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A Study on Nash Min-Max Hybrid Continuous Static Games Under Fuzzy Environment

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Abstract: This paper focuses on hybrid continuous static games (HCSGs) that involve multiple players playing independently using the Nash equilibrium solution (NES) and others playing under a secure concept using Min-Max solution (MMS). All players having fuzzy parameters in both the cost functions and the constraints. These fuzzy parameters are characterized by fuzzy numbers. The concept of the $\alpha-$level set used for converting the fuzzy hybrid continuous static games (F-HCSGs) to a deterministic problem $\alpha-$HCSGs. Here a hybrid Nash Min-Max approach used to solve the deterministic problem. Furthermore, qualitative analysis of some basic notions in parametric fuzzy hybrid continuous static games is presented. These notions are: (i) The set of feasible parameters, (ii) The solvability set of feasible parameters, (iii) The stability set of the first kind, and (iv) The stability set of the second kind..

Keywords: Fuzzy Numbers, Game Theory, Hybrid Continuous Static Games, Min-Max Solution, Nash Equilibrium Solution, Parametric Analysis, Stability Study, $\alpha-$Level Set.

1 Introduction

Since 1920, the development of game theory began and continues to this day. J. Van Neumann in (1928) [1], gave the first proof of the central theorem of matrix games. T. L. Vincent and W. J. Grantham in (1981) [2], presented different formulations of games. The more general case achieved under the assumption that there are multiple decision-makers, each with their cost criterion. We have now arrived in the territory of game theory while the game appears when there exists the more general case of multiple decision-makers [3]. This generalization introduces the possibility of competition among the system controllers, called ”players” and the optimization problem under consideration is therefore termed a ”game”. Each player in the game controls a specified subset of the system parameters , called his control vector, and seeks to minimize his own cost criterion, subject to specified constraints [2].

Game theory applications may be found in engineering, economics, biology, and many other life fields. Games can be classified into three major classes matrix games [4,5,6], continuous static games (CSGs) [2, 7] and differential games [8]. In this paper, hybrid studies on CSGs are provided, in which the decision possibilities need not be discrete and costs also are related in a continuous rather than a discrete manner [2]. The game is called static due to there is no time history involved in the relationship between costs and decisions. In (1994) [9], differential games with vector payoff were studied.

Over the past 30 years, the fuzzy set theory has evolved in several ways in many branches and fields of study such as engineering, economics, and many other disciplines. The first publication in the theory of fuzzy sets was made by L. A. Zadeh in (1965) [10] and J. A. Goguen (1967) [11]. The theory of the fuzzy set is derived for solving the problems that have some uncertion and vague. In (1990) [12], M. Sakawa and H. Yano gave an interactive approach for solving multi-objective non-linear programming problems that include fuzzy parameters in both objective function and constraints. In (1993) [13], M. S. Osman and A. H. El-Banna studied the stability of multi-objective non-linear programming problems with fuzzy parameters. In (1984) [14], M. S. Osman, A. H. El-Banna, and A. H. Amer presented Nash equilibrium fuzzy continuous static games. In (1995)

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[15], A. Dhingra and S. S. Rao introduced a cooperative fuzzy game theoretic approach to multiple objective designs. In addition, in (1995) [16], M. A. Kassem and E. I. Ammar introduced a study of multi-objective fuzzy nonlinear programming problems at which the constraints include fuzzy parameters. Furthermore, in (1997) [17], E. I. Ammar studies the stability of multi-objective non-linear programming problems with fuzzy parameters in both the objectives and the constraints. In (2015) [18], an interactive approach is presented to solve cooperative continuous static games that contained fuzzy parameters in the objective functions by H. Khalifa and R. A. Zeineldin. Additionally, in (2019) [19], H. Khalifa provided a novel study of cooperative continuous static games under fuzzy environment.

