in [a,b]\}.$$

For $0 \leq \beta_j \leq 1$, $j = 1,2$ and the parameter $\rho > 0$, we indicate the weighted space of continuous functions h on $(a,b]$ by

$$C_{\beta_j,\rho}[a,b] = \left\{ h : (a,b] \to \mathbb{R} : \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\beta_j} h(t) \in C[a,b] \right\}, \; j = 1,2$$

with associated norm

$$\|h\|_{C_{\beta_j,\rho}} = \left\| \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\beta_j} h(t) \right\|_C = \max_{t\in[a,b]} \left| \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\beta_j} h(t) \right|. \tag{5}$$

Observe that $C_{0,\rho}[a,b] = C[a,b]$, for any $\rho > 0$.

Let $\rho > 0$ and consider the parameters $\mu_j, \nu_j, \alpha_j, \beta_j$ such that $\beta_j = \mu_j + \nu_j - \mu_j\nu_j$ for $0 < \mu_j, \nu_j, \beta_j < 1$ and $0 \leq \alpha_j < 1$, $j = 1,2$.

Moreover, We define the following weighted spaces

$$C_{1-\beta_j,\alpha_j}^{\mu_j,\nu_j}[a,b] = \left\{ g \in C_{1-\beta_j,\rho}[a,b],\; {}^{\rho}D_{a^+}^{\mu_j,\nu_j} g \in C_{\alpha_j,\rho}[a,b] \right\}$$

and

$$C_{1-\beta_j,\rho}^{\beta_j}[a,b] = \left\{ g \in C_{1-\beta_j,\rho}[a,b],\; {}^{\rho}D_{a^+}^{\beta_j} g \in C_{1-\beta_j,\rho}[a,b] \right\},$$

with the norm given in (5).

Next, we consider the product weighted space $C_{1-\beta_1}^{\mu_1,\nu_1} \times C_{1-\beta_2}^{\mu_2,\nu_2}$ with the norm

$$\|(u,v)\|_{C_{1-\beta_1}^{\mu_1,\nu_1} \times C_{1-\beta_2}^{\mu_2,\nu_2}} = \|u\|_{C_{1-\beta_1}^{\mu_1,\nu_1}} + \|v\|_{C_{1-\beta_2}^{\mu_2,\nu_2}}.$$

As in [11], for $c \in \mathbb{R}$ and $1 \leq p \leq \infty$ consider the space $Z_c^p(a,b)$ of those complex valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{Z_c^p} < \infty$, where

$$\|f\|_{Z_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty, \quad c \in \mathbb{R}, 1 \leq p < \infty$$

and for $p = \infty$

$$\|f\|_{Z_c^{\infty}} = ess \sup_{t\in[a,b]} [t^c |f(t)|], \quad (c \in \mathbb{R}).$$

Definition 1.[8] *(Katugampola fractional integral) Let $\mu, a, b, c \in \mathbb{R}$ with $\mu > 0$ and $0 < a < b$. Let $u \in Z_c^p(a,b)$. The left-sided Katugampola fractional integral of order μ is defined by the formula*

$$\left({}^{\rho}I_{a^+}^{\mu} u\right)(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_a^t \frac{x^{\rho-1} u(x)}{(t^{\rho} - x^{\rho})^{1-\mu}} dx, \quad (t > a), \tag{6}$$

where $\Gamma(\cdot)$ is the Euler's Gamma function.

Definition 2.[9] *(Katugampola fractional derivative) Let $\mu,\rho\in\mathbb{R}$ be such that $\mu,\rho>0$, $\mu\notin\mathbb{N}$ and let $n=[\mu]+1$, where $[\mu]$ is the integer part of μ. The left-sided Katugampola fractional derivative of order α is defined by*

$$\left({}^{\rho}D^{\mu}_{a^+}u\right)(t)=\delta^n_{\rho}\left({}^{\rho}I^{n-\mu}_{a^+}u\right)(t)=\frac{\rho^{1-n+\mu}}{\Gamma(n-\mu)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int\limits_a^t\frac{x^{\rho-1}u(x)}{(t^{\rho}-x^{\rho})^{1-n+\mu}}dx, \tag{7}$$

provided that the integral exists and with $\delta^n_{\rho}=\left(t^{1-\rho}\frac{d}{dt}\right)^n$.

Definition 3.[13] *(Generalized Katugampola fractional derivative) Let the order μ and the type ν be such that $0<\mu\leq 1$ and $0\leq\nu\leq 1$. The generalized Katugampola fractional derivative with respect to t with $\rho>0$ and $u\in C_{1-\beta,\rho}[0,1]$ is defined by the formula*

$$\left({}^{\rho}D^{\mu,\nu}_{a^+}u\right)(t)=\left\{{}^{\rho}I^{\nu(1-\mu)}_{a^+}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}I^{(1-\nu)(1-\mu)}_{a^+}u\right\}(t)=\left\{{}^{\rho}I^{\nu(1-\mu)}_{a^+}\delta_{\rho}\,{}^{\rho}I^{(1-\nu)(1-\mu)}_{a^+}u\right\}(t), \tag{8}$$

where ${}^{\rho}I^{\alpha}_{a^+}$ is the generalized fractional integral defined in (6).

Remark.[13] For $\beta=\mu+\nu-\mu\nu$ the generalized Katugampola fractional derivative operator ${}^{\rho}D^{\mu,\nu}_{a^+}$ can be expressed as

$${}^{\rho}D^{\mu,\nu}_{a^+}={}^{\rho}I^{\nu(1-\mu)}_{a^+}\cdot\delta_{\rho}\cdot{}^{\rho}I^{1-\beta}_{a^+}={}^{\rho}I^{\nu(1-\mu)}_{a^+}\cdot{}^{\rho}D^{\beta}_{a^+}. \tag{9}$$

Lemma 1.[13] *If $\mu>0, 0\leq\beta<1$ and $u\in C_{\beta,\rho}[a,b]$, then*

$$\left({}^{\rho}D^{\mu}_{a^+}\,{}^{\rho}I^{\mu}_{a^+}u\right)(t)=u(t),\quad for\,all\;\;t\in(a,b].$$

