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New Oscillation Results of Second-Order Mixed Nonlinear Neutral Dynamic Equations with Damping on Time Scales

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Abstract: This paper is concerned with the oscillatory behavior of all solutions of the second-order mixed nonlinear neutral dynamic equation with damping

$$(r(t)\phi(z^{\Delta}(t)))^{\Delta}+p(t)\phi(z^{\Delta}(t))+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0,$$

where $\phi(s)=|s|^{\gamma-1}$ and $z(t)=x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t))$. Our results complement and improve the results reported in [[1],[2], [6]-[9], [12], [13], [15]-[20]] because our results can applied to the case where $p_2(t)\neq 0, p(t)\neq 0, g(t,x(\tau_2(t)))\neq 0$ and $\alpha\neq\beta\neq\gamma$, specially the results obtained in [12] and [9] are considered as special cases of our results when taking $\alpha=\beta=\gamma, p(t)=p_2(t)=0$ and $g(t,x(\tau_2(t)))=0$ or either $g(t,x(\tau_2(t)))=0$ or $f(t,x(\tau_1(t)))$ respectively. An example is given to illustrate our main result.

Keywords: Oscillation, neutral dynamic equations, time scales, generalized Riccati technique.

1 Introduction

The theory of time scales was introduced by Hilger [11] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. When the time scale equals to the real numbers, the obtained results represent the classical theories of differential equations while when time scale equals to the integer numbers, our results represent the theories of difference equations. Many other interesting time scales exist and give arise to many applications. The new theory of the so - called " dynamic equation" not only unify the theories of differential equations and difference equations, but also extends these classical cases to the so - called q - difference equations (when $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^t : t\in\mathbb{N}_0$ for $q>1\}$ or $\mathbb{T}=q^{\overline{\mathbb{Z}}}=q^{\mathbb{Z}}\cup\{0\}$) which have important applications in quantum theory (see [14]). Also, it can be applied on different types of time scales like $\mathbb{T}=h\mathbb{Z},\mathbb{T}=\mathbb{N}_0^2$, and the space of the harmonic numbers $\mathbb{T}=\mathbb{T}_n$.
For an introduction to time scale calculus and dynamic equations, see Bohner and Peterson books [3,4]. In recent years, there has been much activities concerning the oscillation and nonoscillation of solution of various equations on time scales. We refer the reader to the papers [[1],[2], [5]-[9], [12], [13], [15]-[20]] and references cited therein.

In this paper, we deal with oscillation of the second order mixed nonlinear neutral dynamic equation with damping on time scales

$$(r(t)\phi(z^{\Delta}(t)))^{\Delta}+p(t)\phi(z^{\Delta}(t))+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0, \quad (1)$$

where

$$\phi(s)=|s|^{\gamma-1}s, \qquad z(t)=x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)) \quad (2)$$

subject to the following hypotheses:
(H_1) $\mathbb{T}$ is an unbounded above time scale and $t_0\in\mathbb{T}$ with $t_0>0$. We define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\bigcap\mathbb{T}$.
(H_2) η_1,η_2,τ_1 and $\tau_2:\mathbb{T}\to\mathbb{T}$ are rd-continuous such that $\eta_1(t)\leq t$, $\eta_2(t)\geq t$, $\tau_1(t)\leq t$, $\tau_2(t)\geq t$;$\lim_{t\to\infty}\tau_1(t)=\infty$

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and $\lim_{t\to\infty} \eta_1(t) = \infty$.
(H_3) p_1, p_2, q_1, p and q_2 are non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$.
(H_4) r is a positive rd-continuous function such that $\frac{-p}{r} \in R^+$, where $R = R(\mathbb{T}, \mathbb{R})$ is defined as the set of all regressive and rd-continuous functions while $R^+ = R^+(\mathbb{T}, \mathbb{R}) = \{l \in R : 1 + \mu(T)l(t) > 0 \text{ for all } t \in \mathbb{T}\}$ and either

$$\int_{t_0}^{\infty} [\frac{1}{r(s)} e_{\frac{-p}{r}}(s,t_0)]^{\frac{1}{\gamma}} \Delta s = \infty, \tag{3}$$

or

$$\int_{t_0}^{\infty} [\frac{1}{r(s)} e_{\frac{-p}{r}}(s,t_0)]^{\frac{1}{\gamma}} \Delta s < \infty, \tag{4}$$

(H_5) $f, g \in C(\mathbb{R}\times\mathbb{T}, \mathbb{R})$ such that $uf(t,u) \geq 0$, $ug(t,u) \geq 0$, $f(t,u) \geq q_1(t)u^{\alpha}$ and $g(t,u) \geq q_2(t)u^{\beta}$ for $u \neq 0$.
(H_6) γ, α and β are quotients of odd positive integers.

By a solution of (1), we mean a nontrivial real valued function x satisfies (1) for $t \in \mathbb{T}$. A solution x of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Eq. (1) is said to be oscillatory if all of its solutions are oscillatory. We concentrate our study to those solutions of Eq. (1) which are not identically vanishing eventually.