In recent years, much of the research has been presented important concepts about the sensitivity and parametric study. M. Osman, in (1975) [20], introduced some of the basic notions that related to sensitivity and parametric study. These notions include: (i) The set of feasible parameters, (ii) The solvability set of feasible parameters, (iii) The stability set of the first kind, (iv) The stability set of the second kind. Qualitative analysis of these notions had been made for convex programs with parameters in the constraints as presented in [21]. Furthermore, the same notions were studied with parameters in the objective function introduced in [22]. As well as, in (1984) [7], the above parametric notions were defined for continuous static games.

In this paper, a fuzzy hybrid continuous static games that consisted of multiple players playing independently using NES [23,2] and others playing under a secure concept using MMS [2]. This game having fuzzy parameters in both the cost functions and the constraints. These fuzzy parameters are characterized by fuzzy numbers. The concept of $\alpha-$level set used for converting F-HCSGs to a deterministic problem $\alpha-$HCSGs. Here a hybrid Nash Min-Max approach, see [24], used to solve the deterministic problem. Furthermore, qualitative analysis of some basic notions in parametric Nash Min-Max fuzzy hybrid continuous static games is also presented.

The remainder of this paper is organized as in the following five sections. In the next section, a formulation of fuzzy hybrid continuous static games and its solution concept is developed. Qualitative analysis of some basic notions in fuzzy hybrid continuous static games is defined and determined in section 3. In section 4, solution procedure is suggested. An illustrative numerical example to clarify the solution aspects is given in section 5. Finally, some concluding remarks are reported in section 6.

2 Problem Formulation and Solution Concept

Fuzzy set theory has developed in various ways for solving problems in which the description of activities and observations are vague and uncertain. This section deals with a class of hybrid continuous static games, in which some players playing independently using NES and others using a MMS, with fuzzy parameters in both the cost functions and the constraints.

2.1 General Structure of Fuzzy Hybrid Continuous Static Games

Firstly, let we introduce the general structure of fuzzy hybrid continuous static games (F-HCSGs) involving fuzzy parameters in both the cost functions and the constraints as follows:
Let $u = (u_1, u_2, ..., u_m) \in \Re^m$ denote the control vectors for player $i \in T_1 = \{1,2...,m\} \subset \{1,2...,m,m+1,...,r\}$ (the set of all players), $\upsilon = (v_1, v_2, ..., v_r,) \in \tau \subset \Re^{s_i-m}$ denote the composite control vectors for player $j \in T_2 = \{m+1,...,r\}$ and $\omega \notin \tau \subset \Re^{s_i-m}$ denote the composite control for other remaining players. Where $(u,\upsilon,\omega) \in \Re^s$, $s = \sum_{e=1}^{r} s_e$ are all composite control. Each player $l = \{1,2...,m,\quad m+1,...,r\}$ selects his control seeking to minimize a scalar-valued criterion

$$F_l(x,u,\upsilon,\omega,\tilde{a}) \tag{1}$$

Subject to n equality constraints

$$g_i(x,u,\upsilon,\omega,\tilde{b}),\ \ i = 1,...,n \tag{2}$$

where $x \in \Re^n$ is the state vector. The composite control is required to be an element of regular control constraint set $\Omega \in \Re^s$, defined by

$$\Omega = \left\{(u,\upsilon,\omega) \in \Re^s \;\middle|\; h_j[\zeta(u,\upsilon,\omega,b),u,\upsilon,\omega,\tilde{c}] \geq 0\right\} \tag{3}$$

where $j = \{1,...,q\}$, $x = \zeta(u,\upsilon,\omega,b)$ is the solution to (2), the functions $F_l(\cdot) : \Re^n \times \Re^s \to \Re^l, g_i(\cdot) : \Re^n \times \Re^s \to \Re^n$, $h(\cdot) : \Re^n \times \Re^s \to \Re^q$ are assumed to be C^1 with $|\,\partial g(x,u,\upsilon,\omega,b)\ /\ \partial x\,| \neq 0$ in a ball about a solution