Lemma 2.[13] *(Semigroup property) Let $\mu>0,\nu>0,1\leq q\leq\infty$. Let $a,b\in(0,\infty)$ be such that $a<b$ and let $\rho,c\in\mathbb{R}$, with $\rho\geq c$. Then for any $u\in Z^q_c(a,b)$ the following property holds:*

$$\left({}^{\rho}I^{\mu}_{a^+}\,{}^{\rho}I^{\nu}_{a^+}u\right)(t)=\left({}^{\rho}I^{\mu+\nu}_{a^+}u\right)(t).$$

Lemma 3.[13] *For any $t>a$, $\mu\geq 0$ and $\nu>0$ we have*

$$\left[{}^{\rho}I^{\mu}_{a^+}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\nu-1}\right](t)=\frac{\Gamma(\nu)}{\Gamma(\mu+\nu)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\nu-1},$$

$$\left[{}^{\rho}D^{\mu}_{a^+}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\mu-1}\right](t)=0,\quad 0<\mu<1.$$

Lemma 4.[13] *Let $\mu,\rho>0$, $0\leq\beta<1$ and let $a,b\in(0,\infty)$ be such that $a<b$ and $u\in C_{\beta,\rho}[a,b]$. Then,*

$$\left({}^{\rho}I^{\mu}_{a^+}u\right)(a)=\lim_{t\to a^+}\left({}^{\rho}I^{\mu}_{a^+}u\right)(t)=0$$

and $\left({}^{\rho}I^{\mu}_{a^+}u\right)$ is continuous on $[a,b]$ if $\beta<\mu$.

Lemma 5.[13] *Let $\mu\in(0,1)$, $\nu\in[0,1]$ and $\beta=\mu+\nu-\mu\nu$. If $u\in C^{\beta}_{1-\beta}[a,b]$, then*

$${}^{\rho}I^{\beta}_{a^+}\,{}^{\rho}D^{\beta}_{a^+}u={}^{\rho}I^{\mu}_{a^+}\,{}^{\rho}D^{\mu,\nu}_{a^+}u$$

and

$${}^{\rho}D^{\beta}_{a^+}\,{}^{\rho}I^{\mu}_{a^+}u={}^{\rho}D^{\nu(1-\mu)}_{a^+}u.$$

Lemma 6.[13] *Let $\mu\in(0,1)$, $0\leq\beta<1$. If $u\in C_{\beta}[a,b]$ and ${}^{\rho}I^{1-\mu}_{a^+}u\in C^1_{\beta}[a,b]$, then*

$$\left({}^{\rho}I^{\mu}_{a^+}\,{}^{\rho}D^{\mu}_{a^+}u\right)(t)=u(t)-\frac{\left({}^{\rho}I^{1-\mu}_{a^+}u\right)(a)}{\Gamma(\mu)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\mu-1},$$

for all $t\in(a,b]$.

Lemma 7.[13] *Let $u \in L^1(a,b)$. If ${}^{\rho}D_{a^+}^{\nu(1-\mu)}u$ is of class $L^1(a,b)$, then*

$$ {}^{\rho}D_{a^+}^{\mu,\nu}\,{}^{\rho}I_{a^+}^{\mu}u = {}^{\rho}I_{a^+}^{\nu(1-\mu)}\,{}^{\rho}D_{a^+}^{\nu(1-\mu)}u. $$

Lemma 8.[13] *Let $f:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function where $f(\cdot,u(\cdot))\in C_{1-\beta}[a,b]$. A function $u\in C_{1-\beta}^{\beta}[a,b]$ is a solution of fractional IVP:*

$$ \begin{aligned} {}^{\rho}D_{a^+}^{\mu,\nu}u(t) &= f(t,u(t)), \qquad \mu\in(0,1),\ \nu\in[0,1] \text{ and } \rho>0,\\ {}^{\rho}I_{a^+}^{1-\beta}u(a^+) &= u_0, \qquad \beta=\mu+\nu-\mu\nu,\ u_0\in\mathbb{R} \end{aligned} $$

if and only if u satisfies the integral equation of Volterra type:

$$ u(t) = \frac{u_0}{\Gamma(\beta)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1} + \frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}f(x,u(x))\,dx. $$

From the above fundamental lemma, we establish the following important Corollary which gives an equivalent mixed type Volterra integral equation for the FDE (1) with the initial condition (3).

Corollary 1.*Let $f_1:(a,b]\times E\times E\to\mathbb{R},(t,u,v)\mapsto f_1(t,u(t),v(t))$ be a function such that for any $u,v\in C_{1-\beta_1}$, $f_1\in C_{1-\beta_1}$ where $\beta_1=\mu_1+\nu_1-\mu_1\nu_1$ with $0<\mu_1<1$, $0\le\nu_1\le1$. A function $u\in C_{1-\beta_1}^{\beta_1}$ is a solution of FDE (1) with initial condition (3) if and only if it satisfies the following mixed type Volterra integral equation*

$$ \begin{aligned} u(t) = {} & \frac{K_1}{\Gamma(\mu_1)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1}\sum_{i=1}^{m}\lambda_{1i}\int_a^{\omega_i}\left\{\left(\frac{\omega_i{}^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\right\}dx\\ & + \frac{1}{\Gamma(\mu_1)}\int_a^t\left\{\left(\frac{t^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\right\}dx, \end{aligned} \tag{10} $$

where $K_1=\left\{\Gamma(\beta_1)-\sum_{i=1}^{m}\lambda_{1i}\left(\frac{\omega_i{}^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1}\right\}^{-1}$.

*Proof.*Let $u\in C_{1-\beta_1}^{\beta_1}[a,b]$ be a solution of the fractional differential equation (1) with the initial condition (3). Then by the Lemma 8 this solution can be written as,

$$ u(t) = \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1}\frac{\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(a)}{\Gamma(\beta_1)} + \frac{1}{\Gamma(\mu_1)}\int_a^t\left(\frac{t^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\,dx. \tag{11} $$

Now substitute $t=\omega_i$ in the above equation

$$ u(\omega_i) = \left(\frac{\omega_i{}^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1}\frac{\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(a)}{\Gamma(\beta_1)} + \frac{1}{\Gamma(\mu_1)}\int_a^{\omega_i}\left(\frac{\omega_i{}^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\,dx. $$

Multiplying by λ_{1i} both sides of the above equation we get

$$ \lambda_{1i}u(\omega_i) = \lambda_{1i}\left(\frac{\omega_i{}^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1}\frac{\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(a)}{\Gamma(\beta_1)} + \frac{\lambda_{1i}}{\Gamma(\mu_1)}\int_a^{\omega_i}\left(\frac{\omega_i{}^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\,dx. $$

