

On the Marei Distribution: A Novel One-Parameter Model with Applications

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Received: 23 Apr. 2026, Revised: 17 Jun. 2026, Accepted: 21 Jun. 2026

Published online: 1 Nov. 2026

Abstract: This paper proposes a one-parameter continuous distribution on the positive half-line, named the Marei distribution, obtained by mapping a symmetric beta random variable U with both shape parameters equal to $(\theta + 1)/2$ through the transformation $X = \sqrt{U/(1-U)}$, $\theta > -1$. The objective is a single-parameter model with a power-law right tail and closed-form properties. The density is decreasing for $\theta \leq 0$ and unimodal for $\theta > 0$, and its right tail decays as $x^{-(\theta+2)}$. Closed-form expressions are derived for the cumulative distribution, survival, hazard rate and quantile functions, raw and incomplete moments, Lorenz and Bonferroni curves, Shannon, Rényi and Tsallis entropies, order statistics, the extreme-value limit and the Fisher information. Every member of the family has unit median and satisfies the reciprocal symmetry $X \stackrel{d}{=} 1/X$. The hazard rate is strictly decreasing for $-1 < \theta \leq -1/5$ and upside-down bathtub shaped for $\theta > 0$, and vanishes at the rate $(\theta + 1)/x$. The model is a regular exponential family, so the maximum likelihood estimator is unique and asymptotically normal for every $\theta > -1$, including parameter values with infinite mean. The parameter is estimated by maximum likelihood with Newton–Raphson iteration and by Bayesian Markov chain Monte Carlo sampling under weak, informative and misspecified priors. A Monte Carlo study covering 25 scenarios confirms consistency, mean squared error decay of order $1/n$ and nominal interval coverage. On carbon-fibre breaking-stress data and electrical-product failure-time data, the proposed model outperforms fifteen one-parameter competitors in information criteria and goodness-of-fit statistics, performs on par with the half-Cauchy and Lomax distributions, and attains a fit comparable to the two-parameter Weibull and Gamma distributions on the failure-time data with half the number of parameters. The model is recommended for heavy-tailed positive data, and its scale extension is outlined.

Keywords: Bayesian estimation; Beta integral transform; Hazard rate function; Heavy-tailed distribution; Power-law tail.

1 Introduction

Modeling positive lifetime and reliability data is central to actuarial science, survival analysis, engineering reliability, and environmental monitoring. While traditional one-parameter continuous distributions offer mathematical simplicity and low computational burden, many widely used families are restricted in their tail decay mechanisms and failure rate characteristics. Specifically, standard one-parameter models based on exponential mixtures—such as the Lindley [19], Shanker [31], Akash [30], Ishita [35], Rani [33],

Pranav [36], Xgamma [29], XLindley [10], Komal [34], Garhy [11], Obulezi [23], extended Lindley [17], inverted Z-Lindley [4], and an updated Lindley [24] distributions—possess exponentially light right tails ($f(x) \sim e^{-cx}$). Consequently, such models underestimate the risk of extreme events when applied to heavy-tailed phenomena, such as extreme insurance claims, financial asset returns, high-stress material fractures, and catastrophic component failures.

On the other hand, classic heavy-tailed distributions on $(0, \infty)$ —including the half-Cauchy, Pareto, Lomax, log-logistic, Burr Type XII, and Beta-prime

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distributions—possess power-law tail decay ($f(x) \sim x^{-\alpha}$). However, standard Pareto and Lomax formulations typically require external scale specifications or two parameters to achieve structural flexibility. The standard half-Cauchy model is parameter-free and thus lacks shape adaptation. This motivates the investigation of simple, single-parameter heavy-tailed models derived from fundamental integral transforms.

In this paper, we study a simple one-parameter heavy-tailed probability model on $(0, \infty)$ generated by mapping a symmetric Beta random variable $U \sim \text{Beta}(a, a)$ onto the positive half-line via the monotone algebraic transform $X = \sqrt{U/(1-U)}$, where $a = (\theta + 1)/2$ for shape parameter $\theta > -1$. This model provides several distinct properties:

- (i) **Power-law tail decay:** The right tail decays algebraically as $x^{-(\theta+2)}$, directly controlled by the single shape parameter θ .
- (ii) **Analytical tractability:** Closed-form expressions for the cumulative distribution function (CDF), quantile function, incomplete moments, and Shannon entropy are established via regularized incomplete Beta and special functions.
- (iii) **Unit median baseline:** The median is identically equal to unity ($M = 1$) for all $\theta > -1$, serving as a normalized location anchor for dimensionless positive lifetime data.
- (iv) **Flexible hazard dynamics:** The hazard rate exhibits a transition from monotonically decreasing failure rates (DFR) for $-1 < \theta \leq -1/5$ to unimodal (upside-down bathtub) failure rates for $\theta > 0$, with $h(x) \sim (\theta + 1)/x \rightarrow 0$ as $x \rightarrow \infty$ (the transitional range $-1/5 < \theta \leq 0$ is described in Proposition 2.1.3).

The model is formulated with a single shape parameter, which yields closed-form expressions for its main properties. In Section 3.2 it is benchmarked against one-parameter and two-parameter competitors (e.g., Weibull and Gamma) under criteria that penalize additional parameters [18, 39], a setting in which odds-based extensions of the Weibull law have recently been proposed for engineering failure-time data [2], and Section 3.3 outlines its scale extension.

Table 1 summarizes key structural attributes of well-known one-parameter continuous lifetime distributions alongside classic heavy-tailed baselines. Existing one-parameter models predominantly rely on exponential mixture mechanisms, resulting in thin-tailed density profiles. In contrast, the proposed symmetric Beta-transformed model produces power-law tails while retaining a simple one-parameter structure. Recent one-parameter lifetime models with decreasing hazard rates and exact inference on restricted supports have also been proposed [27].

To broaden the contextual scope of complex system modeling and transport dynamics, recent literature has explored distributed control and delay differential

formulations [26]. Such methodologies emphasize the necessity of flexible mathematical representations when modeling heavy-tailed physical processes.

The remainder of this article is organized as follows. Section 2 (Materials and Methods) presents the construction of the model and its hazard behavior (Section 2.1), its mathematical properties (Section 2.2) and the estimation procedures (Section 2.3). Section 3 (Results and Discussion) reports the simulation study (Section 3.1), the real-data applications (Section 3.2) and the concluding remarks (Section 3.3).

2 Materials and Methods

2.1 Construction of the Model

Let $U \sim \text{Beta}(a, a)$ with shape parameter $a > 0$ and PDF

$$f_U(u) = \frac{1}{B(a, a)} u^{a-1} (1-u)^{a-1}, \quad 0 < u < 1, \quad (1)$$

where $B(a, a) = \Gamma(a)^2/\Gamma(2a)$. Consider the monotone transformation

$$X = \sqrt{\frac{U}{1-U}} \iff U = \frac{X^2}{1+X^2}. \quad (2)$$

The Jacobian is $du/dx = 2x/(1+x^2)^2$. Applying the change-of-variable formula:

$$\begin{aligned} f_X(x) &= \frac{1}{B(a, a)} \left(\frac{x^2}{1+x^2} \right)^{a-1} \left(\frac{1}{1+x^2} \right)^{a-1} \frac{2x}{(1+x^2)^2} \\ &= \frac{2}{B(a, a)} \frac{x^{2a-1}}{(1+x^2)^{2a}}. \end{aligned}$$

Setting $a = (\theta + 1)/2$, we have $\theta = 2a - 1 > -1$, yielding the probability density function (PDF):

$$f_X(x; \theta) = \frac{2}{B\left(\frac{\theta+1}{2}, \frac{\theta+1}{2}\right)} \frac{x^\theta}{(1+x^2)^{\theta+1}}, \quad x > 0, \theta > -1. \quad (3)$$

Throughout the paper, $a = (\theta + 1)/2$.

2.1.1 Probability Density Function

For $\theta > -1$, the density function $f_X(x; \theta)$ exhibits two distinct shape regimes:

- For $-1 < \theta \leq 0$, $f_X(x; \theta)$ is strictly decreasing on $(0, \infty)$.
- For $\theta > 0$, $f_X(x; \theta)$ is unimodal with unique mode at $x_{\text{mode}} = \sqrt{\theta/(\theta+2)}$.

Table 1: Literature overview of selected one-parameter and baseline continuous distributions.

Distribution	Support	PDF $f(x)$	Tail Type
Lindley [19]	$x > 0$	$\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \theta > 0$	Exponential
Shanker [31]	$x > 0$	$\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}, \theta > 0$	Exponential
Akash [30]	$x > 0$	$\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}, \theta > 0$	Exponential
Ishita [35]	$x > 0$	$\frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-\theta x}, \theta > 0$	Exponential
Rani [33]	$x > 0$	$\frac{\theta^5}{\theta^5+24}(\theta+x^4)e^{-\theta x}, \theta > 0$	Exponential
Pranav [36]	$x > 0$	$\frac{\theta^4}{\theta^4+6}(\theta+x^3)e^{-\theta x}, \theta > 0$	Exponential
Xgamma [29]	$x > 0$	$\frac{\theta^2}{1+\theta}\left(1+\frac{\theta^2 x^2}{2}\right)e^{-\theta x}, \theta > 0$	Exponential
XLindley [10]	$x > 0$	$\frac{\theta^2}{(1+\theta)^2}(2+\theta+x)e^{-\theta x}, \theta > 0$	Exponential
Komal [34]	$x > 0$	$\frac{\theta^2}{\theta^2+\theta+1}(1+\theta+x)e^{-\theta x}, \theta > 0$	Exponential
Half-Cauchy	$x > 0$	$\frac{2}{\pi(1+x^2)}$	Power-law (x^{-2})
1-par. Lomax	$x > 0$	$\frac{\theta}{(1+x)^{\theta+1}}, \theta > 0$	Power-law ($x^{-(\theta+1)}$)
Weibull (2-par.) [39]	$x > 0$	$\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}e^{-(x/\eta)^\beta}, \beta, \eta > 0$	Exponential
Gamma (2-par.) [18]	$x > 0$	$\frac{1}{\Gamma(k)\eta^k}x^{k-1}e^{-x/\eta}, k, \eta > 0$	Exponential
Proposed Model	$x > 0$	$\frac{2x^\theta}{B(a,a)(1+x^2)^{\theta+1}}, a = \frac{\theta+1}{2}, \theta > -1$	Power-law ($x^{-(\theta+2)}$)

2.1.2 Cumulative Distribution Function

Proposition 2.1.1. *The CDF of X is given by*

$$F(x; \theta) = I_{\frac{x^2}{1+x^2}}(a, a), \quad x > 0, \quad (4)$$

where $I_z(a, b) = \frac{1}{B(a, b)} \int_0^z t^{a-1}(1-t)^{b-1} dt$ is the regularized incomplete Beta function.

Proof. Substituting $u = t^2/(1+t^2)$ into $F(x; \theta) = \int_0^x f_X(t; \theta) dt$ yields

$$\begin{aligned} \int_0^x \frac{t^\theta}{(1+t^2)^{\theta+1}} dt &= \frac{1}{2} \int_0^{x^2/(1+x^2)} u^{a-1}(1-u)^{a-1} du \\ &= \frac{1}{2} B_{x^2/(1+x^2)}(a, a). \end{aligned}$$

Multiplying by $2/B(a, a)$ gives $I_{x^2/(1+x^2)}(a, a)$.

2.1.3 Hazard Rate and Failure Rate Behavior

Since $I_z(a, a) = 1 - I_{1-z}(a, a)$ for the symmetric Beta law and $1 - \frac{x^2}{1+x^2} = \frac{1}{1+x^2}$, the survival function is

$$S(x; \theta) = 1 - F(x; \theta) = I_{\frac{1}{1+x^2}}(a, a), \quad x > 0. \quad (5)$$

The hazard rate function (HRF) is therefore

$$h(x; \theta) = \frac{f(x; \theta)}{S(x; \theta)} = \frac{2x^\theta}{B(a, a)(1+x^2)^{\theta+1} I_{\frac{1}{1+x^2}}(a, a)}. \quad (6)$$

Note that the factor $1/(1+x^2)$ enters the denominator *only* through the argument of the regularized incomplete Beta function.