Now, we give some related background details which are strongly motivate our research. Assuming that $p_2(t) = 0$, $p(t) = 0$, $g(t, x(\tau_2(t))) = 0$ and $\alpha = \beta = \gamma$, Jing et al.[12], Li and Saker [13], Saker and O'Regan [15] and Hong-Wu et al.[17] established several oscillation results for (1). Erbe et al.[9] obtained oscillation results for (1) when $p(t) = p_2(t) = 0$, $\alpha = \beta = \gamma$ and either $f(t, x(\tau_1(t))) = 0$ or $g(t, x(\tau_2(t))) = 0$. As a special case where $p(t) = 0$, H. A. Agwa et al [2] investigated oscillation theorems for (1). Chen et al. studied the following second-order dynamic equation with damping

$$((x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(x^{\Delta}(t))^{\gamma} + q(t)f(x^{\sigma}(t)) = 0.$$

Senel [16] studied the second order non linear dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(x^{\Delta}(t))^{\gamma} + f(t, x(g(t))) = 0,$$

where r and p are nonnegative rd-continuous functions and $g(t) \geq t$. Agarwal et al [1] concerned with oscillatory properties of a second-order half-linear dynamic equation

$$(a(x^{\Delta})^{\gamma})^{\Delta}(t) + p(t)(x^{\Delta})^{\gamma}(t) + q(t)x^{\gamma}(\delta(t)) = 0,$$

where $\gamma \geq 1$ and $\delta(t) \leq t$. Bohner and Li [6] and Erbe et al. [8] obtained oscillation results for the equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(x^{\Delta\sigma})^{\gamma}(t) + q(t)f(x(\tau(t))) = 0.$$

Zhang [19] studied the oscillatory behavior of the equation

$$(a(t)\phi(x^{\Delta}(t)))^{\Delta} + p(t)\phi(x^{\Delta}(t)) + q(t)f(\phi(x(\tau(t)))) = 0,$$

which is considered as a special case of (1) by taking $p_1(t) = p_2(t) = 0$, $g(t, x(\tau_2(t))) = 0$ and $\alpha = \beta = \gamma$. Recently Zhang and Lio [20] and Yang and Li [18] established oscillation results of philos type to the following second-order half-linear neutral delay dynamic equation with damping

$$(a(t)\phi(z^{\Delta}(t)))^{\Delta} + p(t)\phi(z^{\Delta}(t)) + q(t)f(\phi(x(\delta(t)))) = 0,$$

where $z(t) = x(t) + r(t)x(\tau(t)$ and assuming that

$$\delta(t) = \tau(t), \qquad 0 \leq r(t) < 1, \qquad \phi(s) = |s|^{\gamma-2}s, \gamma > 1.$$

Recently Yang and Li [18] also studied the previous equation considering

$$\delta(t) \geq \tau(t), 0 \leq r(t) \leq B_0 < +\infty, \phi(s) = |s|^{\gamma-1}s, 0 < \gamma \leq 1$$

$$\int_{t_0}^{\infty} [\frac{1}{r(s)} e_{\frac{-p}{r}}(s,t_0)]^{\frac{1}{\gamma}} \Delta s = \infty. \tag{5}$$

Our principle goal is to establish new results for oscillation of (1) assuming that conditions H_1-H_6 are satisfied. It should be noted that the topic of this paper is new for dynamic equations on time scales due to the fact that the results reported in [[1],[2], [6]-[9], [12], [13], [15]-[20]] can't be applied to equation (1) in case $p_2(t) \neq 0, p(t) \neq 0, g(t, x(\tau_2(t))) \neq 0$ and $\alpha \neq \beta \neq \gamma$.

2 Some Preliminaries on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we define the forward and backward jump operators as:

$$\sigma(t) = inf\{s \in \mathbb{T}, s > t\} \text{ and } \rho(t) = sup\{s \in \mathbb{T}, s < t\}.$$

A point $t \in \mathbb{T}$, $inf\ \mathbb{T} < t < sup\ \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.
A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided that it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The delta derivative of a function f is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

provided f is continuous at t and t is a right-scattered. If t is not a right-scattered, the delta derivative is defined by

$$f^{\Delta}(t) = \lim_{s\to t^+} \frac{f(\sigma(t)) - f(t)}{t - s} = \lim_{s\to t^+} \frac{f(t) - f(s)}{t - s}$$

provided that this limit exists. A function $f : [a,b] \to \mathbb{R}$ is said to be differentiable if its Δ- derivative exists. By $C^1_{rd}(\mathbb{T},\mathbb{R})$, we mean the set of all functions whose delta derivative belong to $C_{rd}(\mathbb{T},\mathbb{R})$. Note that the set $\mathbb{R}$ of f may be actually replaced by any Banach space. The shift operator of a function $f : \mathbb{T} \to \mathbb{R}$ is denoted by f^σ and defined as

$$f^\sigma(t) = f(\sigma(t))$$

The $\Delta-$ derivative f^Δ and the shift operator f^σ of a function f are related as follows

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

The derivative rules of the product and the quotient of two differentiable functions f and g are given by

$$(f.g)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t)$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)},\ g(t)g^\sigma(t) \neq 0.$$

The integration by parts formula reads

$$\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t$$

or,

$$\int_a^b f^\sigma(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t$$

and the infinite integral is defined by

$$\int_b^\infty f(s)\Delta s = \lim_{t\to\infty} \int_b^t f(s)\Delta s.$$

Note that in case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) = 0, \quad f^\Delta(t) = f'(t),$$
$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt$$

and in case $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t+1,\ \rho(t) = t-1, \quad \mu(t) = 1,$$
$$f^\Delta(t) = \Delta f(t) = f(t+1) - f(t).$$

If $a < b$,

$$\int_a^b f(t)\Delta t = \Sigma_{t=a}^{b-1} f(t).$$

3 Basic lemmas

In this section, we give some lemmas that play important roles in the proof of our results.