Thus, we have

$$ \begin{aligned} {}^{\rho}I_{a^+}^{1-\beta_1}u(a) &= \sum_{i=1}^{m}\lambda_{1i}u(\omega_i)\\ &= \frac{\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(a)}{\Gamma(\beta_1)}\sum_{i=1}^{m}\lambda_{1i}\left(\frac{\omega_i{}^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1} + \frac{1}{\Gamma(\mu_1)}\sum_{i=1}^{m}\lambda_{1i}\int_a^{\omega_i}\left(\frac{\omega_i{}^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\,dx. \end{aligned} $$

By choosing ω_i in such a way that $\Gamma(\beta_1) - \sum_{i=1}^{m} \lambda_{1i} \left(\frac{\omega_i{}^{\rho} - a^{\rho}}{\rho}\right)^{\beta_1 - 1} \neq 0$, we have

$$\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(a) = \frac{\Gamma(\beta_1)}{\Gamma(\mu_1)} K_1 \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} \cdot f_1(x, u(x), v(x)) \right\} dx. \tag{12}$$

Substituting (12) in (11) we get (10), which proves that u satisfies the integral equation (10) when it solves FDE (1) with the initial condition (3). This proves the necessity.

Next we prove the sufficiency. By applying ${}^{\rho}I_{a^+}^{1-\beta_1}$ to both sides of the integral equation (10) we get

$$\begin{aligned}\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(t) &= {}^{\rho}I_{a^+}^{1-\beta_1} \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\beta_1 - 1} \frac{K_1}{\Gamma(\mu_1)} \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx \\ &\quad + {}^{\rho}I_{a^+}^{1-\beta_1} {}^{\rho}I_{a^+}^{\mu_1} f_1(x, u(x), v(x)).\end{aligned}$$

Using Lemma 1, Lemma 2 and Lemma 3 we have

$$\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(t) = \frac{\Gamma(\beta_1)}{\Gamma(\mu_1)} K_1 \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx + {}^{\rho}I_{a^+}^{1-\nu_1(1-\mu_1)} f_1(x, u(x), v(x)).$$

Since $1 - \nu_1(1 - \mu_1) > 1 - \beta_1$ then, by taking the limit as $t \to a$ and using Lemma 4, we get

$$\left({}^{\rho}I_{a^+}^{1-\beta_1}u\right)(a) = \frac{\Gamma(\beta_1)}{\Gamma(\mu_1)} K_1 \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx. \tag{13}$$

Now, substituting $t = \omega_i$ in (10), we obtain

$$\begin{aligned}u(\omega_i) &= \frac{K_1}{\Gamma(\mu_1)} \left(\frac{\omega_i{}^{\rho} - a^{\rho}}{\rho}\right)^{\beta_1 - 1} \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx \\ &\quad + \frac{1}{\Gamma(\mu_1)} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sum_{i=1}^{m} \lambda_{1i} u(\omega_i) &= \frac{K_1}{\Gamma(\mu_1)} \sum_{i=1}^{m} \lambda_{1i} \left(\frac{\omega_i{}^{\rho} - a^{\rho}}{\rho}\right)^{\beta_1 - 1} \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx \\ &\quad + \frac{1}{\Gamma(\mu_1)} \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx \\ &= \frac{1}{\Gamma(\mu_1)} \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx \cdot \left\{ K_1 \sum_{i=1}^{m} \lambda_{1i} \left(\frac{\omega_i{}^{\rho} - a^{\rho}}{\rho}\right)^{\beta_1 - 1} + 1 \right\} \\ &= \frac{\Gamma(\beta_1)}{\Gamma(\mu_1)} K_1 \sum_{i=1}^{m} \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^{\rho} - x^{\rho}}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx.\end{aligned} \tag{14}$$

From (13) and (14) it follows that

$${}^{\rho}I_{a^+}^{1-\beta_1}u(a) = \sum_{i=1}^{m} \lambda_{1i} u(\omega_i).$$

From the definition of $C^{\beta_1}_{1-\beta_1}[a,b]$, we have that ${}^{\rho}D^{\beta_1}_{a^+}u \in C^{\beta_1}_{1-\beta_1}[a,b]$. Then, thanks to Lemma 5 (with $\beta = 1$ and $\mu = 1-\nu_1(1-\mu_1)$) we deduce that ${}^{\rho}D^{\nu_1(1-\mu_1)}_{a^+}f_1 = {}^{\rho}D \cdot {}^{\rho}I^{1-\nu_1(1-\mu_1)}_{a^+}f_1 \in C_{1-\beta_1}[a,b]$. Now, by applying ${}^{\rho}D^{\beta_1}_{a^+}$ to equation (10) and using Lemma 3 and Lemma 5 we get

$$ {}^{\rho}D^{\beta_1}_{a^+}u(t) = {}^{\rho}D^{\nu_1(1-\mu_1)}_{a^+}f_1(t,u(t),v(t)). \tag{15}$$

It is obvious that for any $f_1 \in C_{1-\beta_1}[a,b]$, ${}^{\rho}I^{1-\nu_1(1-\mu_1)}_{a^+}f_1 \in C_{1-\beta_1}[a,b]$, then ${}^{\rho}I^{1-\nu_1(1-\mu_1)}_{a^+}f_1 \in C^1_{1-\beta_1}[a,b]$. Thus, both f_1 and ${}^{\rho}I^{1-\nu_1(1-\mu_1)}_{a^+}f_1$ satisfies the conditions of Lemma 6. Now, from Lemma 6 (with $\mu = 1-\nu_1(1-\mu_1)$), by applying ${}^{\rho}I^{1-\nu_1(1-\mu_1)}_{a^+}$ on both sides of (15), we obtain that

$$\left({}^{\rho}D^{\mu_1,\nu_1}_{a^+}u\right)(t) = f_1(t,u(t),v(t)) - \frac{{}^{\rho}I^{1-\nu_1(1-\mu_1)}_{a^+}f_1(a)}{\Gamma(\nu_1(1-\mu_1))}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\nu_1(1-\mu_1)-1}. \tag{16}$$

Finally, Lemma 4 it implies that ${}^{\rho}I^{1-\nu_1(1-\mu_1)}_{a^+}f_1(a) = 0$. Consequently, equation (16) reduces to

$$\left({}^{\rho}D^{\mu_1,\nu_1}_{a^+}u\right)(t) = f_1(t,u(t),v(t)).$$

This completes the proof.