Proposition 2.1.2 (Hazard rate asymptotic behavior). *For any $\theta > -1$, as $x \rightarrow \infty$, the survival function satisfies*

$$1 - F(x; \theta) \sim \frac{1}{aB(a, a)} x^{-2a} = \frac{2}{(\theta+1)B(a, a)} x^{-(\theta+1)}. \quad (7)$$

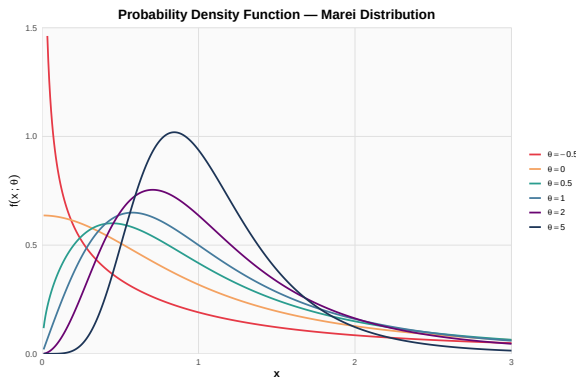


Fig. 1: Probability density function for selected values of θ .

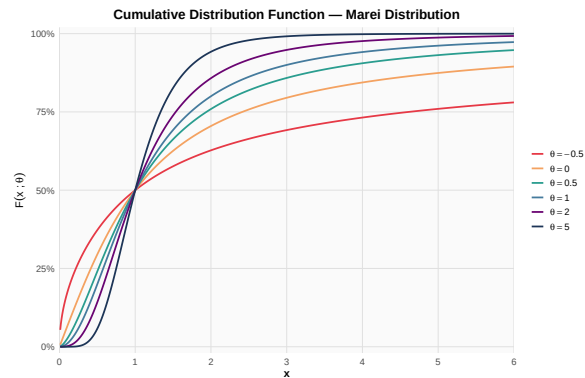


Fig. 2: Cumulative distribution function for selected values of θ .

Consequently, the hazard rate function satisfies

$$h(x; \theta) \sim \frac{\theta + 1}{x} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (8)$$

Proof. The survival function is $1 - F(x; \theta) = I_z(a, a)$ with $z = 1/(1 + x^2) \rightarrow 0^+$ as $x \rightarrow \infty$. Using the standard asymptotic expansion of the incomplete Beta integral near zero, $\int_0^z u^{a-1} (1-u)^{a-1} du = \frac{z^a}{a} [1 + \mathcal{O}(z)]$. Thus,

$$\begin{aligned} 1 - F(x; \theta) &= \frac{1}{B(a, a)} \int_0^{\frac{1}{1+x^2}} u^{a-1} (1-u)^{a-1} du \\ &\sim \frac{1}{aB(a, a)} \left(\frac{1}{1+x^2} \right)^a \sim \frac{1}{aB(a, a)} x^{-2a}. \end{aligned}$$

Since $2a = \theta + 1$, $1 - F(x; \theta) \sim \frac{2}{(\theta+1)B(a, a)} x^{-(\theta+1)}$. Meanwhile, the PDF satisfies $f(x; \theta) \sim \frac{2}{B(a, a)} x^{-(2a+1)} = \frac{2}{B(a, a)} x^{-(\theta+2)}$. Taking the ratio $h(x; \theta) = f(x; \theta)/(1 - F(x; \theta))$:

$$\begin{aligned} h(x; \theta) &\sim \frac{\frac{2}{B(a, a)} x^{-(\theta+2)}}{\frac{2}{(\theta+1)B(a, a)} x^{-(\theta+1)}} \\ &= \frac{\theta + 1}{x} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \end{aligned}$$

This completes the proof.

Proposition 2.1.2 establishes that the hazard rate decays to zero at an algebraic rate $\mathcal{O}(1/x)$ for all $\theta > -1$. The global shape of $h(x; \theta)$ is described next.

Proposition 2.1.3 (Shape of the hazard rate). Let $h(x; \theta)$ be given by (6).

- (i) For $-1 < \theta \leq -1/5$, $h(x; \theta)$ is strictly decreasing on $(0, \infty)$ with $h(0^+) = \infty$ (DFR).
- (ii) For $\theta > 0$, $h(x; \theta)$ is upside-down bathtub shaped: $h(0^+) = 0$, h increases to a unique interior maximum and then decreases to 0.

Proof. We use Glaser's theorem [15], which classifies the shape of $h = f/S$ through the function $\eta(x) = -f'(x)/f(x)$. From (3),

$$\begin{aligned} \eta(x) &= -\frac{\theta}{x} + \frac{2(\theta+1)x}{1+x^2}, \\ \eta'(x) &= \frac{\theta}{x^2} + \frac{2(\theta+1)(1-x^2)}{(1+x^2)^2} = \frac{g(x^2)}{x^2(1+x^2)^2}, \end{aligned}$$

where $g(t) = \theta + (4\theta + 2)t - (\theta + 2)t^2$, $t > 0$. The quadratic g has leading coefficient $-(\theta + 2) < 0$ and discriminant

$$\Delta = (4\theta + 2)^2 + 4\theta(\theta + 2) = 4(5\theta + 1)(\theta + 1).$$

(i) If $-1 < \theta \leq -1/5$, then $\Delta \leq 0$, so $g(t) \leq 0$ for all t (with equality at most at one point). Hence η is decreasing on $(0, \infty)$ and, by Glaser's theorem, h is decreasing. Moreover $f(0^+) = \infty$ while $S(0) = 1$, so $h(0^+) = \infty$.

(ii) If $\theta > 0$, then $g(0) = \theta > 0$ and $g(t) \rightarrow -\infty$ as $t \rightarrow \infty$; since g is a concave quadratic it has exactly one positive root t_0 . Thus $\eta'(x) > 0$ for $0 < x < \sqrt{t_0}$ and $\eta'(x) < 0$ for $x > \sqrt{t_0}$. In addition $f'(y_0) = 0$ at the mode $y_0 = \sqrt{\theta/(\theta+2)}$ (Proposition 2.2.2). By Glaser's theorem, h is upside-down bathtub shaped with a unique change point. Finally $h(0^+) = 0$ because $f(x; \theta) \sim \frac{2}{B(a, a)} x^\theta \rightarrow 0$ and $S(0) = 1$, and $h(x) \rightarrow 0$ as $x \rightarrow \infty$ by Proposition 2.1.2.

Remark 2.1.4 (Transitional range $-1/5 < \theta \leq 0$). For $-1/5 < \theta < 0$ the function η' changes sign twice, so Glaser's theorem does not settle the shape analytically. A numerical evaluation of (6) on a fine logarithmic grid ($10^{-4} \leq x \leq 10^3$) shows that h remains strictly decreasing for $-1 < \theta \leq \theta^* \approx -0.0734$, whereas for $\theta^* < \theta < 0$ it decreases from $h(0^+) = \infty$ to a shallow local minimum, rises to a local maximum and then decays to zero. At $\theta = 0$ (the standard half-Cauchy law) $h(0) = 2/\pi$ and h is upside-down bathtub shaped with its maximum near $x \approx 0.43$ (see the curve $\theta = 0$ in Figure 3). Consequently,

the strict DFR property holds for $-1 < \theta \leq \theta^*$ (analytically for $\theta \leq -1/5$), but not for every $\theta \leq 0$. We also note that the survival function is not log-concave in general (e.g., for $\theta = 1$, $S(x) = 1/(1+x^2)$ and $(\ln S)'' > 0$ for $x > 1$), so arguments for the hazard shape cannot rely on log-concavity of S .

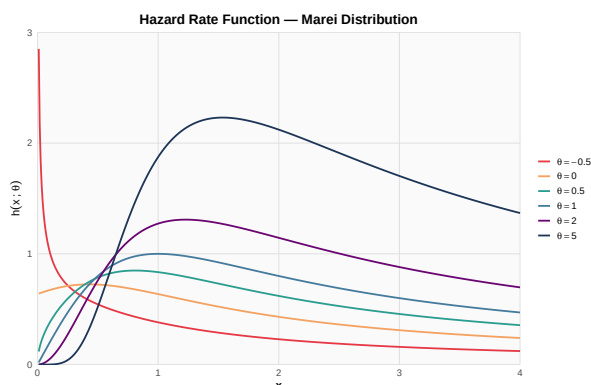


Fig. 3: Hazard rate function for selected values of θ .

2.2 Mathematical Properties

2.2.1 Quantile Function and Shape Measures

Proposition 2.2.1(Quantile function). For $u \in (0, 1)$, the quantile function $Q(u; \theta) = F^{-1}(u; \theta)$ is

$$Q(u; \theta) = \sqrt{\frac{q}{1-q}}, \quad q = I_u^{-1}(a, a), \quad (9)$$

where $I_u^{-1}(a, a)$ denotes the inverse regularized incomplete Beta function. It can be evaluated with standard algorithms for inverting the incomplete Beta function [38].

Proof. Setting $I_{Q^2/(1+Q^2)}(a, a) = u$ yields $q = I_u^{-1}(a, a)$, whence $Q^2/(1+Q^2) = q \implies Q = \sqrt{q/(1-q)}$.

Median. For $u = 0.5$, symmetry of $\text{Beta}(a, a)$ implies $q = 0.5$, hence $M = Q(0.5; \theta) = 1$ for all $\theta > -1$. This unit-median property establishes a normalized scale-free baseline. More generally, since $1 - U \sim \text{Beta}(a, a)$ and $1/X = \sqrt{(1-U)/U}$, the model is *reciprocally symmetric*: $X \stackrel{d}{=} 1/X$, i.e., $f(x; \theta) = x^{-2}f(1/x; \theta)$ and $Q(1-u; \theta) = 1/Q(u; \theta)$. In particular, $\ln X$ is symmetric about 0, $\mathbb{E}[\ln X] = 0$, and $\mathbb{E}[X^k] = \mathbb{E}[X^{-k}]$ whenever these moments exist (compare Theorem 2.2.3).

Bowley's coefficient of skewness.

$$B = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)} = \frac{Q(0.75) - 1}{Q(0.75) + 1} > 0, \quad (10)$$

where $Q(0.25) = 1/Q(0.75)$ due to symmetry of U . Thus, positive skewness holds for all $\theta > -1$.

Moors' coefficient of kurtosis.

$$M_K = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)}. \quad (11)$$

2.2.2 Mode

Proposition 2.2.2(Mode). The mode of X is

$$x_{\text{mode}} = \begin{cases} 0, & -1 < \theta \leq 0, \\ \sqrt{\frac{\theta}{\theta+2}}, & \theta > 0. \end{cases} \quad (12)$$

Proof. Differentiating $\ln f(x; \theta)$ with respect to x gives $\frac{\theta}{x} - \frac{2x(\theta+1)}{1+x^2} = 0 \implies x^2 = \frac{\theta}{\theta+2}$. This root is real and positive if and only if $\theta > 0$. For $\theta \leq 0$, the derivative is strictly negative on $(0, \infty)$, so the mode occurs at $x = 0$.

2.2.3 Moments

Theorem 2.2.3(Raw moments). The k -th raw moment $\mathbb{E}[X^k]$ exists if and only if $-(\theta+1) < k < \theta+1$, and is given by

$$\mathbb{E}[X^k] = \frac{B\left(\frac{\theta+k+1}{2}, \frac{\theta-k+1}{2}\right)}{B\left(\frac{\theta+1}{2}, \frac{\theta+1}{2}\right)}. \quad (13)$$

Proof. Using substitution $u = x^2/(1+x^2)$, $\int_0^\infty x^{\theta+k}(1+x^2)^{-(\theta+1)}dx = \frac{1}{2}B\left(\frac{\theta+k+1}{2}, \frac{\theta-k+1}{2}\right)$. Multiplying by $2/B(a, a)$ yields (13). Convergence requires both Beta arguments to be strictly positive, i.e., $-\theta-1 < k < \theta+1$.

Corollary 2.2.4(Simplified Mean and Variance). For $\theta > 0$, the mean $\mu = \mathbb{E}[X]$ simplifies to

$$\mu = \frac{\theta}{2\pi} \left[B\left(\frac{\theta}{2}, \frac{1}{2}\right) \right]^2. \quad (14)$$

For $\theta > 1$, the variance $\sigma^2 = \text{Var}(X)$ simplifies to

$$\sigma^2 = \frac{\theta+1}{\theta-1} - \frac{\theta^2}{4\pi^2} \left[B\left(\frac{\theta}{2}, \frac{1}{2}\right) \right]^4. \quad (15)$$

Proof. Using Legendre's duplication formula for Gamma functions, $B(a, a) = \frac{\Gamma(a)^2}{\Gamma(2a)} = \frac{\sqrt{\pi}\Gamma(a)}{2^{2a-1}\Gamma(a+1/2)}$. Applying this identity to $E[X] = B(\frac{\theta+2}{2}, \frac{\theta}{2})/B(\frac{\theta+1}{2}, \frac{\theta+1}{2})$ and $E[X^2] = \frac{\theta+1}{\theta-1}$ yields (14) and (15).