Lemma 1.[3] *If $\theta \in R^+$, then the initial value problem $y^\Delta = \theta(t)y,\ y(t_0) = y_0 \in \mathbb{R}$ has the unique positive solution $e_\theta(.,t_0)$ on $[t_0,\infty)_\mathbb{T}$. This solution satisfies the semi group property*

$$e_\theta(a,b)e_\theta(b,c) = e_\theta(a,c)$$

Lemma 2.[3] *If x is a delta differentiable function, then*

$$(x^\gamma)^\Delta = \gamma x^\Delta \int_0^1 [hx^\sigma + (1-h)x]^{\gamma-1} dh \tag{6}$$

Lemma 3.[10] *If X and Y are non negative real numbers, then*

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda-1)Y^\lambda \qquad \text{for all } \lambda > 1,$$

where the equality holds if and only if $X = Y$

Lemma 4. *Let (3) holds. Assume that H_1-H_6 hold and $x(t)$ be an eventually positive solution of (1). Then there exists $t_1 \in [t_0,\infty)_\mathbb{T}$ such that*

$$z(t) > 0, z^\Delta(t) > 0 and [r(t)|z^\Delta(t)|^{\gamma-1}z^\Delta(t)]^\Delta < 0 \tag{7}$$

Proof. Assume that $x(t)$ is a positive solution of (1) on $[t_0,\infty)_\mathbb{T}$ and $t_1 \in [t_0,\infty)_\mathbb{T}$ so that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0, i = 1,2$ on $[t_1,\infty)_\mathbb{T}$. (when $x(t)$ is negative the proof is similar, because the transformation $x(t) = -y(t)$ transforms (1) into the same form). From the definition of $z(t)$, we get $z(t) > 0$ for $t \in [t_1,\infty)_\mathbb{T}$ and from (1) we have

$$(r(t)|z^\Delta(t)|^{\gamma-1}z^\Delta(t))^\Delta + p(t)|z^\Delta(t)|^{\gamma-1}z^\Delta(t) < 0. \tag{8}$$

Hence, from lemma 1 we obtain

$$\left[\frac{r|z^\Delta|^{\gamma-1}z^\Delta}{e_{\frac{-p}{r}}(.,t_0)}\right]^\Delta = \frac{e_{\frac{-p}{r}}(.,t_0)(r|z^\Delta|^{\gamma-1}z^\Delta)^\Delta + pe_{\frac{-p}{r}}(.,t_0)|z^\Delta|^{\gamma-1}z^\Delta}{e_{\frac{-p}{r}}(.,t_0)e^\sigma_{\frac{-p}{r}}(.,t_0)}$$
$$= \frac{(r|z^\Delta|^{\gamma-1}z^\Delta)^\Delta + p|z^\Delta|^{\gamma-1}z^\Delta}{e^\sigma_{\frac{-p}{r}}(.,t_0)} < 0, \tag{9}$$

then $\frac{r|z^\Delta|^{\gamma-1}z^\Delta}{e_{\frac{-p}{r}}(.,t_0)}$ is decreasing for $t \in [t_1,\infty)_\mathbb{T}$ and z^Δ is either eventually positive or eventually negative. We claim that

$$z^\Delta(t) > 0 \tag{10}$$

otherwise, we assume that (10) is not satisfied, then there exists $t_2 \in [t_1,\infty)_\mathbb{T}$ such that $z^\Delta(t) < 0$ for all $t \in [t_1,\infty)_\mathbb{T}$. Using (9) and lemma 1, we obtain

$$\frac{r(t)|z^{\Delta(t)}|^{\gamma-1}z^\Delta(t)}{e_{\frac{-p}{r}}(t,t_0)} \leq \frac{r(t_2)|z^\Delta(t_2)|^{\gamma-1}z^\Delta(t_2)}{e_{\frac{-p}{r}}(t_2,t_0)} \text{for } t \in [t_2,\infty)_\mathbb{T},$$

then

$$z^\Delta(t) \leq -M\left[\frac{1}{r(t)}e_{\frac{-p}{r}}(t,t_2)\right]^{\frac{1}{\gamma}} \qquad \text{for } t \in [t_2,\infty)_\mathbb{T}$$

where $M = (r(t_2))^{\frac{1}{\gamma}}|z^\Delta(t_2)| > 0$.
Now, by integrating both sides of the above inequality from $t_2 \to t$, we have

$$z(t) \leq z(t_2) - M\int_{t_2}^t \left[\frac{1}{r(s)}e_{\frac{-p}{r}}(s,t_2)\right]^{\frac{1}{\gamma}}\Delta s. \tag{11}$$

Taking the limit of both sides of (11) when $t \to \infty$ and using (3), we get $\lim_{t\to\infty} z(t) = -\infty$, therefore $z(t)$ is eventually negative which is a contradiction with $z(t) > 0$, so inequality (10) holds.
Now, using $z^{\Delta}(t) > 0$ in (8), we get

$$[r(t)(z^{\Delta}(t))^{\gamma}]^{\Delta} < 0$$

4 Main Results

Theorem 1. *Assume that H_1-H_6, (3) hold and there exists a positive real-valued Δ -differentiable function $\delta(t)$ such that for all sufficiently large t_1 where $\tau_1(T_0) > t_1$, we have*

$$\limsup_{t\to\infty} \int_{T_0}^{t} \Big[\delta(s)[Q_2(s) + Q_1(s)(\frac{R(\tau_1(t))}{R(t)})^{\alpha} A(s)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(s)(\overline{\delta}_{+}(s))^{\gamma+1}}{\delta^{\gamma}(s) C^{\gamma}(s)}\Big] \Delta s = \infty, \tag{12}$$

and R is a positive real-valued Δ-differentiable function such that

$$\frac{R(t)}{r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s} - R^{\Delta}(t) \le 0. \tag{13}$$