By repeating the process of Corollary 1 for the FDE (2) with initial condition (4) the equivalent mixed type Volterra integral equation can be obtained.

Corollary 2. *Let $f_2 : (a,b] \times E \times E \to \mathbb{R}, (t,u,v) \mapsto f_2(t,u(t),v(t))$ be a function such that for any $u,v \in C_{1-\beta_2}$, $f_2 \in C_{1-\beta_2}$ where $\beta_2 = \mu_2+\nu_2-\mu_2\nu_2$ with $0<\mu_2<1$, $0 \le \nu_2 \le 1$. A function $v \in C^{\beta_2}_{1-\beta_2}$ is a solution of FDE (2) with initial condition (4) if and only if it satisfies the following mixed type Volterra integral equation*

$$\begin{aligned} v(t) = {} & \frac{K_2}{\Gamma(\mu_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta_2-1}\sum_{i=1}^{m}\lambda_{2i}\int_a^{\omega_i}\left\{\left(\frac{\omega_i{}^{\rho}-x^{\rho}}{\rho}\right)^{\mu_2-1}x^{\rho-1}f_2(x,u(x),v(x))\right\}dx \\ & + \frac{1}{\Gamma(\mu_2)}\int_a^t\left\{\left(\frac{t^{\rho}-x^{\rho}}{\rho}\right)^{\mu_2-1}x^{\rho-1}f_2(x,u(x),v(x))\right\}dx, \end{aligned} \tag{17}$$

where $K_2 = \left\{\Gamma(\beta_2) - \sum\limits_{i=1}^{m}\lambda_{2i}\left(\frac{\omega_i{}^{\rho}-a^{\rho}}{\rho}\right)^{\beta_2-1}\right\}^{-1}$.

3 Main result

In the sequel, let us introduce the following hypotheses:

Q1: Let $f_j : (a,b] \times E \times E \to \mathbb{R}$, $j = 1,2$ be continuous functions, such that for all $u,v,\bar{u},\bar{v} \in E$ there exists positive constants $J_{j1}, J_{j2} > 0$ for $j = 1,2$ such that

$$\left|f_j(t,u(t),v(t)) - f_j(t,\bar{u}(t),\bar{v}(t))\right| \le J_{j1}|u(t)-\bar{u}(t)| + J_{j2}|v(t)-\bar{v}(t)|, \ j = 1,2.$$

Q2: For $j = 1,2$ we define the constants σ_j by

$$\sigma_j := 2\frac{J_j\mathbf{B}(\mu_j,\beta_j)}{\Gamma(\mu_j)}\left\{|K_j|\sum_{i=1}^{m}\lambda_{ji}\left(\frac{\omega_i{}^{\rho}-a^{\rho}}{\rho}\right)^{\mu_j+\beta_j-1} + \left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\mu_j}\right\}, \tag{18}$$

where $J_j = \max\{J_{j1}, J_{j2}\}$ and $\mathbf{B}(\cdot,\cdot)$ is the Beta function defined by

$$\mathbf{B}(\mu,\beta) = \int_0^1 t^{\mu-1}(1-t)^{\beta-1}dt,$$

K_1 and K_2 are defined as in Corollary 1 and Corollary 2 respectively. Moreover, we observe that $\sigma_j < 1$.

Now, we will establish our main existence result for the coupled system (1)–(4) using Krasnoselskii fixed point theorem.

Theorem 1. *Assume that the hypothesis $Q1$ and $Q2$ are satisfied. Then, the coupled system (1)–(4) has at least one solution in $C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}$.*

Proof. According to Corollary 1 and Corollary 2, it is sufficient to prove the existence of solutions to Volterra integral equations of mixed type (10) and (17).

Fixed v, we define the operator $\Delta_1 : C_{1-\beta_1}\to C_{1-\beta_1}$ by

$$(\Delta_1 u)(t)=\frac{K_1}{\Gamma(\mu_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\beta_1-1}\sum_{i=1}^{m}\lambda_{1i}\int_a^{\omega_i}\left\{\left(\frac{\omega_i{}^\rho-x^\rho}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\right\}dx$$
$$+\frac{1}{\Gamma(\mu_1)}\int_a^{t}\left\{\left(\frac{t^\rho-x^\rho}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\right\}dx. \tag{19}$$

Fixed u, we define the operator $\Delta_2 : C_{1-\beta_2}\to C_{1-\beta_2}$ by

$$(\Delta_2 v)(t)=\frac{K_2}{\Gamma(\mu_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\beta_2-1}\sum_{i=1}^{m}\lambda_{2i}\int_a^{\omega_i}\left\{\left(\frac{\omega_i{}^\rho-x^\rho}{\rho}\right)^{\mu_2-1}x^{\rho-1}f_2(x,u(x),v(x))\right\}dx$$
$$+\frac{1}{\Gamma(\mu_2)}\int_a^{t}\left\{\left(\frac{t^\rho-x^\rho}{\rho}\right)^{\mu_2-1}x^{\rho-1}f_2(x,u(x),v(x))\right\}dx. \tag{20}$$

Consider the continuous operator $\Delta : C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}\to C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}$ defined by

$$(\Delta(u,v))(t)=((\Delta_1 u)(t),(\Delta_2 v)(t)). \tag{21}$$

It is obvious that the operators $\Delta_j,\ j=1,2$ are well defined and map $C_{1-\beta_j}$ into $C_{1-\beta_j}$. Hence, the operator Δ is also well defined and maps $C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}$ into $C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}$. Clearly, the fixed points of the operator Δ are solutions of the coupled system (1)–(4).

For $j=1,2$, let $\hat{f}_j(x)=f_j(x,0,0)$ and

$$\eta_j:=2\frac{\mathbf{B}(\mu_j,\beta_j)}{\Gamma(\mu_j)}\left\{|K_j|\sum_{i=1}^{m}\lambda_{ji}\left(\frac{\omega_i{}^\rho-a^\rho}{\rho}\right)^{\mu_j+\beta_j-1}+\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\mu_j}\right\}\|\hat{f}_j\|_{C_{1-\beta_j}}. \tag{22}$$

Consider a ball

$$B_s:=B(0,s)=\left\{(u,v)\in C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}:\|(u,v)\|_{C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}}\le s\right\}$$

with $\frac{\eta_j}{1-\sigma_j}\le s$, $(\sigma_j<1)$, $j=1,2$.