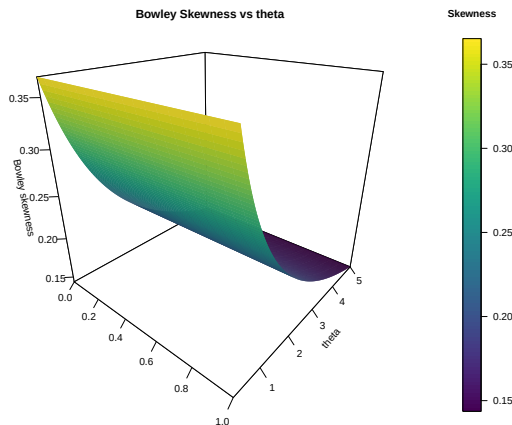


Fig. 4: 3D surface of Bowley's skewness coefficient.

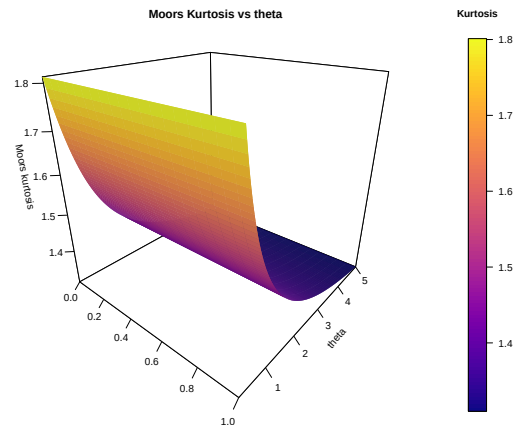


Fig. 5: 3D surface of Moors' kurtosis coefficient.

2.2.4 Incomplete Moments

Proposition 2.2.5(Incomplete moments). For $s > -(\theta + 1)$ and $t > 0$, the s -th incomplete moment $\kappa_s(t) = \int_0^t x^s f(x; \theta) dx$ is

$$\kappa_s(t) = \frac{B_z\left(\frac{\theta+s+1}{2}, \frac{\theta-s+1}{2}\right)}{B\left(\frac{\theta+1}{2}, \frac{\theta+1}{2}\right)}, \quad \text{where } z = \frac{t^2}{1+t^2}. \quad (16)$$

Here $B_z(p, q) = \int_0^z u^{p-1}(1-u)^{q-1} du$ denotes the non-regularized lower incomplete Beta function. Note that the upper bound constraint $s < \theta + 1$ is only required for complete moments ($t \rightarrow \infty$), whereas for finite upper limit $t < \infty$, the integral exists for all $s > -(\theta + 1)$.

Proof. Substituting $u = x^2/(1+x^2)$ into the integrand gives $\int_0^z u^{\frac{\theta+s+1}{2}-1}(1-u)^{\frac{\theta-s+1}{2}-1} du = B_z\left(\frac{\theta+s+1}{2}, \frac{\theta-s+1}{2}\right)$. Dividing by $B(a, a)$ yields (16). Since $z < 1$ for finite t , convergence at $u = 1$ is not required, so $s < \theta + 1$ is unnecessary.

Remark 2.2.6(Incomplete moments with a non-positive second Beta argument). When $s \geq \theta + 1$, the second argument $q = (\theta - s + 1)/2$ of $B_z(p, q)$ in (16) is zero or negative. In this case the complete Beta function $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is either undefined or does not correspond to a convergent integral, so $B_z(p, q)$ must not be computed as $I_z(p, q)B(p, q)$ with the regularized function I_z , which is meaningless here. The integral defining $B_z(p, q)$ is nevertheless finite for every $0 < z < 1$ and every real q , because the factor $(1-u)^{q-1}$ is bounded on $[0, z]$. It can be evaluated through the Gauss hypergeometric representation

$$B_z(p, q) = \frac{z^p}{p} {}_2F_1(p, 1-q; p+1; z),$$

for $0 < z < 1$ and $p > 0$, which is valid for all real q (including $q \leq 0$) and is convenient for numerical evaluation. As $t \rightarrow \infty$ (i.e., $z \rightarrow 1$), $\kappa_s(t)$ diverges when $s \geq \theta + 1$, in agreement with Theorem 2.2.3.

2.2.5 Lorenz and Bonferroni Curves

For $\theta > 0$, the Lorenz $LC(u)$ and Bonferroni $BC(u)$ curves are given by:

$$LC(u) = \frac{\kappa_1(Q(u; \theta))}{\mu} = \frac{B_q\left(\frac{\theta+2}{2}, \frac{\theta}{2}\right)}{B\left(\frac{\theta+1}{2}, \frac{\theta+1}{2}\right)\mu}, \quad (17)$$

$$BC(u) = \frac{LC(u)}{u}, \quad (18)$$

where $q = I_u^{-1}(a, a)$.

2.2.6 Asymptotic Tail Behavior

Proposition 2.2.7(Asymptotics). The PDF exhibits the following limiting behaviors:

$$f(x; \theta) \sim \begin{cases} \frac{2}{B(a, a)} x^\theta, & x \rightarrow 0^+, \\ \frac{2}{B(a, a)} x^{-(\theta+2)}, & x \rightarrow \infty, \end{cases} \quad (19)$$

2.2.7 Shannon Differential Entropy

Theorem 2.2.8(Shannon entropy). The Shannon entropy $H(X) = -\mathbb{E}[\ln f(X; \theta)]$ is

$$H(X) = \ln B(a, a) - \ln 2 + (\theta + 1) [\psi(2a) - \psi(a)], \quad (20)$$

where $\psi(z) = d \ln \Gamma(z) / dz$ is the digamma function.

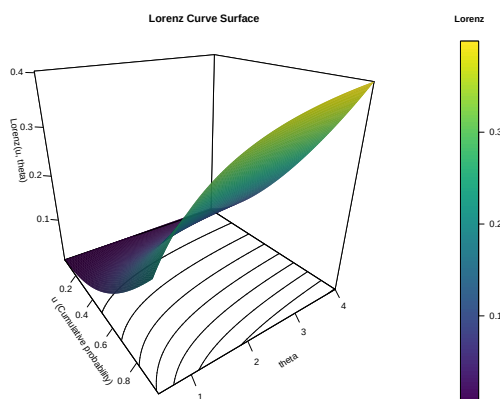


Fig. 6: 3D Lorenz curve surface.

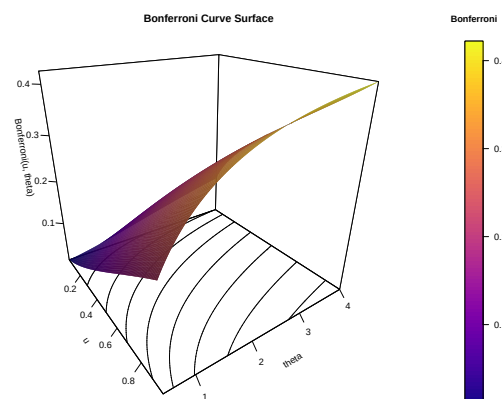


Fig. 7: 3D Bonferroni curve surface.

2.2.8 Rényi and Tsallis Entropies

Theorem 2.2.9(Rényi and Tsallis Entropies). For $\delta > 0$ with $\delta \neq 1$ satisfying

$$\begin{cases} \frac{1}{\theta+2} < \delta, & \theta \geq 0, \\ \frac{1}{\theta+2} < \delta < -\frac{1}{\theta}, & -1 < \theta < 0, \end{cases} \quad (21)$$

the Rényi entropy $H_R(\delta) = \frac{1}{1-\delta} \ln \int_0^\infty [f(x; \theta)]^\delta dx$ is given in closed-form by

$$H_R(\delta) = \frac{1}{1-\delta} \left[(1-\delta) \ln 2 - \delta \ln B(a, a) + \ln B\left(\frac{\delta\theta+1}{2}, \frac{\delta(\theta+2)-1}{2}\right) \right], \quad (22)$$

where $a = (\theta + 1)/2$. The Tsallis entropy $H_T(\delta) = \frac{1}{\delta-1} (1 - \int_0^\infty [f(x; \theta)]^\delta dx)$ is given by

$$H_T(\delta) = \frac{1}{\delta-1} \left[1 - 2^{\delta-1} [B(a, a)]^{-\delta} \times B\left(\frac{\delta\theta+1}{2}, \frac{\delta(\theta+2)-1}{2}\right) \right]. \quad (23)$$

Proof. Consider the integral

$$I(\delta) = \int_0^\infty [f(x; \theta)]^\delta dx = \left(\frac{2}{B(a, a)} \right)^\delta \int_0^\infty \frac{x^{\delta\theta} dx}{(1+x^2)^{\delta(\theta+1)}}.$$

Applying the transformation $u = x^2/(1+x^2)$, we obtain

$$\begin{aligned} & \int_0^\infty \frac{x^{\delta\theta}}{(1+x^2)^{\delta(\theta+1)}} dx \\ &= \frac{1}{2} \int_0^1 u^{\frac{\delta\theta+1}{2}-1} (1-u)^{\frac{\delta(\theta+2)-1}{2}-1} du \\ &= \frac{1}{2} B\left(\frac{\delta\theta+1}{2}, \frac{\delta(\theta+2)-1}{2}\right). \end{aligned}$$

Convergence of the Beta integral requires $\frac{\delta\theta+1}{2} > 0$ and $\frac{\delta(\theta+2)-1}{2} > 0$. The second condition gives $\delta > \frac{1}{\theta+2}$ (integrability at infinity). The first condition (integrability at the origin) holds automatically when $\theta \geq 0$, but for $-1 < \theta < 0$ it requires $\delta\theta > -1$, i.e., $\delta < -1/\theta$; this yields (21). For example, with $\theta = -0.5$ the integral $\int_0^\infty f^\delta dx$ is finite only for $2/3 < \delta < 2$. The interval in (21) is non-empty for every $\theta > -1$ and always contains $\delta = 1$. Substituting $I(\delta)$ into the definitions of Rényi and Tsallis entropies completes the proof. Taking the limit as $\delta \rightarrow 1$ via L'Hôpital's rule recovers the Shannon entropy in Theorem 2.2.8.

2.2.9 Order Statistics and Extreme Value Asymptotics

Theorem 2.2.10(Order Statistics). Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics from an i.i.d. sample of size n from $f_X(x; \theta)$. The PDF of the k -th order statistic $X_{(k)}$ ($1 \leq k \leq n$) is

$$f_{(k)}(x; \theta) = \frac{n!}{(k-1)!(n-k)!} \frac{2}{B(a, a)} \frac{x^\theta}{(1+x^2)^{\theta+1}} \times \left[I_{\frac{x^2}{1+x^2}}(a, a) \right]^{k-1} \left[I_{\frac{1}{1+x^2}}(a, a) \right]^{n-k}, \quad (24)$$

where $a = (\theta + 1)/2$.

Theorem 2.2.11(Extreme Value Limit – Fréchet Domain of Attraction). The maximum order statistic $X_{(n)} = \max(X_1, \dots, X_n)$ belongs to the Fréchet domain of attraction $\mathcal{D}(\Phi_{\theta+1})$. That is, there exist norming constants $a_n = \left[\frac{2n}{(\theta+1)B(a, a)} \right]^{\frac{1}{\theta+1}} > 0$ and $b_n = 0$ such that

$$\lim_{n \rightarrow \infty} P\left(\frac{X_{(n)}}{a_n} \leq x\right) = \exp\left(-x^{-(\theta+1)}\right), \quad (25)$$

for $x > 0$, where the limit $\Phi_{\theta+1}(x) = \exp(-x^{-(\theta+1)})$ is the standard Fréchet distribution with extreme value index $\alpha = \theta + 1$.

Proof. According to Fisher–Tippett–Gnedenko theorem, a distribution F belongs to the Fréchet domain of attraction $\mathcal{D}(\Phi_\alpha)$ if and only if its survival function $1 - F(x)$ is regularly varying at infinity with index $-\alpha$, i.e., $\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$ for $x > 0$. From Proposition 2.1.2, we have $1 - F(x; \theta) \sim \frac{2}{(\theta+1)B(a,a)} x^{-(\theta+1)}$ as $x \rightarrow \infty$. Thus,

$$\lim_{t \rightarrow \infty} \frac{1 - F(tx; \theta)}{1 - F(t; \theta)} = \lim_{t \rightarrow \infty} \frac{(tx)^{-(\theta+1)}}{t^{-(\theta+1)}} = x^{-(\theta+1)}.$$

Hence, $1 - F(x; \theta) \in \mathcal{RV}_{-(\theta+1)}$ with tail index $\alpha = \theta + 1$. Setting $n[1 - F(a_n; \theta)] = 1$ yields the norming constant $a_n = \left[\frac{2n}{(\theta+1)B(a,a)} \right]^{\frac{1}{\theta+1}}$. Consequently, $\lim_{n \rightarrow \infty} [F(a_n x; \theta)]^n = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n} x^{-(\theta+1)} \right]^n = \exp(-x^{-(\theta+1)})$, establishing the Fréchet limit.