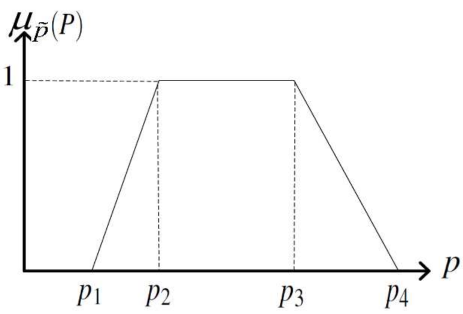


Fig. 1: A special fuzzy number P with continuous trapezoidal membership function

point (x,u,υ,ω,b), and

$$\begin{aligned} \tilde{a} &= (\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m,\tilde{a}_{m+1},...,\tilde{a}_r), \\ \tilde{a}_l &= (\tilde{a}_{l1},\tilde{a}_{l2},...,\tilde{a}_{lm},\tilde{a}_{lm+1},...,\tilde{a}_{lr_l}), \\ \tilde{b} &= (\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_n), \qquad \tilde{b}_e = (\tilde{b}_{e1},\tilde{b}_{e2},...,\tilde{b}_{en_e}) \\ \tilde{c} &= (\tilde{c}_1,\tilde{c}_2,...,\tilde{c}_q), \qquad \tilde{c}_k = (\tilde{c}_{k1},\tilde{c}_{k2},...,\tilde{c}_{kn_k}) \end{aligned}$$

represent a vector of fuzzy parameters in the cost function, equality and inequality constraints respectively. These fuzzy parameters are assumed to be characterized by fuzzy numbers. The real fuzzy numbers form a convex continuous fuzzy subset of the real line whose membership function $\mu_{\tilde{p}}(P)$ are defined as the following (see **Fig. 1**):

1. A continuous mapping from $\Re^1$ to the closed interval $[0,1]$,
2. Constant on $(-\infty,p_1]$: $\mu_{\tilde{p}}(P)=0, \forall\, p\in(-\infty,p_1]$,
3. Strictly increasing on $[p_1,p_2]$,
4. Constant on $[p_2,p_3]$: $\mu_{\tilde{p}}(P)=1, \forall\, p\in[p_2,p_3]$,
5. Strictly decreasing on $[p_3,p_4]$,
6. Constant on $[d,\infty)$: $\mu_{\tilde{p}}(P)=0, \forall\, p\in[p_4,\infty)$.

Then, several F-HCSGs may be formulated depending on the variety chosen of the optimality concepts by the players i and j in the game (1) - (3) as shown in the following subsection.

2.2 Nash Min-Max Fuzzy Hybrid Continuous Static Game

This section deals with an established hybrid Nash Min-Max approach for solving F-HCSG. This approach is based on the fact that the game is presented as a hybrid game between multiple players playing independently using Nash equilibrium solution and others playing under a secure concept using Min-Max solution with fuzzy parameters in both the cost functions and the constraints. The fuzzy parameters are characterized by fuzzy numbers. The concept of $\alpha-$level set used for converting F-HCSG to a deterministic problem $\alpha-$HCSG. Here hybrid Nash Min-Max approach used to solve the deterministic problem $\alpha-$HCSG.

2.2.1 Problem Formulation

Consider the following Nash Min-Max fuzzy hybrid continuous static game (NMMFHCSGs) problem that involving fuzzy parameters in both the cost functions and the constraints. The game consists of two teams, the first team decided to play independently under NES with players set $i\in T_1=\{1,2...,m\}\subset\{1,2...,m,m+1,...,r\}$ (the set of all players). On the other hand, the second one use a MMS between the players $j\in T_2=\{m+1,...,r\}$ as:

NMMFHCSG:

$$\begin{aligned} \text{Min} \quad & F_i(x,u,\upsilon,\tilde{a}),\ i=1,2,...,m \\ & F_j(x,u,\upsilon,\tilde{b}),\ j\in\tau\subset\{m+1,...,r\} \end{aligned} \tag{4}$$