Now, let us subdivide the operator Δ_1 into two operators F_1 and G_1 as follows,

$$(F_1 u)(t)=\frac{K_1}{\Gamma(\mu_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\beta_1-1}\sum_{i=1}^{m}\lambda_{1i}\int_a^{\omega_i}\left\{\left(\frac{\omega_i{}^\rho-x^\rho}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\right\}dx \tag{23}$$

and

$$(G_1 u)(t)=\frac{1}{\Gamma(\mu_1)}\int_a^{t}\left\{\left(\frac{t^\rho-x^\rho}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\right\}dx. \tag{24}$$

Similarly, subdivide the operator Δ_2 into two operators F_2 and G_2 as follows,

$$(F_2 v)(t)=\frac{K_2}{\Gamma(\mu_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\beta_2-1}\sum_{i=1}^{m}\lambda_{2i}\int_a^{\omega_i}\left\{\left(\frac{\omega_i{}^\rho-x^\rho}{\rho}\right)^{\mu_2-1}x^{\rho-1}f_2(x,u(x),v(x))\right\}dx \tag{25}$$

and

$$(G_2 v)(t) = \frac{1}{\Gamma(\mu_2)} \int_a^t \left\{ \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu_2 - 1} x^{\rho-1} f_2(x, u(x), v(x)) \right\} dx. \tag{26}$$

Consequently, from equation (21) we have

$$\begin{aligned}(\Delta(u,v))(t) &= ((\Delta_1 u)(t), (\Delta_2 v)(t)) \\ &= ((F_1 + G_1)(u), (F_2 + G_2)(v)) \\ &= ((F_1) u + (G_1) u, (F_2) v + (G_2) v) \\ &= ((F_1) u, (F_2) v) + ((G_1) u, (G_2) v) \\ &= \overline{F}(u,v) + \overline{G}(u,v),\end{aligned} \tag{27}$$

where $\overline{F}(u,v) = ((F_1)u, (F_2)v)$ and $\overline{G}(u,v) = ((G_1)u, (G_2)v)$.

The proof is subdivided into following steps.

Step I: Δ maps the ball B_s into itself, that is, $\overline{F}(u,v) + \overline{G}(u,v) \in B_s$, for all $(u,v) \in B_s$.
Now, for each $t \in (a,b]$ we have

$$(F_1 u)(t) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\beta_1} = \frac{K_1}{\Gamma(\mu_1)} \sum_{i=1}^m \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^\rho - x^\rho}{\rho} \right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x)) \right\} dx$$

Then,

$$\begin{aligned}\left| (F_1 u)(t) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\beta_1} \right| &\leq \frac{|K_1|}{\Gamma(\mu_1)} \sum_{i=1}^m \lambda_{1i} \int_a^{\omega_i} \left\{ \left(\frac{\omega_i{}^\rho - x^\rho}{\rho} \right)^{\mu_1 - 1} x^{\rho-1} |f_1(x, u(x), v(x))| \right\} dx \\ &\leq \frac{|K_1|}{\Gamma(\mu_1)} \sum_{i=1}^m \lambda_{1i} \int_a^{\omega_i} \left(\frac{\omega_i{}^\rho - x^\rho}{\rho} \right)^{\mu_1 - 1} x^{\rho-1} (|f_1(x, u(x), v(x)) - f_1(x,0,0)| + |f_1(x,0,0)|) dx \\ &\leq \frac{|K_1|}{\Gamma(\mu_1)} \sum_{i=1}^m \lambda_{1i} \int_a^{\omega_i} \left(\frac{\omega_i{}^\rho - x^\rho}{\rho} \right)^{\mu_1 - 1} x^{\rho-1} \left(J_{11} |u(x)| + J_{12} |v(x)| + \left| \hat{f}_1(x) \right| \right) dx.\end{aligned}$$

Here we use the fact that

$$\begin{aligned}\int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu_1 - 1} x^{\rho-1} (|u(x)| + |v(x)|) dx &\leq \left\{ \int_a^t \left(\frac{t^\rho - x^\rho}{\rho} \right)^{\mu_1 - 1} \left(\frac{x^\rho - a^\rho}{\rho} \right)^{\beta_1 - 1} x^{\rho-1} dx \right\} \left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}} \right) \\ &= \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\mu_1 + \beta_1 - 1} \mathbf{B}(\mu_1, \beta_1) \left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}} \right).\end{aligned} \tag{28}$$

Thus, we have

$$\left| (F_1 u)(t) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\beta_1} \right| \leq \frac{|K_1|}{\Gamma(\mu_1)} \sum_{i=1}^m \lambda_{1i} \left\{ \left(\frac{\omega_i^\rho - a^\rho}{\rho} \right)^{\mu_1 + \beta_1 - 1} \mathbf{B}(\mu_1, \beta_1) \right\} \left(J_1 \left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}} \right) + \|f_1\|_{C_{1-\beta_1}} \right),$$

where $J_1 = \max\{J_{11}, J_{12}\}$ which gives

$$\|(F_1 u)\|_{C_{1-\beta_1}} \leq \frac{|K_1| \mathbf{B}(\mu_1, \beta_1)}{\Gamma(\mu_1)} \sum_{i=1}^m \lambda_{1i} \left\{ \left(\frac{\omega_i^\rho - a^\rho}{\rho} \right)^{\mu_1 + \beta_1 - 1} \left(J_1 \left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}} \right) + \|f_1\|_{C_{1-\beta_1}} \right) \right\}. \tag{29}$$

Similarly, it can be shown that

$$\|(F_2 v)\|_{C_{1-\beta_2}} \leq \frac{|K_2| \mathbf{B}(\mu_2, \beta_2)}{\Gamma(\mu_2)} \sum_{i=1}^m \lambda_{2i} \left\{ \left(\frac{\omega_i^\rho - a^\rho}{\rho} \right)^{\mu_2 + \beta_2 - 1} \left(J_2 \left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}} \right) + \|f_2\|_{C_{1-\beta_2}} \right) \right\}, \tag{30}$$

where $J_2 = \max\{J_{21}, J_{22}\}$.