2.2.10 Stress-Strength Reliability Analysis

Stress-strength reliability has been studied extensively for one-parameter and related lifetime models under complete, record and censored sampling schemes [1, 5, 13, 28].

Theorem 2.2.12 (Stress-Strength Reliability). Let $X \sim f(x; \theta_1)$ (strength) and $Y \sim f(y; \theta_2)$ (stress) be independent. Then, for all $\theta_1, \theta_2 > -1$,

$$R = P(Y < X) = \frac{1}{2}. \quad (26)$$

Proof. By the reciprocal symmetry of Section 2.2 (Median paragraph), $X \stackrel{d}{=} 1/X$ and $Y \stackrel{d}{=} 1/Y$; by independence, $(X, Y) \stackrel{d}{=} (1/X, 1/Y)$. Hence $P(Y < X) = P(1/Y < 1/X) = P(X < Y)$. Since X and Y are continuous and independent, $P(X = Y) = 0$, so $2R = 1$. Equivalently, writing $R = \int_0^1 I_u(a_2, a_2) B(a_1, a_1)^{-1} u^{a_1-1} (1-u)^{a_1-1} du$ with $a_j = (\theta_j + 1)/2$ and substituting $u \mapsto 1-u$ gives $R = 1 - R$.

Remark 2.2.13. Theorem 2.2.12 shows that, because every member of the family has median one and $\ln X$ is symmetric about zero, the shape parameter alone cannot separate stress from strength: the two variables differ only in tail heaviness, not in location. A non-trivial stress-strength model requires the scale extension $X_j = \sigma_j Z_j$ with $Z_j \sim f(\cdot; \theta_j)$ discussed in Section 3.3; then

$$R = P(\sigma_2 Z_2 < \sigma_1 Z_1) = \int_0^\infty F\left(\frac{\sigma_1}{\sigma_2} x; \theta_2\right) f(x; \theta_1) dx,$$

which equals $1/2$ when $\sigma_1 = \sigma_2$ and increases strictly with σ_1/σ_2 .

2.3 Estimation

In this section, three different methods of estimation, such as; maximum likelihood estimation, quantile matching estimation, and Bayesian estimation [7–9, 12, 16] [4, 6], will be discussed.

2.3.1 Maximum Likelihood Estimation

Let $x_1, \dots, x_n$ be an i.i.d. sample from $f_X(x; \theta)$. The log-likelihood function is

$$\begin{aligned} \ell(\theta) &= n \ln 2 - n \ln B(a, a) + \theta \sum_{i=1}^n \ln x_i \\ &\quad - (\theta + 1) \sum_{i=1}^n \ln(1 + x_i^2). \end{aligned} \quad (27)$$

The score function $U(\theta) = d\ell(\theta)/d\theta$ is (see Appendix A for the derivation)

$$U(\theta) = -n[\psi(a) - \psi(2a)] + \sum_{i=1}^n [\ln x_i - \ln(1 + x_i^2)] = 0. \quad (28)$$

Proposition 2.3.1 (Fisher information and log-concavity). The expected Fisher information per observation is

$$\mathcal{J}(\theta) = \frac{1}{2} [\psi_1(a) - 2\psi_1(2a)] > 0 \quad \text{for all } \theta > -1, \quad (29)$$

where $\psi_1(z) = d\psi(z)/dz$ is the trigamma function. Thus, $\ell(\theta)$ is strictly concave and the maximum likelihood estimator (MLE) $\hat{\theta}$ is unique.

Proof. Differentiating $-U(\theta)$ with respect to θ gives $-U'(\theta) = n[\frac{1}{2}\psi_1(a) - \psi_1(2a)]$, which does not depend on the data; hence the observed and expected information coincide and $\mathcal{J}(\theta) = \frac{1}{2}[\psi_1(a) - 2\psi_1(2a)]$. By the trigamma duplication formula, $\psi_1(a) + \psi_1(a + 1/2) = 4\psi_1(2a)$. Since ψ_1 is strictly decreasing, $\psi_1(a) > \psi_1(a + 1/2)$, so $2\psi_1(a) > 4\psi_1(2a) \implies \psi_1(a) > 2\psi_1(2a)$, proving $\mathcal{J}(\theta) > 0$.

Remark 2.3.2 (Exponential-family structure).

Equation (27) can be written as $\ell(\theta) = \theta T_n - n \ln B(a, a) - \sum_{i=1}^n \ln(1 + x_i^2) + n \ln 2$ with $T_n = \sum_{i=1}^n \ln\{x_i/(1 + x_i^2)\} = -\sum_{i=1}^n \ln(x_i + x_i^{-1})$. Hence the model is a regular one-parameter exponential family with natural parameter θ and sufficient statistic T_n . Each summand $\ln(X + X^{-1}) \geq \ln 2$ has finite moments of all orders for every $\theta > -1$ (its tail is logarithmic in that of X), with $\text{Var}\{\ln(X + X^{-1})\} = \mathcal{J}(\theta)$. This is the reason why the MLE is consistent and asymptotically normal, $\sqrt{n}(\hat{\theta} - \theta) \rightarrow N(0, \mathcal{J}(\theta)^{-1})$, for all $\theta > -1$, including values such as $\theta = -0.5$ for which $\mathbb{E}[X] = \infty$ (Section 3.1).

Numerical evaluation of the MLE is implemented using a standard Newton–Raphson iteration, which here coincides with Fisher scoring because the observed information equals $n\mathcal{J}(\theta)$:

$$\begin{aligned}\theta^{(k+1)} &= \theta^{(k)} + \frac{U(\theta^{(k)})}{n\mathcal{J}(\theta^{(k)})} \\ &= \theta^{(k)} + \frac{-[\psi(a_k) - \psi(2a_k)] + \bar{T}}{\frac{1}{2}[\psi_1(a_k) - 2\psi_1(2a_k)]},\end{aligned}\quad (30)$$

where $a_k = (\theta^{(k)} + 1)/2$ and $\bar{T} = T_n/n$, with U and $\mathcal{J}$ given by (28) and (29). We initialize at $\theta^{(0)} = 0$ (the half-Cauchy member) and halve the step whenever an update would leave the parameter space $(-1, \infty)$. Because $\ell(\theta)$ is strictly concave, the iteration converges to the unique MLE; in our Python 3.11 implementation it reached $|\theta^{(k+1)} - \theta^{(k)}| < 10^{-10}$ in every one of 25,000 samples generated under the design of Section 3.1 and in both real datasets (median 5–9 iterations, at most 19 iterations, the largest counts occurring for $\theta = -0.5$ where step-halving is occasionally activated).

2.3.2 Quantile Matching Estimation

As an alternative non-iterative/quantile-based estimation procedure, the Quantile Matching Estimator (QME) estimates parameter θ by minimizing the weighted squared error between sample quantiles $\hat{Q}(p_j)$ and theoretical quantiles $Q(p_j; \theta)$:

$$\hat{\theta}_{\text{QME}} = \arg \min_{\theta > -1} \sum_{j=1}^k w_j \left[\hat{Q}(p_j) - \sqrt{\frac{I_{p_j}^{-1}(a, a)}{1 - I_{p_j}^{-1}(a, a)}} \right]^2, \quad (31)$$

where $p_j = j/(k+1)$ are uniform quantile probabilities and $w_j > 0$ are weights. When $k = 1$ and $p_1 = 0.5$, the shape-free unit median $Q(0.5; \theta) = 1$ provides a normalized benchmark.

2.3.3 Bayesian Estimation

We reparameterize $\phi = \theta + 1 > 0$ with a Gamma prior $\phi \sim \text{Gamma}(\alpha_0, \beta_0)$, inducing:

$$\pi(\theta) \propto (\theta + 1)^{\alpha_0 - 1} e^{-\beta_0(\theta + 1)}, \quad \theta > -1. \quad (32)$$

We consider three prior specifications to assess estimation robustly:

- (i) **Weak (Non-informative):** $\alpha_0 = \beta_0 = 0.01$.
- (ii) **Informative (Centered):** $\alpha_0 = 5(\theta_{\text{true}} + 1)$, $\beta_0 = 5$.
- (iii) **Misspecified Informative:** $\alpha_0 = 5(1.3\theta_{\text{true}} + 1)$, $\beta_0 = 5$ (reflecting a +30% prior bias).

Posterior sampling is carried out via Metropolis–Hastings Markov chain Monte Carlo (MCMC) with a Gaussian proposal centered at the current value, using 4,000 iterations, 1,000 burn-in draws, and target acceptance rates between 20% and 40%.

Proposal scale and convergence diagnostics. The random-walk proposal is $\theta^* \sim N(\theta^{(t)}, \tau^2)$ with $\tau = 3\widehat{\text{SE}}(\hat{\theta}) = 3/\sqrt{n\mathcal{J}(\hat{\theta})}$, i.e., the proposal scale is tied to the asymptotic posterior standard deviation in the spirit of Gelman, Roberts and Gilks [14]; proposals with $\theta^* \leq -1$ are rejected. Chains are started at the MLE in the simulation study. For the real datasets we run four chains started at $\hat{\theta} \pm 2\widehat{\text{SE}}$ and $\hat{\theta} \pm 4\widehat{\text{SE}}$ and assess convergence with (i) the acceptance rate, (ii) the split- $\hat{R}$ potential scale reduction factor (target $\hat{R} < 1.01$), (iii) the effective sample size (ESS) and Monte Carlo standard error (MCSE) of the posterior mean, (iv) Geweke's z -scores comparing the first 10% and last 50% of each post-burn-in chain, and (v) trace and autocorrelation plots. The results are reported in Table 8 and Figure 17.

3 Results and Discussion

3.1 Simulation Study

3.1.1 Design

To evaluate estimator performance under small, moderate, and large samples, we consider $n \in \{50, 100, 200, 500, 1000\}$ and true parameter values $\theta \in \{-0.5, 0.5, 1.0, 2.0, 5.0\}$ across $N = 1,000$ Monte Carlo replications. The boundary value $\theta = -0.5$ is specifically included to assess numerical MLE stability when theoretical population moments do not exist ($E[X] = \infty$). Evaluation metrics include Bias, Mean Squared Error (MSE), Root MSE (RMSE), and 95% empirical coverage. Samples are generated as $X = \sqrt{U/(1-U)}$ with $U = G_1/(G_1 + G_2)$, $G_j \sim \text{Gamma}(a, 1)$ independent; to avoid floating-point overflow when U is extremely close to 0 or 1 (which occurs frequently for small a , e.g. $a = 0.25$ when $\theta = -0.5$), all computations are carried out on the log scale, using $\ln\{x/(1+x^2)\} = \frac{1}{2}\ln\{U(1-U)\}$. MLE

intervals are Wald intervals $\hat{\theta} \pm 1.96/\sqrt{n\mathcal{J}(\hat{\theta})}$ with $\mathcal{J}$ from (29); Bayesian point estimates are posterior means and intervals are equal-tailed 95% credible intervals from the 3,000 post-burn-in draws. The full study (25 scenarios, four estimators) was run with a fixed random seed, and the code is available from the corresponding author.

3.1.2 Results

Figure 8 summarizes the scenario $\theta = 1$; the remaining scenarios in Table 2 show the same pattern. Empirical coverage of the MLE and Bayes.Weak intervals lies between 93.9% and 96.1% in all 25 scenarios. The informative prior (Bayes.Inf) yields the smallest MSE in small samples, at the cost of conservative coverage (up to

Table 2: Monte Carlo simulation results ($N = 1,000$ replications) for all scenarios $\theta \in \{-0.5, 0.5, 1, 2, 5\}$ and $n \in \{50, 100, 200, 500, 1000\}$: bias, mean squared error (MSE; the RMSE is $\sqrt{\text{MSE}}$) and empirical coverage (Cov.) of the 95% confidence/credible intervals. The mean Metropolis–Hastings acceptance rate lies between 0.27 and 0.37 in all scenarios.