Subject to

$$\Omega=\left\{x\in\Re^n,(u,\upsilon)\in\Re^s \;\middle|\; \begin{array}{l} g_I(x,u,\upsilon,\tilde{c})=0, I=\{1,...,n\}, \\ h_J(x,u,\upsilon,\tilde{d})\geq 0, J=\{1,...,q\}, \\ \varphi_K(x,u,\upsilon,\tilde{e})=0, K=\{1,...,\ell\}, \\ \psi_L(x,u,\upsilon,\tilde{f})\geq 0, L=\{1,...,\varsigma\} \end{array}\right\}$$

and

$$\text{Max} \quad F_j(x,u,\upsilon,\tilde{b}),\ j\notin\tau$$

Subject to

$$\Omega=\left\{x\in\Re^n,(u,\upsilon)\in\Re^s \;\middle|\; \begin{array}{l} g_I(x,u,\upsilon,\tilde{c})=0, I=\{1,...,n\}, \\ h_J(x,u,\upsilon,\tilde{d})\geq 0, J=\{1,...,q\}, \\ \varphi_K(x,u,\upsilon,\tilde{e})=0, K=\{1,...,\ell\}, \\ \psi_L(x,u,\upsilon,\tilde{f})\geq 0, L=\{1,...,\varsigma\} \end{array}\right\}$$

where $(u,\upsilon)\in\Re^s, s=s_1+s_2+...+s_r$ are all composite control. The functions $F_i(\cdot), i=1,...,m, F_j(\cdot), j=m+1,...,r$, $g_I(\cdot), I=\{1,...,n\}, h_J(\cdot), J=\{1,...,q\}, \varphi_K(\cdot), K=\{1,...,\ell\}, \psi_L(\cdot), L=\{1,...,\varsigma\}$ are assumed to be C^1 with $|\,\partial g(x,u,\upsilon,c)/\partial x\,|\neq 0$ and $|\,\partial\varphi(x,u,\upsilon,e)/\partial x\,|\neq 0$ in a ball about a solution points (x,u,υ,c) and (x,u,υ,e) respectively, also

$$\begin{aligned} \tilde{a} &= (\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m), & \tilde{a}_i &= (\tilde{a}_{i1},\tilde{a}_{i2},...,\tilde{a}_{im_i}), \\ \tilde{b} &= (\tilde{b}_{m+1},...,\tilde{b}_r), & \tilde{b}_j &= (\tilde{b}_{j(m+1)},...,\tilde{b}_{jr_j}), \\ \tilde{c} &= (\tilde{c}_1,\tilde{c}_2,...,\tilde{c}_n), & \tilde{c}_I &= (\tilde{c}_{I1},\tilde{c}_{I2},...,\tilde{c}_{In_I}), \\ \tilde{d} &= (\tilde{d}_1,\tilde{d}_2,...,\tilde{d}_q), & \tilde{d}_J &= (\tilde{d}_{J1},\tilde{d}_{J2},...,\tilde{d}_{Jq_J}), \\ \tilde{e} &= (\tilde{e}_1,\tilde{e}_2,...,\tilde{e}_\ell), & \tilde{e}_K &= (\tilde{e}_{K1},\tilde{e}_{K2},...,\tilde{e}_{K\ell_K}), \\ \tilde{f} &= (\tilde{f}_1,\tilde{f}_2,...,\tilde{f}_\varsigma), & \tilde{f}_L &= (\tilde{f}_{L1},\tilde{f}_{L2},...,\tilde{f}_{L\varsigma_L}) \end{aligned}$$

represent a vector of fuzzy parameters in the cost function, equality and inequality constraints respectively. These fuzzy parameters are characterized by fuzzy numbers. The real fuzzy numbers form a convex continuous fuzzy subset of the real line whose membership functions $\mu_{\tilde{p}}(P)$ as shown in **Fig. 1**.