For $t \in (a,b]$ we have

$$(G_1u)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1} = \frac{1}{\Gamma(\mu_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\int_a^t \left\{\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} f_1(x, u(x), v(x))\right\} dx.$$

Then

$$\begin{aligned}
\left|(G_1u)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\right| &\leq \frac{1}{\Gamma(\mu_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} |f_1(x, u(x), v(x))| \, dx \\
&\leq \frac{1}{\Gamma(\mu_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} \left(|f_1(x, u(x), v(x)) - f_1(x, 0, 0)|\right. \\
&\quad \left. + |f_1(x, 0, 0)|\right) dx \\
&\leq \frac{1}{\Gamma(\mu_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu_1 - 1} x^{\rho-1} \left(J_{11}|u(x)| + J_{12}|v(x)| + \left|\hat{f}_1(x)\right|\right) dx.
\end{aligned}$$

Again, using (28), we have

$$\begin{aligned}
\left|(G_1u)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\right| &\leq \frac{1}{\Gamma(\mu_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\left\{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\mu_1+\beta_1-1} \mathbf{B}(\mu_1, \beta_1)\left(J_1\left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}}\right)\right.\right. \\
&\quad \left.\left. + \|f_1\|_{C_{1-\beta_1}}\right)\right\} \\
&\leq \frac{\mathbf{B}(\mu_1, \beta_1)}{\Gamma(\mu_1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu_1}\left(J_1\left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}}\right) + \|f_1\|_{C_{1-\beta_1}}\right),
\end{aligned}$$

which gives

$$\|(G_1u)\|_{C_{1-\beta_1}} \leq \frac{\mathbf{B}(\mu_1, \beta_1)}{\Gamma(\mu_1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu_1}\left(J_1\left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}}\right) + \|f_1\|_{C_{1-\beta_1}}\right). \tag{31}$$

Similarly, we get

$$\|(G_2v)\|_{C_{1-\beta_2}} \leq \frac{\mathbf{B}(\mu_2, \beta_2)}{\Gamma(\mu_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu_2}\left(J_2\left(\|u\|_{C_{1-\beta_1}} + \|v\|_{C_{1-\beta_2}}\right) + \|f_2\|_{C_{1-\beta_2}}\right). \tag{32}$$

Thus, for every $(u,v) \in B_s$,

$$\begin{aligned}
\|\Delta(u,v)\|_{C_{1-\beta_1}^{\mu_1,\nu_1} \times C_{1-\beta_2}^{\mu_2,\nu_2}} &= \|\Delta_1 u\|_{C_{1-\beta_1}} + \|\Delta_2 v\|_{C_{1-\beta_2}} \\
&= \|(F_1 + G_1)u\|_{C_{1-\beta_1}} + \|(F_2 + G_2)v\|_{C_{1-\beta_2}} \\
&\leq \left(\|(F_1)u\|_{C_{1-\beta_1}} + \|(G_1)u\|_{C_{1-\beta_2}}\right) + \left(\|(F_2)v\|_{C_{1-\beta_1}} + \|(G_2)v\|_{C_{1-\beta_2}}\right).
\end{aligned}$$

Using equations (29), (30), (31) and (32) we obtain

$$\begin{aligned}
\|\Delta(u,v)\|_{C_{1-\beta_1}^{\mu_1,\nu_1} \times C_{1-\beta_2}^{\mu_2,\nu_2}} &\leq \left\{\frac{\sigma_1}{2}s + \frac{\eta_1}{2}\right\} + \left\{\frac{\sigma_2}{2}s + \frac{\eta_2}{2}\right\} \\
&= \left\{\frac{\sigma_1 s + \eta_1}{2}\right\} + \left\{\frac{\sigma_2 s + \eta_2}{2}\right\} \\
&\leq \frac{s}{2} + \frac{s}{2} = s,
\end{aligned}$$

which implies that $\Delta(u,v) \in B_s$. Therefore, we have proved that Δ maps the ball B_s into itself.

Step II: The operator $\overline{F}$ is a contraction mapping.
For any $(u,v),(\overline{u},\overline{v}) \in B_s$ and the operator F_1 we have

$$\{(F_1 u)(t) - (F_1 \bar{u})(t)\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1} = \frac{K_1}{\Gamma(\mu_1)}\sum_{i=1}^{m}\lambda_{1i}\int_a^{\omega_i}\left\{\left(\frac{\omega_i{}^\rho - x^\rho}{\rho}\right)^{\mu_1-1} x^{\rho-1} \cdot (f_1(x,u(x),v(x)) - f_1(x,\bar{u}(x),v(x)))\right\}dx.$$

Then,

$$\begin{aligned}\left|\{(F_1 u)(t) - (F_1 \bar{u})(t)\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\right| &\le \frac{|K_1|}{\Gamma(\mu_1)}\sum_{i=1}^{m}\lambda_{1i}\int_a^{\omega_i}\left(\frac{\omega_i{}^\rho - x^\rho}{\rho}\right)^{\mu_1-1} x^{\rho-1} \\ &\quad \cdot (|f_1(x,u(x),v(x)) - f_1(x,\bar{u}(x),v(x))|)\,dx \\ &\le \frac{|K_1|}{\Gamma(\mu_1)}\sum_{i=1}^{m}\lambda_{1i}\int_a^{\omega_i}\left(\frac{\omega_i{}^\rho - x^\rho}{\rho}\right)^{\mu_1-1} x^{\rho-1}(J_{11}|u(x)-\bar{u}(x)|)\,dx \\ &\le \frac{|K_1|}{\Gamma(\mu_1)}\sum_{i=1}^{m}\lambda_{1i}\left(\frac{\omega_i{}^\rho - a^\rho}{\rho}\right)^{\mu_1+\beta_1-1}\mathbf{B}(\mu_1,\beta_1)\left(J_{11}\|u-\overline{u}\|_{C_{1-\beta_1}}\right). \qquad (33)\end{aligned}$$

Similarly, it can be shown that

$$\left|\{(F_2 v)(t) - (F_2 \overline{v})(t)\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_2}\right| \le \frac{|K_2|}{\Gamma(\mu_1)}\sum_{i=1}^{m}\lambda_{2i}\left(\frac{\omega_i{}^\rho - a^\rho}{\rho}\right)^{\mu_1+\beta_1-1}\mathbf{B}(\mu_1,\beta_1)\left(J_{22}\|v-\overline{v}\|_{C_{1-\beta_2}}\right). \qquad (34)$$