θ	n	MLE			Bayes.Weak			Bayes.Inf			Bayes.Miss		
		Bias	MSE	Cov.	Bias	MSE	Cov.	Bias	MSE	Cov.	Bias	MSE	Cov.
-0.5	50	0.0155	0.0064	0.953	0.0163	0.0065	0.950	0.0146	0.0057	0.961	0.0066	0.0053	0.955
	100	0.0083	0.0031	0.946	0.0087	0.0031	0.944	0.0082	0.0029	0.946	0.0043	0.0028	0.948
	200	0.0029	0.0014	0.951	0.0030	0.0014	0.951	0.0030	0.0014	0.951	0.0010	0.0014	0.951
	500	0.0011	0.0006	0.950	0.0012	0.0006	0.947	0.0012	0.0005	0.954	0.0003	0.0005	0.949
	1000	0.0006	0.0003	0.952	0.0007	0.0003	0.945	0.0006	0.0003	0.948	0.0002	0.0003	0.949
0.5	50	0.0340	0.0667	0.955	0.0378	0.0677	0.947	0.0245	0.0445	0.969	0.0495	0.0478	0.966
	100	0.0244	0.0322	0.949	0.0265	0.0324	0.948	0.0219	0.0261	0.953	0.0356	0.0275	0.951
	200	0.0095	0.0144	0.961	0.0107	0.0145	0.950	0.0095	0.0131	0.965	0.0166	0.0134	0.955
	500	0.0011	0.0061	0.948	0.0014	0.0061	0.951	0.0015	0.0059	0.948	0.0045	0.0059	0.951
	1000	0.0014	0.0027	0.958	0.0017	0.0027	0.956	0.0016	0.0027	0.961	0.0030	0.0027	0.957
1	50	0.0581	0.1384	0.943	0.0635	0.1409	0.941	0.0350	0.0763	0.973	0.1028	0.0916	0.971
	100	0.0361	0.0642	0.940	0.0390	0.0648	0.939	0.0303	0.0478	0.951	0.0672	0.0538	0.949
	200	0.0146	0.0277	0.955	0.0161	0.0277	0.952	0.0139	0.0240	0.964	0.0338	0.0257	0.963
	500	0.0064	0.0121	0.942	0.0068	0.0121	0.939	0.0068	0.0114	0.950	0.0149	0.0117	0.950
	1000	0.0045	0.0060	0.947	0.0048	0.0060	0.943	0.0048	0.0059	0.944	0.0087	0.0059	0.943
2	50	0.0833	0.3000	0.959	0.0902	0.3015	0.959	0.0377	0.1233	0.978	0.2288	0.1930	0.978
	100	0.0393	0.1492	0.944	0.0431	0.1501	0.946	0.0269	0.0948	0.969	0.1387	0.1217	0.960
	200	0.0292	0.0708	0.949	0.0306	0.0710	0.944	0.0249	0.0561	0.962	0.0879	0.0664	0.957
	500	0.0005	0.0293	0.942	0.0010	0.0294	0.939	0.0008	0.0266	0.946	0.0272	0.0281	0.945
	1000	0.0035	0.0132	0.955	0.0037	0.0132	0.956	0.0038	0.0126	0.955	0.0170	0.0131	0.957
5	50	0.2489	1.5459	0.954	0.2515	1.5310	0.949	0.0615	0.3066	0.997	0.8337	1.0842	0.921
	100	0.0995	0.6886	0.952	0.1015	0.6881	0.952	0.0414	0.2845	0.981	0.5562	0.6447	0.929
	200	0.0507	0.3354	0.948	0.0512	0.3352	0.941	0.0313	0.2055	0.961	0.3430	0.3464	0.927
	500	0.0146	0.1245	0.954	0.0144	0.1246	0.952	0.0120	0.1024	0.960	0.1528	0.1308	0.943
	1000	0.0154	0.0605	0.955	0.0154	0.0607	0.953	0.0145	0.0550	0.959	0.0889	0.0639	0.946

99.7% for $n = 50$, $\theta = 5$). The misspecified prior (Bayes.Miss) has the largest bias for small n and large θ (e.g., bias 0.834 and coverage 92.1% for $n = 50$, $\theta = 5$), but its influence fades as n grows.

3.1.3 Discussion

1.Consistency: All estimation methods demonstrate asymptotic unbiasedness and $\mathcal{O}(1/n)$ MSE decay as n increases. For $n = 1000$, $n \times \text{MSE}(\hat{\theta}_{\text{MLE}})$ equals 0.279, 2.741, 5.999, 13.198 and 60.525 for $\theta = -0.5, 0.5, 1, 2, 5$, against the theoretical values $1/\mathcal{I}(\theta) = 0.273, 2.975, 5.633, 13.799$ and 61.942, which also confirms the corrected Fisher information (29) numerically.

2.Boundary behavior ($\theta = -0.5$): The stability of the MLE when $\mathbb{E}[X] = \infty$ is assessed by four explicit criteria: (i) convergence of the Newton–Raphson iteration (30) in every replication (no failures, no estimates on the boundary $\theta = -1$); (ii) the bias, which is below 0.016 for $n = 50$ and below 0.001 for

$n = 1000$; (iii) the agreement between the empirical MSE and the asymptotic variance $1/\{n\mathcal{I}(\theta)\}$ (ratio 1.02 at $n = 1000$); and (iv) the empirical coverage of the Wald intervals, which lies between 94.6% and 95.3%. The theoretical reason is given in Remark 2.3.2: the MLE depends on the data only through $T_n = -\sum \ln(X_i + X_i^{-1})$, whose summands have finite moments of all orders even when X has no finite mean, so infinite population moments of X do not affect likelihood-based inference on θ .

3.Sensitivity to prior misspecification: Mildly misspecified priors (Bayes.Miss) increase finite-sample bias for small samples, particularly for large θ where the prior shift 0.3θ is largest, but their influence quickly diminishes as likelihood information dominates (for $n \geq 500$ the coverage of Bayes.Miss is at least 94.3% in every scenario).

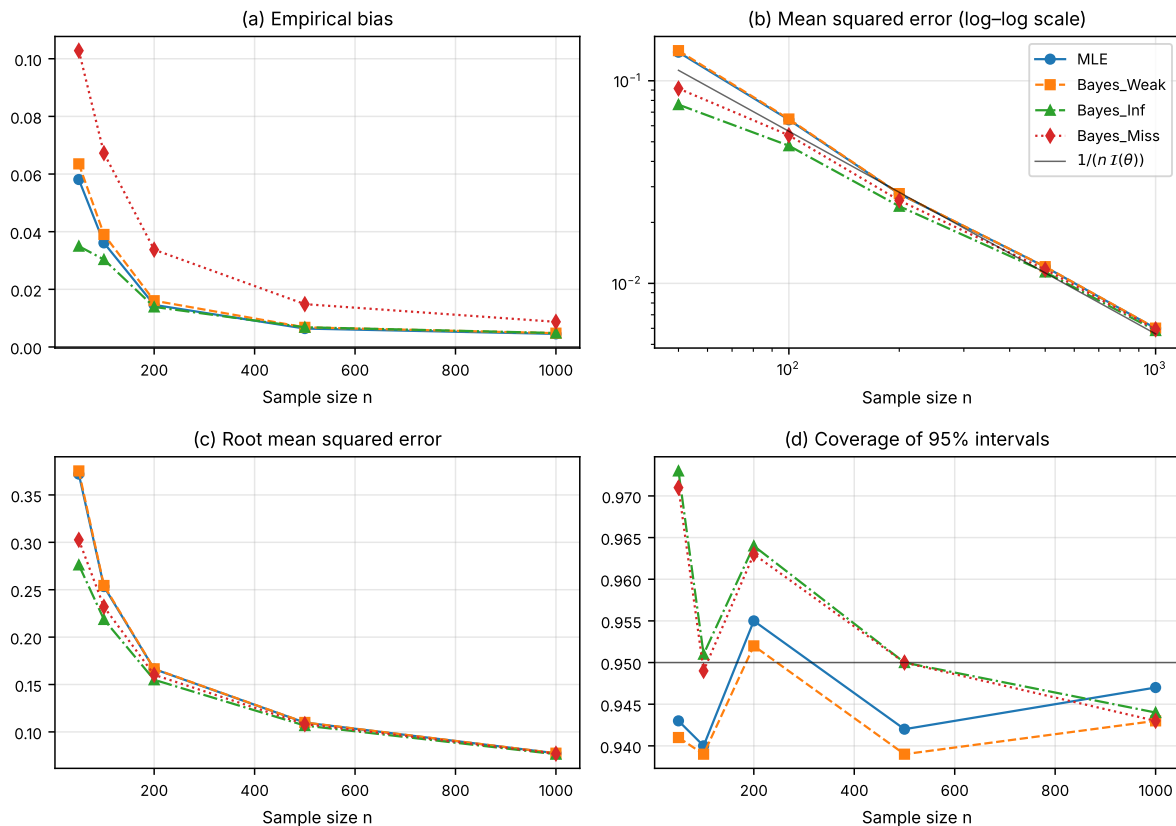


Fig. 8: Finite-sample performance of the four estimators for $\theta = 1$: (a) empirical bias, (b) MSE on log–log scale together with the asymptotic variance $1/\{n\mathcal{I}(\theta)\}$ (solid black line), (c) RMSE and (d) empirical coverage of the 95% confidence/credible intervals (horizontal line at the nominal level).

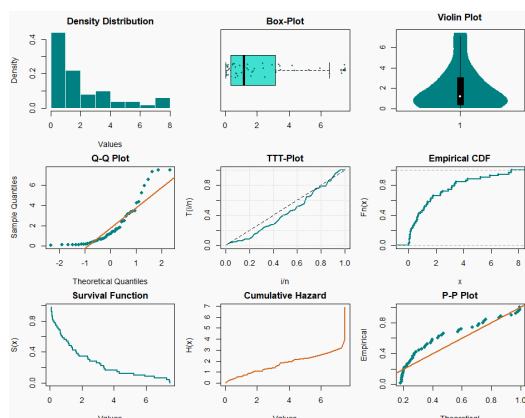


Fig. 9: Exploratory data analysis for the carbon fibre dataset.

The empirical histogram and QQ-plot reveal sharp right-skew and pronounced heavy upper tails which are incompatible with exponential assumptions. The Marei density overlays closely align with empirical frequencies and capture early-mode concentration as well as extreme-value tail behavior.

Table 3: Descriptive statistics for the carbon fibre dataset.

n	Min	Max	Mean	Median	Std. Dev	Skewness
50	0.04	7.49	1.975	1.190	2.116	1.366

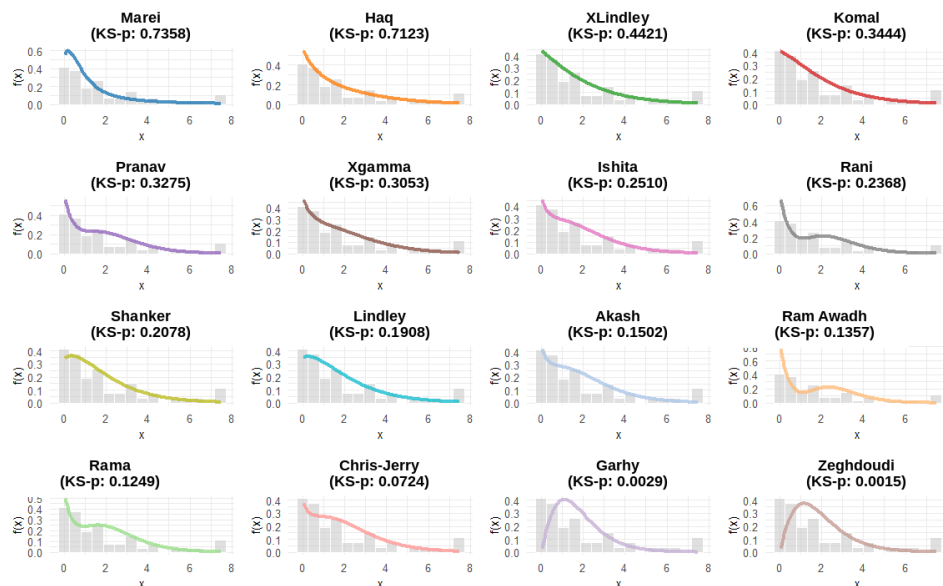
3.2 Applications

Fifteen one-parameter competitor distributions (Lindley [19], Garhy [11], Shanker [31], Akash [30], Ishita [35], Rani [33], Pranav [36], Rama [32], Xgamma [29], Chris-Jerry [25], XLindley [10], Komal [34], Ram Awadh [37], Zeghdoudi [20], and Haq [3]) are compared against the Marei distribution in Tables 4 and 6, so that sixteen one-parameter models are compared in each table. In addition, Section 3.2.3 compares the Marei model with two one-parameter heavy-tailed baselines (half-Cauchy with scale σ and one-parameter Lomax) and two classical two-parameter models (Weibull and Gamma).

Their fit is measured using the Akaike (AIC), Bayesian (BIC) and Hannan–Quinn (HQIC) information

Table 4: MLEs, SEs, and goodness-of-fit for the carbon fibre dataset.