2.2.2 Solution Concept

Let we now introduce solution concept formal definitions for solving NMMFHCSGs problem (4) as follows:

Definition 2.1. ($\alpha-$ Level set [12]) The $\alpha-$level set ($\alpha-$cut) of fuzzy numbers $\tilde{a}_i,\tilde{b}_j,\tilde{c}_I,\tilde{d}_J,\tilde{e}_K,$and $\tilde{f}_L$ are defined as the ordinary set $\Gamma_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f})$ for which the degree of their membership function exceeds the level α where $\Gamma_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f})$ defined by

$$\Gamma_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f})=\left\{(a,b,c,d,e,f) \;\middle|\; \begin{array}{l} \mu_{\tilde{a}_i}(a_i)\geq\alpha, \\ \mu_{\tilde{b}_j}(b_j)\geq\alpha, \\ \mu_{\tilde{c}_I}(c_I)\geq\alpha, \\ \mu_{\tilde{d}_J}(d_J)\geq\alpha, \\ \mu_{\tilde{e}_K}(e_K)\geq\alpha, \\ \mu_{\tilde{f}_L}(f_L)\geq\alpha \end{array}\right\}$$

where $i = 1,...,m, j = m+1,...,r, I = \{1,...,n\}, J = \{1,...,q\}, K = \{1,...,\ell\}$, and $L = \{1,...,\varsigma\}$.
By using **Definition 2.1.** and for a certain degree of α, NMMFHCSG problem (4) can be transferred into non-fuzzy $\alpha-$NMMHCSG as follows:

α- NMMHCSG:

$$\text{Min} \quad F_i(x,u,\upsilon,a), \; i = 1,2,...,m$$
$$F_j(x,u,\upsilon,b), \; j \in \tau \subset \{m+1,...,r\} \tag{5}$$

Subject to

$$\Omega = \left\{ x \in \Re^n, (u,\upsilon) \in \Re^s \left| \begin{array}{l} g_I(x,u,\upsilon,c) = 0, I = \{1,...,n\}, \\ h_J(x,u,\upsilon,d) \geq 0, J = \{1,...,q\}, \\ \varphi_K(x,u,\upsilon,e) = 0, K = \{1,...,\ell\}, \\ \psi_L(x,u,\upsilon,f) \geq 0, L = \{1,...,\varsigma\} \end{array} \right. \right\}$$

$$(\, a,b,c,d,e,f \,) \in \Gamma_\alpha(\, \tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f} \,)$$

and

$$\text{Max} \quad F_j(x,u,\upsilon,b), \; j \notin \tau$$

Subject to

$$\Omega = \left\{ x \in \Re^n, (u,\upsilon) \in \Re^s \left| \begin{array}{l} g_I(x,u,\upsilon,c) = 0, I = \{1,...,n\}, \\ h_J(x,u,\upsilon,d) \geq 0, J = \{1,...,q\}, \\ \varphi_K(x,u,\upsilon,e) = 0, K = \{1,...,\ell\}, \\ \psi_L(x,u,\upsilon,f) \geq 0, L = \{1,...,\varsigma\} \end{array} \right. \right\}$$

$$(\, a,b,c,d,e,f \,) \in \Gamma_\alpha(\, \tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f} \,)$$

Definition 2.2. ($\alpha-$Nash Min-Max solution) A point $u^* = (u_i^*, v_j^*, \omega^*) \in \Omega$ is an $\alpha-$Nash Min-Max point if and only if for each $i = 1,2,...,m$ and $j = m+1,...,r$

$$F_i[\zeta(u^*,c_I^*),u^*,a_i^*] \leq F_i\left[\zeta(u_i,v_j^*,\omega^*),u_i,v_j^*,\omega^*,a_i\right] \tag{6}$$

and

$$\begin{cases} F_j[\zeta(u_i^*,v_j^*,\omega,e_K^*),u_i^*,v_j^*,\omega,b_j] \leq \\ \qquad F_j\left[\zeta(u^*,e_K^*),u^*,b_j^*\right] \leq \\ \qquad\qquad F_j\left[\zeta(u_i^*,v_j,\omega^*,e_K^*),u_i^*,v_j,\omega^*,b_j\right] \end{cases} \tag{7}$$