Where

$$1 - p_1(t) - p_2(t) \frac{R(\eta_2(t))}{R(t)} > 0, \tag{14}$$

$$\overline{\delta} = \delta^{\Delta} - \frac{p\delta}{r} \qquad , \qquad \overline{\delta}_{+}(t) := \max\{0, \overline{\delta}(t)\},$$

$$A(t) := \begin{cases} b_0^{\alpha-\beta} & \alpha \ge \beta \\ \frac{1}{[b_2 m(t)]^{\beta-\alpha}} & \alpha < \beta, \end{cases} \tag{15}$$

$$C(t) := \begin{cases} b_0^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} \ge 1 \\ [\frac{1}{b_2 m^{\sigma}(t)}]^{1-\frac{\beta}{\gamma}} & \frac{\beta}{\gamma} < 1, \end{cases} \tag{16}$$

$$Q_1(t) := (1 - p_1(\tau_1(t)) - p_2(\tau_1(t)) \frac{R(\eta_2(\tau_1(t)))}{R(\tau_1(t))})^{\alpha} q_1(t),$$

and

$$Q_2(t) := (1 - p_1(\tau_2(t)) - p_2(\tau_2(t)) \frac{R(\eta_2(\tau_2(t)))}{R(\tau_2(t))})^{\beta} q_2(t).$$

Then, every solution of (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that $x(t)$ is a positive solution of (1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0, i = 1, 2$ on $[t_1, \infty)_{\mathbb{T}}$.(when $x(t)$ is negative, the proof is similar). Using Lemma 4, we see that

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{(r(s)(z^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \ge r^{\frac{1}{\gamma}}(t) z^{\Delta}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

Since

$$(\frac{z(t)}{R(t)})^{\Delta} = \frac{z^{\Delta}(t)R(t) - z(t)R^{\Delta}(t)}{R(t)R^{\sigma}(t)} \le \frac{z(t)}{R(t)R^{\sigma}(t)} \Big[\frac{R(t)}{r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s} - R^{\Delta}(t)\Big] \le 0, \tag{17}$$

then z/R is a non-increasing function. Now from the definition of $z(t)$, we see that

$$\begin{aligned} x(t) &= z(t) - p_1(t)x(\eta_1(t)) - p_2(t)x(\eta_2(t)) \\ &\ge z(t) - p_1(t)z(\eta_1(t)) - p_2(t)z(\eta_2(t)) \\ &\ge (1 - p_1(t)\frac{z(\eta_1(t))}{z(t)} - p_2(t)\frac{z(\eta_2(t))}{z(t)})z(t) \\ &\ge (1 - p_1(t) - p_2(t)\frac{R(\eta_2(t))}{R(t)})z(t) \ \text{ for all} \end{aligned} \tag{18}$$

Choosing t_4 sufficiently large such that $t_4 > t_1$, $\tau_1(t) > t_1$ for all $t \in [t_4, \infty)_{\mathbb{T}}$, then

$$x(\tau_i(t)) \ge [1 - p_1(\tau_i(t)) - p_2(\tau_i(t)) \frac{R(\eta_2(\tau_i(t)))}{R(\tau_i(t))}] z(\tau_i(t)). \tag{19}$$

Now from lemma 4, we have

$$z(t) \le z(t_1) + r^{\frac{1}{\gamma}}(t_1) z^{\Delta}(t_1) \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

Thus there exist $t_5 \in \mathbb{T}$ and suitable constants b_0 and $b_2 > 0$ such that

$$b_0 \le z(t) \le b_2 \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} := b_2 m(t), \ \text{for all} t \ge t_5. \tag{20}$$

From lemma 4, we obtain that

$$\frac{1}{z(t)} > \frac{1}{z(\sigma(t))} \ \text{and} \ r(t)(z^{\Delta}(t))^{\gamma} > r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}, \tag{21}$$

hence

$$z^{\Delta}(t) > \frac{(r(\sigma(t))^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} z^{\Delta}(\sigma(t)). \tag{22}$$

Also, from lemma 2, we have

$$(z^{\beta}(t))^{\Delta} \geq \begin{cases} \beta z^{\Delta}(t) z^{\beta-1}(t), & \beta \geq 1 \\ \beta z^{\Delta}(t)(z(\sigma(t)))^{\beta-1}, & \beta \leq 1 \end{cases} \tag{23}$$

Define the function $w(t)$ by

$$w(t) = \delta(t)\frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(t)}. \tag{24}$$

Then $w(t) > 0$, on $[T_0, \infty)_{\mathbb{T}}$, where $T_0 = max\{t_4, t_5\}$ and

$$w^{\Delta}(t) = (\frac{\delta}{z^{\beta}}(t))(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(\frac{\delta(t)}{z^{\beta}(t)})^{\Delta}.$$
$$= (\frac{\delta(t)}{z^{\beta}(t)})(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}.$$
$$\frac{z^{\beta}(t)\delta^{\Delta}(t) - \delta(t)(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}.$$

From (1) and H_4 we have

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(z^{\Delta}(t))^{\gamma} + q_1(t)(x(\tau_1(t)))^{\alpha} + q_2(t)(x(\tau_2(t)))^{\beta} \leq 0.$$