Model	$\hat{\theta}$ (SE)	−LogLik	AIC	BIC	HQIC	KS (p)	CvM (p)	AD (p)
Marei	0.0710 (0.1681)	87.2858	176.5716	178.4836	177.2997	0.0938 (0.7358)	0.1003 (0.5854)	0.7291 (0.5343)
Haq	0.8108 (0.0833)	87.4258	176.8517	178.7637	177.5798	0.0958 (0.7123)	0.1089 (0.5445)	0.7853 (0.4911)
XLindley	0.6870 (0.0745)	87.8901	177.7802	179.6922	178.5083	0.1192 (0.4421)	0.1984 (0.2711)	1.4240 (0.1957)
Komal	0.7356 (0.0752)	87.5329	177.0657	178.9777	177.7938	0.1292 (0.3444)	0.2370 (0.2061)	1.7204 (0.1318)
Pranav	1.4365 (0.0788)	87.7512	177.5024	179.4144	178.2305	0.1311 (0.3275)	0.2674 (0.1675)	2.0268 (0.0890)
Xgamma	0.9836 (0.0966)	88.3054	178.6109	180.5229	179.3390	0.1337 (0.3053)	0.2770 (0.1571)	1.7805 (0.1219)
Ishita	1.1307 (0.0770)	87.3294	176.6587	178.5708	177.3868	0.1407 (0.2510)	0.3022 (0.1331)	2.2413 (0.0682)
Rani	1.7522 (0.0807)	89.2173	180.4346	182.3466	181.1627	0.1427 (0.2368)	0.2525 (0.1852)	2.0354 (0.0881)
Shanker	0.8323 (0.0754)	87.3296	176.6592	178.5712	177.3873	0.1471 (0.2078)	0.3168 (0.1211)	2.4452 (0.0532)
Lindley	0.7891 (0.0831)	87.7670	177.5340	179.4461	178.2621	0.1499 (0.1908)	0.3259 (0.1143)	2.3679 (0.0584)
Akash	1.1260 (0.0894)	87.8208	177.6416	179.5537	178.3698	0.1575 (0.1502)	0.3855 (0.0786)	2.6726 (0.0405)
Rama	1.4871 (0.0941)	89.6144	181.2289	183.1409	181.9570	0.1631 (0.1249)	0.4269 (0.0610)	2.9480 (0.0293)
Ram Awadh	2.0756 (0.0824)	91.6304	185.2607	187.1727	185.9888	0.1606 (0.1357)	0.2600 (0.1761)	2.2353 (0.0687)
Chris-Jerry	1.1026 (0.1030)	88.3711	178.7423	180.6543	179.4704	0.1786 (0.0724)	0.5158 (0.0359)	3.3396 (0.0186)
Garhy	1.4465 (0.1002)	108.7814	219.5627	221.4748	220.2908	0.2513 (0.0029)	0.9718 (0.0027)	11.4250 (0.0000)
Zeghdoudi	1.3176 (0.1105)	106.5900	215.1800	217.0920	215.9081	0.2630 (0.0015)	1.0653 (0.0016)	11.5568 (0.0000)

**Fig. 10:** Estimated PDFs for the carbon fibre dataset.

The Marei PDF provides a more accurate representation of empirical density peaks and tail decay. In contrast, competitors based on an exponential mixture approach tend to systematically underestimate extreme stresses. This showcases the urgent need of using power-law models to analyze the reliability of composite materials.

criteria and the Kolmogorov–Smirnov (KS), Cramér–von Mises (CvM) and Anderson–Darling (AD) statistics (with p -values); KS p -values are computed from the exact finite-sample null distribution of the Kolmogorov–Smirnov statistic, and CvM and AD p -values from their null distributions with parameters treated as known. For the comparison with the additional baselines we also report the small-sample corrected AIC, $AICc = AIC + 2k(k + 1)/(n - k - 1)$, and the

leave-one-out cross-validated log score $LOOCV = -\sum_{i=1}^n \ln f(x_i; \hat{\theta}_{(-i)})$, where $\hat{\theta}_{(-i)}$ is the MLE computed without the i -th observation and k is the number of parameters. Smaller values are better for all criteria.

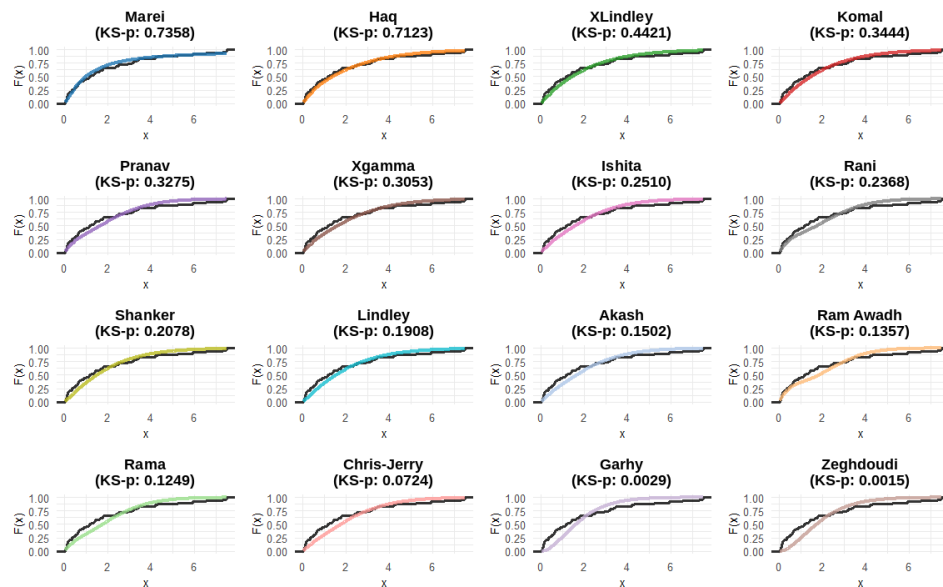


Fig. 11: Estimated CDFs for the carbon fibre dataset.

The CDF overlay shows close agreement of the Marei model with the empirical CDF, including the upper quantiles, where several exponential-mixture competitors deviate more visibly.

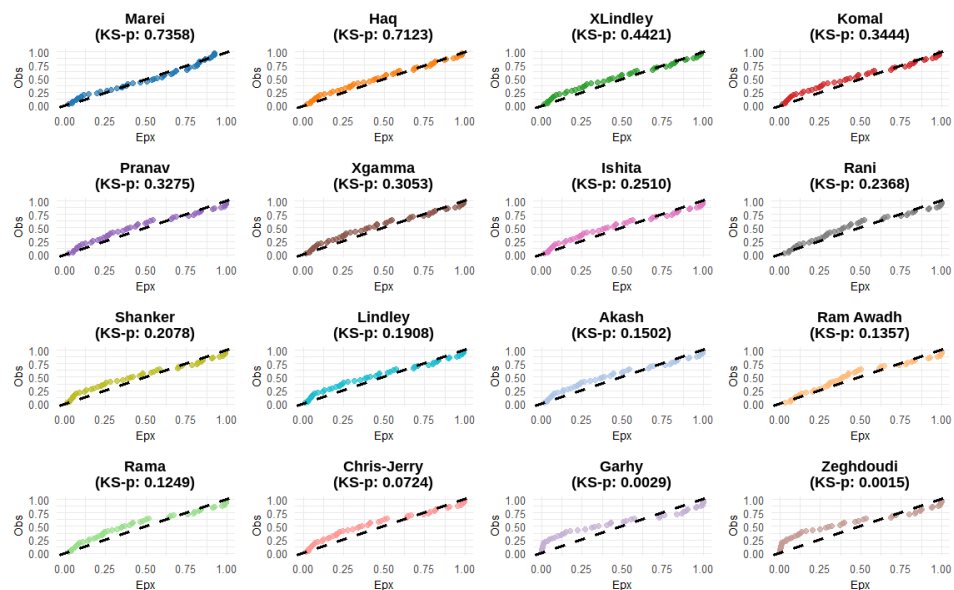


Fig. 12: P-P plots for the carbon fibre dataset.

Points cluster along the diagonal for Marei; the maximal vertical deviation is 0.084 (squared correlation 0.98), with no systematic pattern of departure.

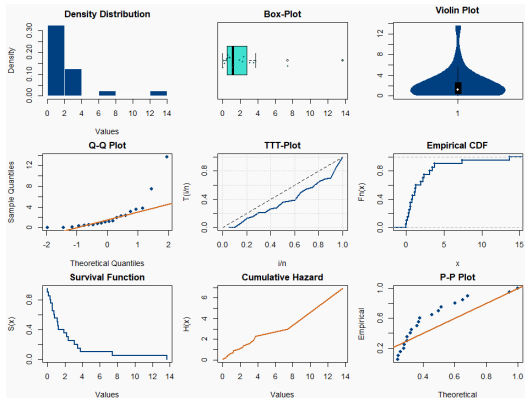
3.2.1 Dataset 1: Carbon Fibre Breaking Stress

$n = 50$ observations of carbon fibre breaking stress (GPa) from [22]: 0.37, 1.12, 1.22, 4.32, 0.17, 0.64, 1.42, 1.16,

1.67, 0.09, 0.13, 0.25, 0.08, 0.04, 2.35, 0.20, 0.78, 0.34, 1.02, 0.17, 1.76, 2.39, 0.50, 1.35, 3.36, 0.45, 0.90, 2.92, 6.53, 1.62, 7.46, 3.19, 2.49, 1.40, 7.49, 0.57, 0.14, 0.63,

Table 5: Descriptive statistics for the electrical products dataset.

n	Min	Max	Mean	Median	Std. Dev	Skewness
20	0.0003	13.6866	2.2662	1.1762	3.2189	2.5389

**Fig. 13:** Exploratory data analysis for the electrical products dataset.

Extreme right skewness ($\gamma_1 = 2.54$) and long tails of distribution suggests catastrophic failure modes. This regime is easily integrated with the Marei framework without any arbitrary truncations or ad-hoc thresholding.

5.23, 0.71, 0.68, 0.12, 0.09, 3.47, 5.93, 1.82, 4.20, 7.29, 3.13, 3.41.

The Marei distribution achieves the lowest AIC (176.57), BIC (178.48), KS (0.094, $p = 0.736$), CvM (0.100), and AD (0.729) among all sixteen one-parameter models of Table 4. Figures 10–12 confirm the fit graphically.

3.2.2 Dataset 2: Electrical Products Failure Times

$n = 20$ failure times (years) from [21]: 0.0003, 0.0298, 0.1648, 0.3529, 0.4044, 0.5712, 0.5808, 0.7607, 0.8188, 1.1296, 1.2228, 1.2773, 1.9115, 2.2333, 2.3791, 3.0916, 3.4999, 3.7744, 7.4339, 13.6866.

The Marei model ranks first on all criteria among the sixteen one-parameter models of Table 6: AIC = 72.34, BIC = 73.34, KS = 0.106 ($p = 0.960$), CvM = 0.046, AD = 0.444.

3.2.3 Comparison with Heavy-Tailed and Two-Parameter Baselines

Table 7 compares the Marei model with the half-Cauchy distribution with scale parameter, $f(x; \sigma) = 2/[\pi\sigma(1 + x^2/\sigma^2)]$, the one-parameter Lomax distribution of Table 1, and the two-parameter Weibull

and Gamma distributions, on both datasets and under the same estimation and testing procedures.

The results in Table 7 can be summarized as follows:

- (i) **One-parameter heavy-tailed baselines.** On Dataset 1 the Marei model and the half-Cauchy model are practically indistinguishable (AIC 176.57 versus 176.55); the fitted value $\hat{\theta} = 0.071$ is close to $\theta = 0$, for which the Marei law reduces to the standard half-Cauchy. The Marei model has the better LOOCV score (87.83 versus 88.44), and the half-Cauchy model has smaller CvM and AD statistics. The Marei model outperforms the Lomax model on all criteria (AIC 176.57 versus 176.98, LOOCV 87.83 versus 88.03, AD 0.729 versus 1.056). On Dataset 2 the Marei model has the smallest AIC, AICc and BIC of the three one-parameter heavy-tailed models (72.34 versus 73.35 and 73.92); the half-Cauchy model has smaller KS and CvM statistics, the AD statistics of the two models are equal to two decimals (0.444 and 0.443), and the LOOCV scores are 38.11 (Marei), 36.71 (half-Cauchy) and 36.61 (Lomax).
- (ii) **Two-parameter models, Dataset 1.** The Weibull and Gamma models attain AIC 170.67 and 170.77 (Marei: 176.57), AICc 170.93 and 171.02 (Marei: 176.65), LOOCV 84.98 for both (Marei: 87.83) and AD 0.319 and 0.339 (Marei: 0.729), reflecting the contribution of a free scale parameter for this dataset, which the scale extension of Section 3.3 provides.
- (iii) **Two-parameter models, Dataset 2.** Gamma and Weibull have smaller AIC (71.18 and 71.69 versus 72.34) and AICc (71.89 and 72.39 versus 72.57); Gamma also has a smaller BIC (73.17 versus 73.34), whereas the Marei BIC is smaller than that of Weibull (73.68). Both two-parameter models have smaller AD statistics (0.313 and 0.322 versus 0.444). Their LOOCV scores (37.43 and 37.08) are also better than that of the Marei model (38.11). The differences in AIC and AICc are small (below 1.2 and 0.7, respectively), so the one-parameter Marei model attains a fit comparable to the two-parameter models with half the number of parameters.