for all (u_i, v_j, ω) such that $(u_i, \hat{v}_j, \hat{\omega}) \in \Omega$, $(\hat{u}_i, v_j, \hat{\omega}) \in \Omega$ and $(\hat{u}_i, \hat{v}_j, \omega) \in \Omega$ defined by:

$$\Omega = \left\{ (u_i,v_j,\omega) \in \Re^s \left| \begin{array}{l} h\left[\zeta_1(u_i,v_j,\omega,c),u_i,v_j,\omega,d\right] \geq 0, \\ \psi\left[\zeta_2(u_i,v_j,\omega,e),u_i,v_j,\omega,f\right] \geq 0 \end{array} \right. \right\}$$

where $x = \zeta_1(u_i,v_j,c)$, $x = \zeta_2(u_i,v_j,e)$ are the solution to $g(x,u_i,v_j,c) = 0$ and $\varphi(x,u_i, \; v_j,e) = 0$ respectively, and the corresponding values of parameters a_i^*, b_j^*, c_I^* and e_K^* are called $\alpha-$level optimal parameters. Now, under calling the **definition 2.1** and **2.2**, then $\alpha-$NMMHCSG problem (5) can be transformed into the following form

α- NMMHCSG*:

$$\text{Min} \; F_i(x,u,\upsilon,a), \; i = 1,2,...,m$$
$$F_j(x,u,\upsilon,b), \; j \in \tau \subset \{m+1,...,r\} \tag{8}$$

Subject to

$$\Omega = \left\{ x \in \Re^n, (u,\upsilon) \in \Re^s \left| \begin{array}{l} g_I(x,u,\upsilon,c) = 0, I = \{1,...,n\}, \\ h_J(x,u,\upsilon,d) \geq 0, J = \{1,...,q\}, \\ \varphi_K(x,u,\upsilon,e) = 0, K = \{1,...,\ell\}, \\ \psi_L(x,u,\upsilon,f) \geq 0, L = \{1,...,\varsigma\} \end{array} \right. \right\}$$

$$\begin{array}{ll} \underline{A_i} \leq a_i \leq \overline{A_i}, & i = 1,2,...,m, \\ \underline{B_j} \leq b_j \leq \overline{B_j}, & j = m+1,...,r, \\ \underline{C_I} \leq c_I \leq \overline{C_I}, & I = \{1,2,...,n\}, \\ \underline{D_J} \leq d_J \leq \overline{D_J}, & J = \{1,2,...,q\}, \\ \underline{E_K} \leq e_K \leq \overline{E_K}, & K = \{1,2,...,\ell\}, \\ \underline{F_L} \leq f_L \leq \overline{F_L}, & L = \{1,2,...,\varsigma\} \end{array}$$

and

$$\text{Max} \quad F_j(x,u,\upsilon,b), \; j \notin \tau$$

Subject to

$$\Omega = \left\{ x \in \Re^n, (u,\upsilon) \in \Re^s \left| \begin{array}{l} g_I(x,u,\upsilon,c) = 0, I = \{1,...,n\}, \\ h_J(x,u,\upsilon,d) \geq 0, J = \{1,...,q\}, \\ \varphi_K(x,u,\upsilon,e) = 0, K = \{1,...,\ell\}, \\ \psi_L(x,u,\upsilon,f) \geq 0, L = \{1,...,\varsigma\} \end{array} \right. \right\}$$