Thus, by the hypothesis $Q2$ we have

$$\begin{aligned}\left\|\overline{F}(u,v) - \overline{F}(\overline{u},\overline{v})\right\|_{C_{1-\beta_1}^{\mu_1,\nu_1}\times C_{1-\beta_2}^{\mu_2,\nu_2}} &= \|F_1 u - F_1\overline{u}\|_{C_{1-\beta_1}} + \|F_2 v - F_2\overline{v}\|_{C_{1-\beta_2}} \\ &\le \frac{\sigma_1}{2}\left\{\|u-\overline{u}\|_{C_{1-\beta_1}}\right\} + \frac{\sigma_2}{2}\left\{\|v-\overline{v}\|_{C_{1-\beta_2}}\right\} \\ &\le \sigma\left\{\|u-\overline{u}\|_{C_{1-\beta_1}} + \|v-\overline{v}\|_{C_{1-\beta_2}}\right\},\end{aligned}$$

where $\sigma = \max\{\sigma_1,\sigma_2\}$.
Hence, the operator $\overline{F}$ is a contraction mapping.
Step III: The operator $\overline{G}$ is compact and continuous.
Let $\{(u_n,v_n)\}_{n=1}^{\infty}$ be a sequence in B_s such that $(u_n,v_n) \to (u,v)$. Then, from the definitions of the spaces $C_{1-\beta_j,\rho}$, we have

$$\begin{aligned}\left\|\overline{G}(u_n,v_n) - \overline{G}(u,v)\right\|_{C_{1-\beta_1}^{\mu_1,\nu_1}\times C_{1-\beta_2}^{\mu_2,\nu_2}} &= \|G_1 u_n - G_1 u\|_{C_{1-\beta_1}^{\mu_1,\nu_1}} + \|G_2 v_n - G_2 v\|_{C_{1-\beta_2}^{\mu_2,\nu_2}} \\ &= \max_{t\in[a,b]}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}((G_1 u_n)(t) - (G_1 u)(t))\right| \\ &\quad + \max_{t\in[a,b]}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_2}((G_2 v_n)(t) - (G_2 v)(t))\right|. \qquad (35)\end{aligned}$$

Next,

$$\begin{aligned}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}((G_1 u_n)(t) - (G_1 u)(t))\right| &\le \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\frac{1}{\Gamma(\mu_1)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu_1-1}x^{\rho-1} \\ &\quad \cdot (|f_1(x,u_n(x),v(x)) - f_1(x,u(x),v(x))|)\,dx \\ &\le \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\beta_1}\frac{1}{\Gamma(\mu_1)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu_1-1}x^{\rho-1}(J_{11}|u_n(x)-u(x)|)\,dx.\end{aligned}$$

Again, using (30), we have

$$\begin{aligned}\left|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\beta_1}\left((G_1u_n)(t)-(G_1u)(t)\right)\right| &\leq \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\beta_1}\frac{1}{\Gamma(\mu_1)}\left\{\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\mu_1+\beta_1-1}\mathbf{B}(\mu_1,\beta_1)\right.\\ &\qquad \left.\cdot J_1\left(\|u_n-u\|_{C_{1-\beta_1}}\right)\right\}\\ &\leq \frac{\mathbf{B}(\mu_1,\beta_1)}{\Gamma(\mu_1)}\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\mu_1}J_1\left(\|u_n-u\|_{C_{1-\beta_1}}\right),\end{aligned}$$

that tends to 0 as $n\to\infty$.

Thus, $\max\limits_{t\in[a,b]}\left|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\beta_1}\left((G_1u_n)(t)-(G_1u)(t)\right)\right|$ tends to 0 as $n\to\infty$. Similarly, it can be shown that $\max\limits_{t\in[a,b]}\left|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\beta_2}\left((G_2v_n)(t)-(G_2v)(t)\right)\right|$ tends to 0 as $n\to\infty$.

Thus, from (35) we obtain $\left\|\overline{G}(u_n,v_n)-\overline{G}(u,v)\right\|_{C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}}$ tends to 0 as $n\to\infty$.

This proves the continuity of $\overline{G}$. Next, we prove the compactness of $\overline{G}$. For any $a<t_1<t_2\leq b$, we have

$$\begin{aligned}\left|(G_1u)(t_1)-(G_1u)(t_2)\right| &= \left|\frac{1}{\Gamma(\mu_1)}\int_a^{t_1}\left(\frac{t_1^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\,dx\right.\\ &\qquad\left.-\frac{1}{\Gamma(\mu_1)}\int_a^{t_2}\left(\frac{t_2^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}x^{\rho-1}f_1(x,u(x),v(x))\,dx\right|\\ &\leq \frac{\|f_1\|_{C_{1-\beta_1}}}{\Gamma(\mu_1)}\left|\int_a^{t_1}\left(\frac{t_1^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1}x^{\rho-1}dx\right.\\ &\qquad\left.-\int_a^{t_2}\left(\frac{t_2^{\rho}-x^{\rho}}{\rho}\right)^{\mu_1-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\beta_1-1}x^{\rho-1}dx\right|\\ &\leq \frac{\|f_1\|_{C_{1-\beta_1}}}{\Gamma(\mu_1)}\mathbf{B}(\mu_1,\beta_1)\left|\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\mu_1+\beta_1-1}-\left(\frac{t_2^{\rho}-a^{\rho}}{\rho}\right)^{\mu_1+\beta_1-1}\right|,\end{aligned} \tag{36}$$

that tends to 0 as $t_2\to t_1$, whether $\mu_1+\beta_1-1\geq 0$ or $\mu_1+\beta_1-1<0$. Thus, G_1 is equicontinuous.

Also, we have that

$$\left|(G_2v)(t_1)-(G_2v)(t_2)\right| \leq \frac{\|f_2\|_{C_{1-\beta_2}}}{\Gamma(\mu_2)}\mathbf{B}(\mu_2,\beta_2)\left|\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\mu_2+\beta_2-1}-\left(\frac{t_2^{\rho}-a^{\rho}}{\rho}\right)^{\mu_2+\beta_2-1}\right|, \tag{37}$$

that goes to 0 as $t_2\to t_1$, whether $\mu_2+\beta_2-1\geq 0$ or $\mu_2+\beta_2-1<0$. Thus G_2 is equicontinuous.