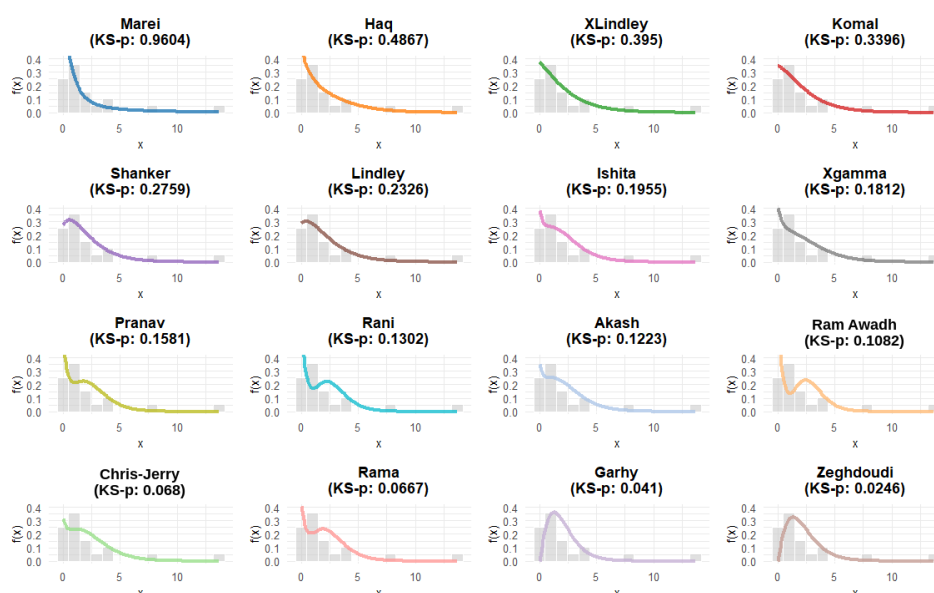
Overall, the Marei model is a strong one-parameter heavy-tailed model: it is clearly preferable to the exponential-mixture one-parameter families for these data, it is on par with the half-Cauchy and Lomax models, and it is comparable to the two-parameter Weibull and Gamma models on Dataset 2. Its scale extension (Section 3.3) adds the flexibility of a scale parameter while preserving the analytical results of this paper.

3.2.4 MCMC Convergence Diagnostics for the Real Datasets

Table 8 reports the convergence diagnostics of the Metropolis–Hastings sampler (weak prior $\alpha_0 = \beta_0 = 0.01$; four chains; 4,000 iterations per chain of

Table 6: MLEs, SEs, and goodness-of-fit for the electrical products dataset.

Model	$\hat{\theta}$ (SE)	−LogLik	AIC	BIC	HQIC	KS (p)	CvM (p)	AD (p)
Marei	−0.2185 (0.1884)	35.1719	72.3438	73.3395	72.5381	0.1061 (0.9604)	0.0464 (0.9028)	0.4444 (0.8020)
Haq	0.7206 (0.1163)	36.9657	75.9314	76.9272	76.1258	0.1792 (0.4867)	0.1310 (0.4562)	1.1090 (0.3041)
XLindley	0.6152 (0.1044)	37.3458	76.6916	77.6874	76.8860	0.1930 (0.3950)	0.1579 (0.3687)	1.3298 (0.2227)
Komal	0.6618 (0.1061)	37.8511	77.7023	78.6980	77.8967	0.2023 (0.3396)	0.1794 (0.3129)	1.4988 (0.1769)
Shanker	0.7566 (0.1077)	39.0762	80.1524	81.1481	80.3467	0.2143 (0.2759)	0.2139 (0.2429)	1.8474 (0.1122)
Lindley	0.7008 (0.1158)	38.5755	79.1509	80.1467	79.3453	0.2236 (0.2326)	0.2356 (0.2084)	1.8746 (0.1083)
Ishita	1.0500 (0.1126)	40.4936	82.9872	83.9829	83.1816	0.2327 (0.1955)	0.2342 (0.2104)	1.9640 (0.0967)
Xgamma	0.8637 (0.1339)	38.7181	79.4361	80.4319	79.6305	0.2365 (0.1812)	0.2635 (0.1720)	1.8772 (0.1080)
Pranav	1.3522 (0.1173)	42.2944	86.5887	87.5845	86.7831	0.2433 (0.1581)	0.2394 (0.2030)	2.0334 (0.0887)
Rani	1.6633 (0.1216)	44.5351	91.0702	92.0660	91.2646	0.2526 (0.1302)	0.2513 (0.1869)	2.1792 (0.0740)
Akash	1.0215 (0.1274)	40.5813	83.1626	84.1583	83.3570	0.2555 (0.1223)	0.3023 (0.1328)	2.3010 (0.0638)
Ram Awadh	1.9828 (0.1256)	47.1728	96.3455	97.3412	96.5399	0.2612 (0.1082)	0.2689 (0.1658)	2.3832 (0.0577)
Chris-Jerry	0.9691 (0.1432)	40.5728	83.1456	84.1414	83.3400	0.2815 (0.0680)	0.3972 (0.0726)	2.7586 (0.0369)
Rama	1.3688 (0.1370)	43.2352	88.4705	89.4662	88.6649	0.2823 (0.0667)	0.3575 (0.0931)	2.6917 (0.0399)
Garhy	1.3111 (0.1449)	56.0492	114.0984	115.0941	114.2928	0.3019 (0.0410)	0.5178 (0.0348)	6.0715 (0.0009)
Zeghdoudi	1.1617 (0.1535)	54.3848	110.7695	111.7652	110.9639	0.3212 (0.0246)	0.6016 (0.0212)	6.3679 (0.0007)

**Fig. 14:** Estimated PDFs for the electrical products dataset.

The negative $\hat{\theta}$ estimate inducing the DFR regime accurately describes the infant-mortality failures observed in electronic components. Competing models over-smooth the peak and subsequently underestimate the early-life risk.

which 1,000 are burn-in; proposal standard deviation $\tau = 3\text{SE}$) and Figure 17 shows the trace and autocorrelation plots. All acceptance rates lie in the target range 20–40%, $\hat{R} \leq 1.003$, the effective sample sizes exceed 2,300 out of 12,000 retained draws, and the Monte Carlo standard errors of the posterior means are below 0.004, i.e., about 2% of the posterior standard deviation. Only one of the eight Geweke z -scores slightly exceeds 2 in absolute value (2.06), which is consistent with chance

variation among eight tests. The posterior means agree with the MLEs to within 0.007.

3.3 Concluding Remarks

This article studied a continuous one-parameter heavy-tailed probability distribution derived from a symmetric Beta transformation $X = \sqrt{U/(1-U)}$. The model features a power-law right tail of order $x^{-(\theta+2)}$,

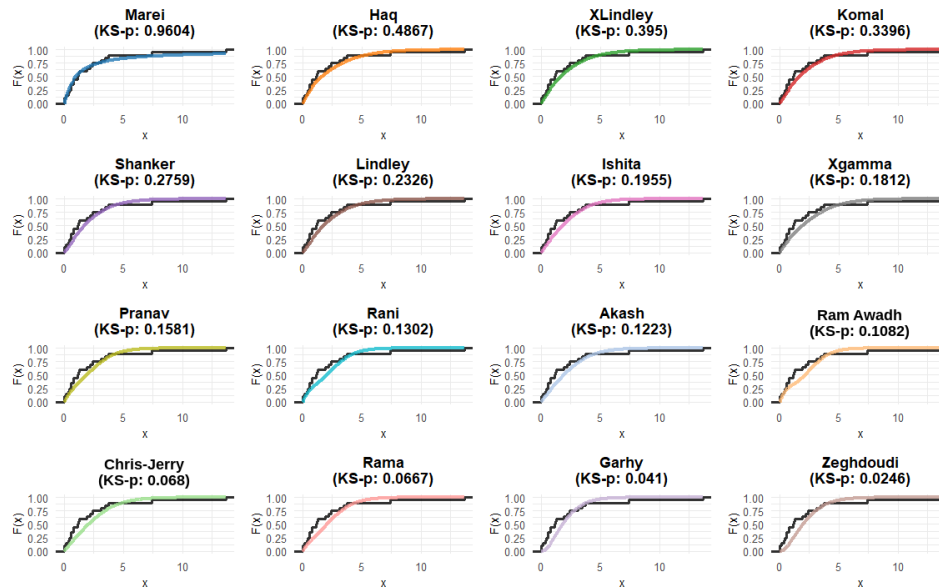


Fig. 15: Estimated CDFs for the electrical products dataset.

Convergence of the CDF in the upper tail (e.g., $x > 5$) is essential for forecasting. According to Marei, adjusting for divergence in exponential families is likely to lead to better long-horizon reliability forecasting.

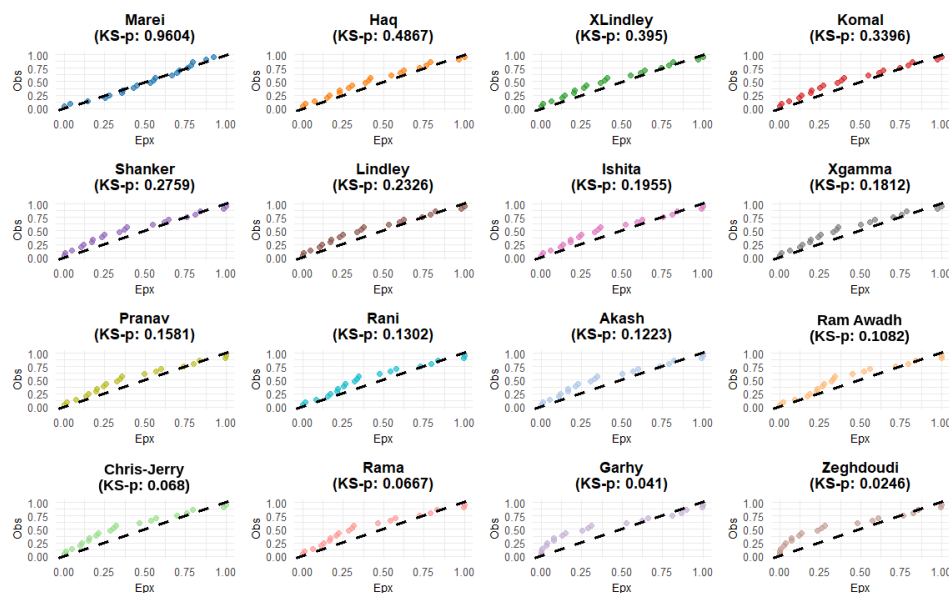


Fig. 16: P–P plots for the electrical products dataset.

The Marei P–P points follow the diagonal closely (maximal vertical deviation 0.081, squared correlation 0.98), supporting the adequacy of the model for this dataset.

closed-form CDF and quantile functions via incomplete Beta functions, a shape-invariant unit median ($M = 1$), and a failure rate transition from DFR ($-1 < \theta \leq -1/5$,

and numerically up to $\theta^* \approx -0.073$) to unimodal upside-down bathtub HRF ($\theta > 0$) with $h(x) \sim (\theta + 1)/x \rightarrow 0$ as $x \rightarrow \infty$. Parameter estimation via

Table 7: MLEs (SEs), information criteria, goodness-of-fit statistics (p -values) and leave-one-out cross-validated log score for the Marei model and four baseline models; k is the number of parameters. Bold entries mark the best value in each column within each dataset.