Consequently, from (19) and (25) we get

$$w^{\Delta}(t) \leq \frac{-p(t)\delta(t)}{z^{\beta}(t)}(z^{\Delta}(t))^{\gamma} - \frac{\delta(t)}{z^{\beta}(t)}Q_1(t)(z(\tau_1(t)))^{\alpha}$$
$$- \frac{\delta(t)}{z^{\beta}(t)}Q_2(t) - (z(\tau_2(t)))^{\beta} + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))$$
$$\frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}$$
$$= -\delta(t)Q_1(t)(\frac{z(\tau_1(t))}{z(t)})^{\alpha}z^{\alpha-\beta}(t) - \delta(t)Q_2(t)(\frac{z(\tau_2(t))}{z(t)})^{\beta}$$
$$- \frac{p(t)}{r(t)}w(t) + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))$$
$$- \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}. \tag{25}$$

using (21) and (24), we get

$$\frac{w(t)}{\delta(t)} = \frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(t)} \geq \frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\sigma(t))} = \frac{w(\sigma(t))}{\delta(\sigma(t))},$$

hence

$$w(t) > \frac{\delta(t)}{\delta(\sigma(t))}w(\sigma(t)).$$

Substituting from the above inequality, using (17) and (20) in (25), we obtain

$$w^{\Delta}(t) \leq -\delta(t)Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(t) - \delta(t)Q_2(t)$$
$$+ (\delta^{\Delta}(t) - \frac{p(t)\delta(t)}{r(t)})\frac{w(\sigma(t))}{\delta(\sigma(t))}$$
$$- \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}$$
$$\leq -\delta(t)Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(t) - \delta(t)Q_2(t) + \frac{\overline{\delta}_+(t)}{\delta(\sigma(t))}w(\sigma(t))$$
$$- \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))},$$

using (23), (22) and (21) in the last term of the previous inequality, we obtain

$$w^{\Delta}(t) \leq -\delta(t)Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(t) - \delta(t)Q_2(t)$$
$$+ \frac{\overline{\delta}_+(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)(r(\sigma(t)))^{1+\frac{1}{\gamma}}(z^{\Delta}(\sigma(t)))^{\gamma+1}}{r^{\frac{1}{\gamma}}(t)(z(\sigma(t)))^{\beta+1}}$$
$$= -\delta(t)Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(t) - \delta(t)Q_2(t) + \frac{\overline{\delta}_+(t)}{\delta(\sigma(t))}w(\sigma(t))$$
$$- \frac{\beta\delta(t)(r(\sigma(t)))^{1+\frac{1}{\gamma}}(z^{\Delta}(\sigma(t)))^{\gamma+1}}{r^{\frac{1}{\gamma}}(t)(z(\sigma(t)))^{\beta+\frac{\beta}{\gamma}}}(z(\sigma(t)))^{\frac{\beta}{\gamma}-1}$$
$$\leq -\delta(t)Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(t) - \delta(t)Q_2(t) + \frac{\overline{\delta}_+(t)}{\delta(\sigma(t))}w(\sigma(t))$$
$$- \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}, \tag{26}$$

where

$$C(t) := \begin{cases} b_0^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} \geq 1 \\ [\frac{1}{b_2 m^{\sigma}(t)}]^{1-\frac{\beta}{\gamma}} & \frac{\beta}{\gamma} < 1, \end{cases}$$

taking $\lambda = \frac{\gamma+1}{\gamma}$ and using lemma 3 with

$$X = [\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}]^{\frac{1}{\lambda}}w(\sigma(t))$$

and

$$Y = [\frac{\overline{\delta}_+(t)}{\lambda\delta(\sigma(t))}[\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}]^{\frac{-1}{\lambda}}]^{\frac{1}{\lambda-1}},$$

we have

$$\frac{\overline{\delta}_+(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}$$
$$\leq \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(t)(\overline{\delta}_+(t))^{\gamma+1}}{\delta^{\gamma}(t)C^{\gamma}(t)}. \tag{27}$$

Substituting from (27) into (26), we obtain

$$w^{\Delta}(t) \leq -\delta(t)Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(t) - \delta(t)Q_2(t) + \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(t)(\overline{\delta}_+(t))^{\gamma+1}}{\delta^{\gamma}(t)1C^{\gamma}(t)}.$$

Integrating the above inequality from T_0 to t, we get

$$\int_{T_0}^{t} [\delta(s)[Q_2(s)+Q_1(s)(\frac{R(\tau_1(t))}{R(t)})^{\alpha}A(s)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\overline{\delta}_+(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s \leq w(T_0) - w(t) \leq w(T_0),$$

which is a contradiction with (12). Then every solution of (1) is oscillatory.

Theorem 2. *Assume that H_1-H_6,(14), (3) hold and there exist functions H,h such that for each fixed t, $H(t,s)$ and $h(t,s)$ are rd-continuous functions with respect to s on $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$,*

$$H(t,t) = 0,\ t \geq t_0, H(t,s) > 0,\ t > s \geq t_0, \tag{28}$$

and H has a non-positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ satisfying

$$-H^{\Delta_s}(t,s) = h(t,s)(H(t,s))^{\frac{\gamma}{\gamma+1}}. \tag{29}$$

If there exists a positive and differentiable function $\delta : \mathbb{T} \to \mathbb{R}$ such that for all sufficiently large t_1 where $\tau_1(T_0) > t_1$, $i = 1,2$, for all $t \in [T_0,\infty)_{\mathbb{T}}$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T_0)}\int_{T_0}^{t} [H(t,s)\delta(s)[Q_2(s)+Q_1(s)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(s)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(G_+(t,s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s = \infty, \tag{30}$$

where

$$G(t,s) = \overline{\delta}(s)H^{1-\frac{1}{\lambda}}(t,s) - \delta(\sigma(s))h(t,s),\ G_+(t,s) = max\{0,G(t,s)\}.$$

Then every solution of (1) is oscillatory.