Now, from the definition of the spaces $C_{1-\beta_j,\rho}$, $\overline{G}(u,v)$ and using (36) and (37) we get that

$$\left\|\overline{G}(u,v)(t_1)-\overline{G}(u,v)(t_2)\right\|_{C^{\mu_1,\nu_1}_{1-\beta_1}\times C^{\mu_2,\nu_2}_{1-\beta_2}} = \left\|(G_1u)(t_1)-(G_1u)(t_2)\right\|_{C^{\mu_1,\nu_1}_{1-\beta_1}}+\left\|(G_2v)(t_1)-(G_2v)(t_2)\right\|_{C^{\mu_2,\nu_2}_{1-\beta_2}}$$

which tends to 0 as $t_2\to t_1$. Therefore, $\overline{G}$ is equicontinuous. From equation (31) and (32) we deduce that $G_j,\ j=1,2$ are uniformly bounded. Consequently, $\overline{G}$ is uniformly bounded. Hence, by Arzela–Ascoli theorem the operator $\overline{G}$ is compact on B_s.

It follows from Krasnoselskii fixed point theorem that the operator Δ has at least on fixed point in B_s which is a solution of system (1)–(4).

3.1 Examples

*Example 1.*Consider the following coupled system of fractional differential equation of the form

$$\begin{cases} {}^{\rho}D_{0^+}^{\mu_1,\nu_1}u(t) = \dfrac{|u(t)|+|v(t)|}{75e^{t+7}(1+|u(t)|+|v(t)|)}, \\ {}^{\rho}D_{0^+}^{\mu_2,\nu_2}v(t) = \dfrac{|u(t)|\sin t}{100} + \dfrac{|v(t)|}{100(2+|v(t)|)}, \end{cases} \tag{38}$$

with the initial conditions

$$\begin{cases} {}^{\rho}I_{0^+}^{1-\beta_1}u(0) = 8u\left(\dfrac{1}{2}\right)+5u\left(\dfrac{3}{4}\right), & \beta_1=\mu_1+\nu_1-\mu_1\nu_1, \\ {}^{\rho}I_{0^+}^{1-\beta_2}v(0) = 9v\left(\dfrac{5}{6}\right)+15v\left(\dfrac{2}{3}\right), & \beta_2=\mu_2+\nu_2-\mu_2\nu_2, \end{cases} \tag{39}$$

where $\mu_1=\frac{1}{4}, \nu_1=\frac{3}{5}, \beta_1=\frac{7}{10}$ and $\mu_2=\frac{1}{2}, \nu_2=\frac{2}{3}, \beta_2=\frac{5}{6}$.
Set $f_1(t,u,v)=\frac{|u|+|v|}{75e^{t+7}(1+|u|+|v|)}$ and $f_2(t,u,v)=\frac{|u|\sin t}{100}+\frac{|v|}{100(2+|v|)}$ for $t\in(0,1]$. It is obvious that the functions f_1 and f_2 are continuous. For any $u,v,\overline{u},\overline{v}\in\mathbb{R}$ and $t\in(0,1]$, we have

$$|f_1(t,u,v)-f_1(t,\overline{u},\overline{v})| \le \frac{1}{75e^7}|u-\overline{u}|+\frac{1}{75e^7}|v-\overline{v}|$$

and

$$|f_2(t,u,v)-f_2(t,\overline{u},\overline{v})| \le \frac{1}{100}|u-\overline{u}|+\frac{1}{50}|v-\overline{v}|.$$

Thus, the condition Q_1 of Theorem 1 is satisfied for f_1 with $J_{11}=J_{12}=\frac{1}{75e^7}$ and for f_2 with $J_{21}=\frac{1}{100}$ and $J_{22}=\frac{1}{50}$. Moreover, with some elementary computation, for $\rho>0$ we have

$$|K_1| = \left|\left\{\Gamma\left(\frac{7}{10}\right)-\left[8\left(\frac{(1/2)^{\rho}-0^{\rho}}{\rho}\right)^{-\frac{3}{10}}+5\left(\frac{(3/4)^{\rho}-0^{\rho}}{\rho}\right)^{-\frac{3}{10}}\right]^{-1}\right\}\right| < 1,$$

$$|K_2| = \left|\left\{\Gamma\left(\frac{5}{6}\right)-\left[9\left(\frac{(5/6)^{\rho}-0^{\rho}}{\rho}\right)^{-\frac{1}{6}}+15\left(\frac{(2/3)^{\rho}-0^{\rho}}{\rho}\right)^{-\frac{1}{6}}\right]^{-1}\right\}\right| < 1$$

and

$$\sigma_1 = \frac{2}{75e^7}\frac{\mathbf{B}(1/4,7/10)}{\Gamma(1/4)}\left\{|K_1|\left[8\left(\frac{(1/2)^{\rho}-0^{\rho}}{\rho}\right)^{-\frac{1}{20}}+5\left(\frac{(3/4)^{\rho}-0^{\rho}}{\rho}\right)^{-\frac{1}{20}}\right]+\left(\frac{1^{\rho}-0^{\rho}}{\rho}\right)^{\frac{1}{4}}\right\} < 1,$$

$$\sigma_2 = \frac{2}{50}\frac{\mathbf{B}(1/2,5/6)}{\Gamma(1/2)}\left\{|K_2|\left[9\left(\frac{(5/6)^{\rho}-0^{\rho}}{\rho}\right)^{\frac{1}{3}}+15\left(\frac{(2/3)^{\rho}-0^{\rho}}{\rho}\right)^{\frac{1}{3}}\right]+\left(\frac{1^{\rho}-0^{\rho}}{\rho}\right)^{\frac{1}{2}}\right\} < 1$$

Hence, the condition Q_2 of Theorem 1 is satisfied. It follows from Theorem 1 that the coupled system (38)–(39) has at least one solution defined on $[0,1]$.

4 Conclusion

In the present work, the sufficient conditions for the existence of solutions to a coupled system of fractional differential equations involving generalized Katugampola derivative with non local initial conditions were obtained. We have used Krasnoselskii fixed point theorem to develop the existence results. Finally, as an application, a suitable example is given to demonstrate our main results.

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