$$\begin{array}{ll} \underline{A_i} \leq a_i \leq \overline{A_i}, & i = 1,2,...,m, \\ \underline{B_j} \leq b_j \leq \overline{B_j}, & j = m+1,...,r, \\ \underline{C_I} \leq c_I \leq \overline{C_I}, & I = \{1,2,...,n\}, \\ \underline{D_J} \leq d_J \leq \overline{D_J}, & J = \{1,2,...,q\}, \\ \underline{E_K} \leq e_K \leq \overline{E_K}, & K = \{1,2,...,\ell\}, \\ \underline{F_L} \leq f_L \leq \overline{F_L}, & L = \{1,2,...,\varsigma\} \end{array}$$

where $(\underline{A_i},\overline{A_i})$, $\left(\underline{B_j},\overline{B_j}\right)$, $(\underline{C_I},\overline{C_I})$, $(\underline{D_J},\overline{D_J})$, $(\underline{E_K},\overline{E_K})$, and $(\underline{F_L},\overline{F_L})$ are the $\alpha-$level set of the fuzzy parameters a_i, b_j, c_I, d_J, e_K and f_L respectively.
The necessary optimal conditions usable for determining $\alpha-$Nash Min-Max solution points to $\alpha-$NMMHCSG problem (8) may be stated in the following theorem.

Theorem 2.1. If $(u^*, \upsilon^*) = (u_i^*, v_j^*) \in \Omega$ is a completely regular local $\alpha-$Nash Min-Max point and if $x^* = \zeta_1(u^*, \upsilon^*, c^*)$ is solution to $g(x,u^*,\upsilon^*,c^*) = 0$, and $x^* = \zeta_2(u^*,\upsilon^*,e^*)$ is solution to $\varphi(x,u^*,\upsilon^*,e^*) = 0$ then for each $i = 1,2,...,m$ and $j = m+1,...,r$ there exist vectors $\lambda(i) \in \Re^{n_1}, \mu(i) \in \Re^{q_1}, \rho(j) \in \Re^{n_2}, \beta(j) \in \Re^{q_2}$, $\bar{\rho}(j) \in \Re^{n_2}$, $\bar{\beta}(j) \in \Re^{q_2}, \gamma(i) \in \Re^m, \grave{\gamma}(i) \in \Re^m, \delta(j) \in \Re^{r-m}, \grave{\delta}(j) \in \Re^{r-m}$, $\vartheta(i) \in \Re^n, \grave{\vartheta}(i) \in \Re^n, \theta(i) \in \Re^q$, $\grave{\theta}(i) \in \Re^q, \sigma(j) \in \Re^\ell, \grave{\sigma}(j) \in \Re^\ell, \pi(j) \in \Re^\varsigma$ and $\grave{\pi}(j) \in$

$\Re^{\varsigma}$ such that

$$\frac{\partial L_i}{\partial x} = 0,\ i = 1,2,...,m \quad (9)$$

$$\frac{\partial L_i}{\partial u_i} = 0,\ i = 1,2,...,m \quad (10)$$

$$\frac{\partial L_j}{\partial x} = 0,\ j \in \tau \subset \{m+1,...,r\} \quad (11)$$

$$\frac{\partial L_j}{\partial x} = 0,\ j \notin \tau \quad (12)$$

$$\frac{\partial L_j}{\partial v_j} = 0,\ j \in \tau \subset \{m+1,...,r\} \quad (13)$$

$$\frac{\partial L_j}{\partial v_j} = 0,\ j \notin \tau \quad (14)$$

$$\frac{\partial L_i}{\partial a, \partial b, \partial c, \partial d, \partial e, \partial f} = 0,\ i = 1,2,...,m \quad (15)$$

$$\frac{\partial L_j}{\partial a, \partial b, \partial c, \partial d, \partial e, \partial f} = 0,\ j \in \tau \subset \{m+1,...,r\} \quad (16)$$

$$\frac{\partial L_j}{\partial a, \partial b, \partial c, \partial d, \partial e, \partial f} = 0,\ j \notin \tau \quad (17)$$