Proof. Let x be an eventually positive solution of (1). Then proceeding as in the proof of first part of Theorem 1 until we get (26), it follows that

$$w^{\Delta}(t) \leq -\delta(t)Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(t) - \delta(t)Q_2(t) + \frac{\overline{\delta}_+(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}.$$

Multiplying both sides of the previous inequality by $H(t,s)$, we get

$$H(t,s)\delta(t)\ [\ Q_1(t)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}]A(t)+Q_2(t)] \leq -H(t,s)w^{\Delta}(t) + \frac{\overline{\delta}(t)H(t,s)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)C(t)H(t,s)}{\delta^{\lambda}(\sigma(t))r^{\frac{1}{\gamma}}(t)}w^{\lambda}(\sigma(t)).$$

Integrating both sides of the above inequality from $T_0 \to t$ and using integration by parts, we get

$$\begin{aligned}
&\int_{T_0}^{t} H(t,s)\delta(s)[Q_1(s)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}]A(s)+Q_2(s)]\Delta s \leq \\
&H(t,T_0)w(T_0) - \int_{T_0}^{t}[-H^{\Delta_s}(t,s)]w(\sigma(s))\Delta S + \int_{T_0}^{t}\frac{\overline{\delta}(s)H(t,s)}{\delta(\sigma(s))}w(\sigma(s))\Delta s \\
&- \int_{T_0}^{t}\frac{\beta\delta(s)C(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s))\Delta s \\
&= H(t,T_0)w(T_0) + \int_{T_0}^{t}\frac{\overline{\delta}(s)H(t,s)-\delta(\sigma(s))h(t,s)H^{\frac{1}{\lambda}}(t,s)}{\delta(\sigma(s))}w(\sigma(s))\Delta s \\
&- \int_{T_0}^{t}\frac{\beta\delta(s)C(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s))\Delta s \\
&\leq H(t,T_0)w(T_0) + \int_{T_0}^{t}\frac{G_+(t,s)}{\delta(\sigma(s))}H^{\frac{1}{\lambda}}(t,s)w(\sigma(s))\Delta s \\
&- \int_{T_0}^{t}\frac{\beta\delta(s)C(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s))\Delta s
\end{aligned} \tag{31}$$

where

$$\begin{aligned} G(t,s) &= \overline{\delta}(s)H^{1-\frac{1}{\lambda}}(t,s) - \delta(\sigma(s))h(t,s) \text{ and } G_+(t,s) \\ &= max\{0,G(t,s)\}. \end{aligned}$$

Using lemma 3 with

$$X = [\frac{\beta\delta(s)C(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}]^{\frac{1}{\lambda}}w(\sigma(s))$$

and

$$Y = [\frac{G_+(t,s)}{\lambda}[\frac{\beta\delta(s)C(s)}{r^{\frac{1}{\gamma}}}]^{\frac{-1}{\lambda}}]^{\frac{1}{\lambda-1}},$$

we get

$$\frac{G_+(t,s)}{\delta(\sigma(s))}H^{\frac{1}{\lambda}}(t,s)w(\sigma(s)) - \frac{\beta\delta(s)C(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s)) \leq \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(G_+(t,s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}. \tag{32}$$

Substituting from (32) into (31), we get

$$\frac{1}{H(t,T_0)}\int_{T_0}^{t}[\ H(t,s)\delta(s)[Q_2(s)+Q_1(s)[\frac{R(\tau_1(t))}{R(t)}]^{\alpha}A(s)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(G_+(t,s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s \leq w(T_0),$$

which is a contradiction with (30). Then every solution of (1) is oscillatory.

Theorem 3. *let (4) and all assumptions of Theorem 1 or Theorem 2 except (3) hold. If*

$$\int_{T_0}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(t)} \Big[\int_{T_0}^{t} e_{\frac{p}{r}}(t,\sigma(u))[q_1(u)+q_2(u)]\Delta u\Big]^{\frac{1}{\gamma}} \Delta t = \infty, \quad (33)$$

for all sufficiently large T_0, then every solution x of equation (1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that (1) has a non-oscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, we assume that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0$, $i = 1,2$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$. In view of (2), we have $z(t) > x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. From Lemma 4, we find that $z^{\Delta}(t)$ is either eventually positive or eventually negative. Thus we consider the following two cases:

Case 1. $z^{\Delta}(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$.

As in the proof of Theorem 1, we obtain a contradiction to (12).

Case 2. $z^{\Delta}(t) < 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$.

In this case, we have

$$\lim_{t\to\infty} z(t) = l, \ l \geq 0.$$

Now, let $l = 0$. If this is not true, then for any $\varepsilon > 0$, we have $l < z(t) < l+\varepsilon$, eventually.