Model	k	Estimates (SE)	–LogLik	AIC	AICc	BIC	KS (p)	CvM (p)	AD (p)	LOOCV
<i>Dataset 1: carbon fibre ($n = 50$)</i>										
Marei	1	0.0710 (0.1681)	87.2858	176.5716	176.6549	178.4836	0.0938 (0.7359)	0.1003 (0.5851)	0.7294 (0.5341)	87.8252
Half-Cauchy (σ)	1	1.1004 (0.2279)	87.2735	176.5471	176.6304	178.4591	0.0929 (0.7467)	0.0623 (0.8020)	0.6108 (0.6370)	88.4361
Lomax (θ)	1	1.1378 (0.1609)	87.4892	176.9784	177.0617	178.8904	0.1147 (0.4907)	0.1784 (0.3143)	1.0558 (0.3289)	88.0259
Weibull (β, η)	2	0.8792 (0.0984), 1.8531 (0.3145)	83.3363	170.6727	170.9280	174.4967	0.0683 (0.9614)	0.0360 (0.9539)	0.3190 (0.9228)	84.9814
Gamma (k, η)	2	0.8246 (0.1427), 2.3955 (0.5577)	83.3834	170.7668	171.0222	174.5909	0.0675 (0.9650)	0.0397 (0.9365)	0.3390 (0.9058)	84.9789
<i>Dataset 2: electrical products ($n = 20$)</i>										
Marei	1	-0.2185 (0.1885)	35.1719	72.3438	72.5660	73.3395	0.1061 (0.9604)	0.0464 (0.9028)	0.4444 (0.8020)	38.1137
Half-Cauchy (σ)	1	1.1126 (0.3510)	35.6751	73.3502	73.5724	74.3459	0.0830 (0.9972)	0.0204 (0.9974)	0.4433 (0.8031)	36.7069
Lomax (θ)	1	1.1092 (0.2480)	35.9575	73.9150	74.1372	74.9107	0.1442 (0.7477)	0.0840 (0.6741)	0.6340 (0.6144)	36.6094
Weibull (β, η)	2	0.6974 (0.1237), 1.8224 (0.6105)	33.8440	71.6881	72.3939	73.6795	0.1226 (0.8895)	0.0412 (0.9313)	0.3222 (0.9197)	37.0768
Gamma (k, η)	2	0.5644 (0.1490), 4.0153 (1.5975)	33.5910	71.1819	71.8878	73.1734	0.1260 (0.8699)	0.0443 (0.9148)	0.3129 (0.9274)	37.4261

Table 8: MCMC convergence diagnostics for the Marei model under the weak prior (four chains). τ : proposal standard deviation; Acc.: mean acceptance rate; ESS: bulk effective sample size; $|z|_{\max}$: largest absolute Geweke z -score over the four chains.

Dataset	τ	Acc.	$\hat{R}$	ESS	MCSE	$ z _{\max}$	Posterior mean	Posterior SD	95% credible interval
Dataset 1	0.504	0.368	1.001	2304	0.0034	2.06	0.0725	0.1630	[−0.220, 0.413]
Dataset 2	0.565	0.373	1.002	2506	0.0039	1.15	−0.2122	0.1951	[−0.537, 0.218]

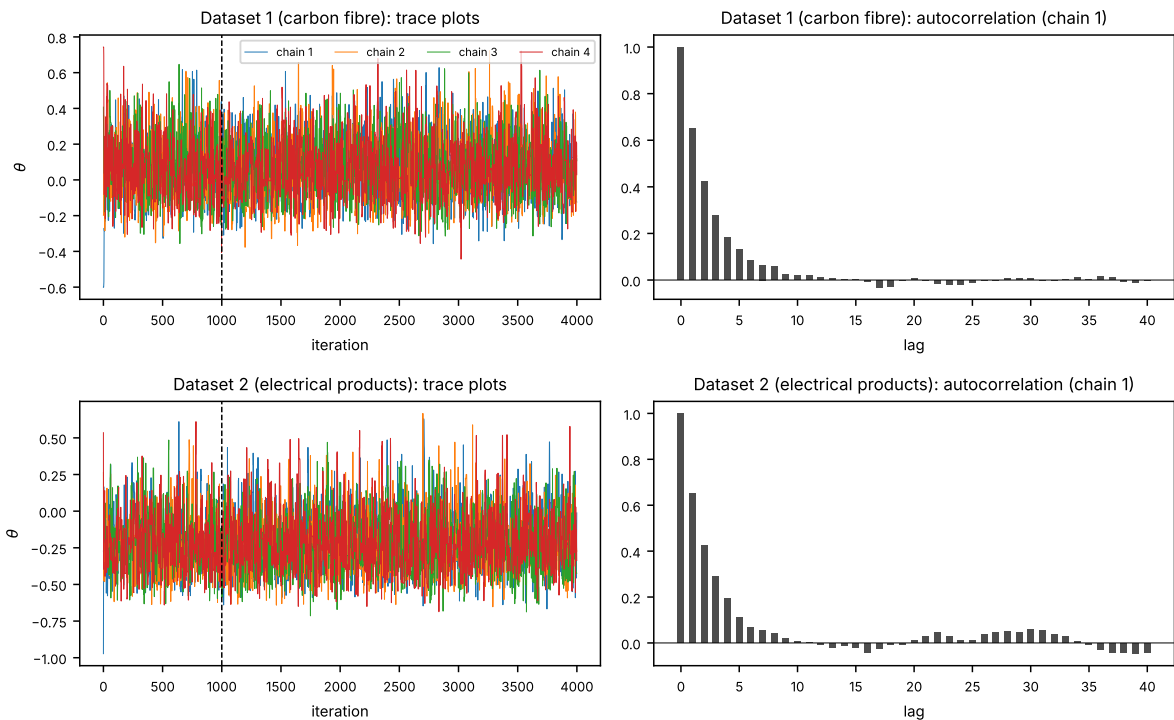


Fig. 17: Trace plots of four Metropolis–Hastings chains (left; the dashed line marks the end of burn-in) and autocorrelation function of the first chain after burn-in (right) for Dataset 1 (top) and Dataset 2 (bottom).

MLE and Bayesian MCMC demonstrated consistency and numerical stability, even in boundary cases with

infinite population moments, which is explained by the exponential-family structure of the model.

Empirically, on both benchmark datasets the Marei model attained the smallest AIC and BIC and the smallest KS, CvM and AD statistics among the sixteen one-parameter models of Tables 4 and 6 (the Marei model and fifteen exponential-mixture-type competitors). Table 7 extends the comparison to the one-parameter heavy-tailed half-Cauchy and Lomax models and to the two-parameter Weibull and Gamma models, giving twenty models in total. Among the heavy-tailed one-parameter models, on Dataset 1 the Marei model is tied with the half-Cauchy model in AIC (176.57 versus 176.55) and has the best LOOCV score (87.83, versus 88.44 for half-Cauchy and 88.03 for Lomax); on Dataset 2 it has the smallest AIC, AICc and BIC (72.34 versus 73.35 and 73.92 in AIC), with LOOCV scores of 38.11, 36.71 and 36.61 and AD statistics of 0.444, 0.443 and 0.634 for the Marei, half-Cauchy and Lomax models, respectively. For the two-parameter models, on Dataset 1 ($n = 50$) Weibull and Gamma attain AIC 170.67 and 170.77, LOOCV 84.98 and AD 0.319 and 0.339 (Marei: 176.57, 87.83 and 0.729), whereas on Dataset 2 ($n = 20$) the one-parameter Marei model attains a comparable fit (AIC 72.34 versus 71.18 and 71.69; AICc 72.57 versus 71.89 and 72.39; LOOCV 38.11 versus 37.43 and 37.08; AD 0.444 versus 0.313 and 0.322) and a lower BIC than the Weibull model (73.34 versus 73.68). The proposed family thus provides a parsimonious and analytically tractable heavy-tailed model that is competitive with standard two-parameter lifetime models.

The single shape parameter yields the distinguishing closed-form, shape-invariant properties of the model (unit median, reciprocal symmetry, tail index $\theta + 1$ of the survival function, and the hazard-shape transition). A natural direction for future work is the *scale extension* $Y = \sigma X$, $\sigma > 0$, with density

$$f_Y(y; \theta, \sigma) = \frac{2}{\sigma B(a, a)} \frac{(y/\sigma)^\theta}{\{1 + (y/\sigma)^2\}^{\theta+1}}, \quad y > 0,$$

for which the following properties carry over directly: (i) $F_Y(y) = I_{y^2/(\sigma^2+y^2)}(a, a)$ and $Q_Y(u) = \sigma Q(u; \theta)$, so that σ is the median of Y ; (ii) the tail index $\theta + 1$, the Fréchet domain of attraction and the hazard-shape classification are unchanged, since $h_Y(y) = h(y/\sigma; \theta)/\sigma$; (iii) $\mathbb{E}[Y^k] = \sigma^k \mathbb{E}[X^k]$; and (iv) the stress-strength reliability $P(Y_2 < Y_1)$ becomes informative, depending on the ratio σ_1/σ_2 (Theorem 2.2.12 and the subsequent remark). Moreover, $\ln Y = \ln \sigma + \ln X$ is a location family with symmetric error, so σ admits simple robust initial estimates (e.g., the sample median). Topics to be developed include the joint likelihood equations and Fisher information matrix of (θ, σ) , the existence and uniqueness of the joint MLE, Bayesian inference with a joint prior, regression modeling through $\ln \sigma_i = \mathbf{z}_i^\top \boldsymbol{\beta}$, and a renewed comparison with the Weibull, Gamma,

log-logistic and Burr XII families. Further extensions include censored and truncated samples.

Data Availability Statement

All datasets analyzed during this study are publicly available in the published literature cited in the references (Nichols and Padgett [22]; Murthy et al. [21]) and fully reproduced within the text.

Conflicts of Interest

The authors declare no conflict of interest.

Funding Statement

This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4340323GSSR02.

Acknowledgments

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4340323GSSR02.

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A Derivation of the Score Function and Fisher Information

Let $a = (\theta + 1)/2$, so that $da/d\theta = 1/2$, and recall $B(a, a) = \Gamma(a)^2/\Gamma(2a)$, whence

$$\ln B(a, a) = 2\ln \Gamma(a) - \ln \Gamma(2a).$$

Score. By the chain rule,

$$\begin{aligned} \frac{d}{d\theta} \ln B(a, a) &= \frac{da}{d\theta} [2\psi(a) - 2\psi(2a)] \\ &= \psi(a) - \psi(2a). \end{aligned}$$

Differentiating (27) term by term,

$$U(\theta) = \frac{d\ell}{d\theta} = -n[\psi(a) - \psi(2a)] + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \ln(1 + x_i^2),$$

which is (28). Since $\psi(a) < \psi(2a)$, the first term is positive. Because $\mathbb{E}[U(\theta)] = 0$, we also obtain the identity $\mathbb{E}[\ln\{X/(1+X^2)\}] = \psi(a) - \psi(2a)$.

Fisher information. Differentiating once more,

$$\frac{d}{d\theta} [\psi(a) - \psi(2a)] = \frac{1}{2} \psi_1(a) - \psi_1(2a),$$

so that

$$-\frac{d^2\ell}{d\theta^2} = n \left[\frac{1}{2} \psi_1(a) - \psi_1(2a) \right] = \frac{n}{2} [\psi_1(a) - 2\psi_1(2a)].$$

This quantity does not depend on the observations; hence the observed and expected information coincide and the expected Fisher information per observation is

$$\mathcal{J}(\theta) = \frac{1}{2} [\psi_1(a) - 2\psi_1(2a)],$$

which is (29). Its positivity follows from the duplication formula $4\psi_1(2a) = \psi_1(a) + \psi_1(a + \frac{1}{2})$, which gives $\mathcal{J}(\theta) = \frac{1}{4} [\psi_1(a) - \psi_1(a + \frac{1}{2})] > 0$ because ψ_1 is strictly decreasing on $(0, \infty)$. The asymptotic standard error of $\hat{\theta}$ is $\{n\mathcal{J}(\hat{\theta})\}^{-1/2}$; for example, for Dataset 1 ($n = 50$, $\hat{\theta} = 0.0710$) $\mathcal{J}(\hat{\theta}) = 0.7074$ and the standard error is 0.1681, as reported in Table 4.

Newton–Raphson / Fisher scoring update.

Combining the two expressions, the update (30) reads

$$\theta^{(k+1)} = \theta^{(k)} + \frac{\bar{T} - [\psi(a_k) - \psi(2a_k)]}{\frac{1}{2} [\psi_1(a_k) - 2\psi_1(2a_k)]},$$

where $\bar{T} = n^{-1} \sum_{i=1}^n \ln\{x_i/(1+x_i^2)\}$ and $a_k = (\theta^{(k)} + 1)/2$.

As a numerical check, the simulation study gives $n \times \text{MSE}(\hat{\theta}) \approx 1/\mathcal{J}(\theta)$ for $n = 1000$ in all five scenarios (subsection 3.1).