If $p_1(t) + p_2(t) \leq \rho < 1$ and $0 < \varepsilon < l(1-\rho)/\rho$, then

$$\begin{aligned} x(t) &= z(t) - p_1(t)x(\eta_1(t)) - p_2(t)x(\eta_2(t)) \\ &\geq z(t) - p_1(t)z(\eta_1(t)) - p_2(t)z(\eta_2(t)) \\ &\geq l - p_1(t)(l+\varepsilon) - p_2(t)(l+\varepsilon) \\ &\geq l - \rho(l+\varepsilon), \end{aligned}$$

hence

$$x(t) \geq m(l+\varepsilon) > mz(t), \quad (34)$$

where

$$m := \frac{l}{l+\varepsilon} - \rho = \frac{l(1-\rho)-\varepsilon\rho}{l+\varepsilon} > 0.$$

Take $T_0 \in [t_2,\infty)_{\mathbb{T}}$ such that

$$z(\tau_1(t)) \geq z(t) \geq z(\tau_2(t)) \geq l. \quad (35)$$

Consequently, from (34) we have

$$x(t) > ml \text{ for all } t \in [T_0,\infty)_{\mathbb{T}}. \quad (36)$$

Defining $u(t) = r(t)|z^{\Delta}(t)|^{\gamma-1}z^{\Delta}(t) = -r(t)|z^{\Delta}(t)|^{\gamma} = r(t)(z^{\Delta}(t))^{\gamma}$, then (1), (36) and (35) yield to

$$\begin{aligned} u^{\Delta}(t) &= \frac{-p(t)}{r(t)}u(t) - f(t,x(\tau_1(t)) - g(t,x(\tau_2(t))) \\ &\leq \frac{-p(t)}{r(t)}u(t) - q_1(t)x^{\alpha}(\tau_1(t)) - q_2(t)x^{\beta}(\tau_2(t)) \\ &\leq \frac{-p(t)}{r(t)}u(t) - (ml)^{\alpha}q_1(t) - (ml)^{\beta}q_2(t) \\ &\leq \frac{-p(t)}{r(t)}u(t) - N[q_1(t)+q_2(t)], \end{aligned} \quad (37)$$

where $N = min\{(ml)^{\alpha},(ml)^{\beta}\}$. The inequality (37) is the assumed inequality of Theorem 6.1 in [3], see also Lemma 1 in [5]. Hence

$$\begin{aligned} u(t) &\leq u(T_0)e_{\frac{-p}{r}}(t,T_0) - N\int_{T_0}^{t} e_{\frac{-p}{r}}(t,\sigma(s))[q_1(s)+q_2(s)]\Delta s \\ &\leq -N\int_{T_0}^{t} e_{\frac{-p}{r}}(t,\sigma(s))[q_1(s)+q_2(s)]\Delta s, \\ &\quad \text{for all } t \in [T_0,\infty)_{\mathbb{T}} \end{aligned} \quad (38)$$

then

$$z^{\Delta}(t) \leq -N^{\frac{1}{\gamma}}\Big[\frac{1}{r(t)}\int_{T_0}^{t} e_{\frac{-p}{r}}(t,\sigma(s))[q_1(s)+q_2(s)]\Delta s\Big]^{\frac{1}{\gamma}}.$$

Integrating from T_0 to t, we get

$$z(t) \leq z(T_0) - N^{\frac{1}{\gamma}}\int_{T_0}^{t}\Big[\frac{1}{r(u)}\int_{T_0}^{u} e_{\frac{-p}{r}}(t,\sigma(s))[q_1(s)+q_2(s)]\Delta s\Big]^{\frac{1}{\gamma}}\Delta u, \quad (39)$$

from which we get $lim_{t\to\infty}z(t) = -\infty$. This is a contradiction. Hence, $lim_{t\to\infty}z(t) = 0$. Since $0 < x(t) \leq z(t)$, then $lim_{t\to\infty}x(t) = 0$. This completes the proof.

5 Examples

In this section, we give an example to illustrate our main result.

*Example 1.*Consider the equation

$$\begin{aligned} &\Big[\frac{1}{t^2}|z^{\Delta}(t)|^{\gamma-1}z^{\Delta}(t)\Big]^{\Delta} + \frac{1}{t^4}|z^{\Delta}(t)|^{\gamma-1}z^{\Delta}(t) + f(t,x(t)) + \\ &g(t,x(t+1)) = 0, \qquad t \in \overline{2^{\mathbb{Z}}},\ t \geq t_0 := 2, \end{aligned} \quad (40)$$

where

$$z(t) = x(t) + \frac{1}{2}x(\eta_1(t)) + \frac{1}{t+2}x(t),$$

$$f(t,u) = \Big(\tfrac{2(t+2)}{t}\Big)^{\alpha}\frac{1}{b_0^{\alpha-\beta}t^2}u^{\alpha} \text{ and} g(t,u) = \Big(\tfrac{2(t+3)}{t+1}\Big)^{\beta}\frac{1}{t^2}u^{\beta}.$$

Here

$$q_1(t) = \Big(\frac{2(t+2)}{t}\Big)^{\alpha}\frac{1}{b_0^{\alpha-\beta}t^2},\ q_2(t) = \Big(\frac{2(t+3)}{t+1}\Big)^{\beta}\frac{1}{t^2},$$

$$r(t) = \frac{1}{t^2},\ p(t) = \frac{1}{t^4}, \eta_2(t) = t = \tau_1(t),\ \tau_2(t) = t+1,$$

$$\eta_1(t) \leq t, p_1(t) = \frac{1}{2},\ p_2(t) = \frac{1}{t+2}.$$

Taking $R(t) = \int_{t_1}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$, hence (13) holds. Taking, $\alpha \geq \beta \geq \gamma$, we obtain

$$A(t)=b_0^{\alpha-\beta},C(t)=b_0^{\frac{\beta}{\gamma}-1}.$$

Since $\mathbb{T}=2^{\mathbb{Z}}$, $\sigma(t)=2t$ and $\mu(t)=t$,then

$$1-\mu(t)\frac{p(t)}{r(t)}=1-t\frac{1}{t^2}=\frac{t-1}{t}>0 \qquad \text{for all } t\in[2,\infty)_{\mathbb{T}}.$$

Using Lemma 2 in [5], we obtain

$$e_{\frac{-p}{r}}(t,2)\geq 1-\int_2^t \frac{p(s)}{r(s)}\Delta s=1-\int_2^t \frac{1}{s^2}\Delta s=\frac{2}{t} \qquad \text{for all } t\in[2,\infty)_{\mathbb{T}},$$

so

$$\int_2^t [\frac{1}{r(s)}e_{\frac{-p}{r}}(s,2)]^{\frac{1}{\gamma}}\Delta s\geq\int_2^t [s^2\frac{2}{s}]^{\frac{1}{\gamma}}\Delta s=2^{\frac{1}{\gamma}}\int_2^t s^{\frac{1}{\gamma}}\Delta s\to\infty \text{ as } t\to\infty.$$

Hence H_1-H_5 and (3) are satisfied. Moreover,

$$\begin{aligned}1-p_1(t)-p_2(t)\frac{R(\eta_2(t))}{R(t)}&=1-\frac{1}{2}-\frac{1}{t+2}\\&=\frac{t}{2(t+2)}>0,\end{aligned}$$

so, condition (14) is also satisfied. Now, assume that $\delta(t)=t$. Hence $\overline{\delta(t)}=1-\frac{1}{t}>0$, $\overline{\delta_+(t)}=\frac{t-1}{t}$. Since,

$$\begin{aligned}Q_1(t)&=(1-p_1(\tau_1(t))-p_2(\tau_1(t))\frac{R(\eta_2(\tau_1(t)))}{R(\tau_1(t))})^{\alpha}q_1(t)\\&=[1-\frac{1}{2}-\frac{1}{t+2}]^{\alpha}q_1(t)\\&=[\frac{t}{2(t+2)}]^{\alpha}q_1(t)\\&=\frac{1}{t^2b_0^{\alpha-\beta}},\end{aligned}$$

and

$$\begin{aligned}Q_2(t)&=(1-p_1(\tau_2(t))-p_2(\tau_2(t))\frac{R(\eta_2(\tau_2(t)))}{R(\tau_2(t))})^{\beta}q_2(t)\\&=[1-\frac{1}{2}-\frac{1}{t+3}]^{\beta}q_2(t)\\&=[\frac{t+1}{2(t+3)}]^{\beta}q_2(t)\\&=\frac{1}{t^2},\end{aligned}$$

then

$$\limsup_{t\to\infty}\int_{T_0}^t [\delta(s)[Q_2(s)+Q_1(s)[\frac{R(\tau_1(t)))}{R(t)}A(s)]$$
$$-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\overline{\delta}_+(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s=$$
$$\limsup_{t\to\infty}\int_{T_0}^t [\frac{2}{s}-\frac{\gamma^{\gamma}}{\beta^{\gamma}b_0^{\frac{\beta}{\gamma}-1}(\gamma+1)^{\gamma+1}}\frac{(s-1)^{\gamma+1}}{s^{2\gamma+3}}]\Delta s=\infty.$$

Hence by Theorem 1, equation (40) is oscillatory.

6 Conclusion

In this paper, we use Riccati transformation technique and the generalized exponential function to establish some new oscillation results of second-order mixed nonlinear neutral dynamic equations with damping on time scales. Our results not only unify the oscillation of differential equations and difference equations but also improve the results established in [[1],[2], [6]-[9], [12], [13], [15]-[20]] that can not be applied to (40), but according to Theorem 1 we obtain that every solution of (40) is oscillatory. Also, we found that Yang and Li[18] and Zhang and Liu [20] imposed many conditions on the function τ, like $\tau([t_0,+\infty)_{\mathbb{T}}=[\tau(t_0),+\infty)_{\mathbb{T}}$, $\tau^{\Delta}(t)\geq\tau_0>0$ and $\tau\circ\delta=\delta\circ\tau$, and they wished to get other methods in studying their equations without requiring these restrictive assumptions (see[Remark 3.1, [18]]). Our results achieve this goal. beside that the results in Theorem 2.1 and Theorem 2.3 of Jing and Li [12] can be achieved when $p(t)=p_2(t)=0$, $\alpha=\beta=\gamma$ and $g(t,\tau_2(t))=0$. Also when $p(t)=p_2(t)=0$, $\alpha=\beta=\gamma$, choosing $R(t)=\int_{t_1}^t\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$ and either $g(t,\tau_2(t))=0$ or $f(t,\tau_1(t))=0$, we obtain the same results of theorem 2.1 and Theorem 2.2 of Erbe et al. [9] . So the results of [12] and [9] can be considered as special cases of our results. So that by using the results in section 4, we can obtain some suffcient conditions for oscillation of all solutions of Eq. (1) which are essentially new.

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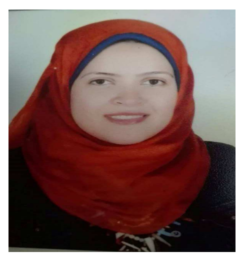

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