$$g(x^*,u^*,\upsilon^*,c^*) = 0 \quad (18)$$

$$\varphi(x^*,u^*,\upsilon^*,e^*) = 0 \quad (19)$$

$$h(x^*,u^*,\upsilon^*,d^*) \geq 0 \quad (20)$$

$$\psi(x^*,u^*,\upsilon^*,f^*) \geq 0 \quad (21)$$

$$\mu^t(i)\ h(x^*,u^*,\upsilon^*,d^*) = 0,\ i = 1,2,...,m \quad (22)$$

$$\beta^t(j)\psi(x^*,u^*,\upsilon^*,f^*) = 0,\ j \in \tau \subset \{m+1,...,r\} \quad (23)$$

$$\bar{\beta}^t(j)\ \psi(x^*,u^*,\upsilon^*,f^*) = 0,\ j \notin \tau \quad (24)$$

$$\gamma^t(i)\ (a_i - \underline{A_i}) = 0,\ i = 1,2,...,m \quad (25)$$

$$\grave{\gamma}^t(i)\ (\overline{A_i} - a_i) = 0,\ i = 1,2,...,m \quad (26)$$

$$\delta^t(j)\ (b_j - \underline{B_j}) = 0,\ j = m+1,...,r \quad (27)$$

$$\grave{\delta}^t(j)\ (\overline{B_j} - b_j) = 0,\ j = m+1,...,r \quad (28)$$

$$\vartheta^t(i)\ (c_I - \underline{C_I}) = 0,\ I = 1,2,...,n \quad (29)$$

$$\grave{\vartheta}^t(i)\ (\overline{C_I} - c_I) = 0,\ I = 1,2,...,n \quad (30)$$

$$\theta^t(i)\ (d_J - \underline{D_J}) = 0,\ J = 1,2,...,q \quad (31)$$

$$\grave{\theta}^t(i)\ (\overline{D_J} - d_J) = 0,\ J = 1,2,...,q \quad (32)$$

$$\sigma^t(j)\ (e_K - \underline{E_K}) = 0,\ K = 1,2,...,\ell \quad (33)$$

$$\grave{\sigma}^t(j)\ (\overline{E_K} - e_K) = 0,\ K = 1,2,...,\ell \quad (34)$$

$$\pi^t(j)\ (f_L - \underline{F_L}) = 0,\ L = 1,2,...,\varsigma \quad (35)$$

$$\grave{\pi}^t(j)\ (\overline{F_L} - f_L) = 0,\ L = 1,2,...,\varsigma \quad (36)$$

$$\mu(i) \geq 0,\ i = 1,2,...,m \quad (37)$$

$$\beta(j) \geq 0,\ j \in \tau \subset \{m+1,...,r\} \quad (38)$$

$$\gamma(i),\ \grave{\gamma}(i) \geq 0,\ i = 1,2,...,m \quad (39)$$

$$\delta(j),\ \grave{\delta}(j) \geq 0,\ j = m+1,...,r \quad (40)$$

$$\vartheta(i),\ \grave{\vartheta}(i) \geq 0,\ i = 1,2,...,m \quad (41)$$

$$\theta(i),\ \grave{\theta}(i) \geq 0,\ i = 1,2,...,m \quad (42)$$

$$\sigma(j),\ \grave{\sigma}(j) \geq 0,\ j = m+1,...,r \quad (43)$$

$$\pi(j),\ \grave{\pi}(j) \geq 0,\ j = m+1,...,r \quad (44)$$

$$\bar{\beta}(j) \leq 0,\ j \notin \tau \quad (45)$$

where,

$$\begin{aligned} L_i[x,u,\upsilon,a,b,c,d,e,f,\lambda,\mu,\rho,\beta,\gamma,\delta,\vartheta,\theta,\sigma,\pi] &= F_i(x,u,\upsilon,a) \\ -\lambda^t(i)g(x,u,\upsilon,c) - \mu^t(i)h(x,u,\upsilon,d) - \rho^t(j)\varphi(x,u,\upsilon,e) \\ -\beta^t(j)\psi(x,u,\upsilon,f) - \gamma^t(i)(a_i - \underline{A_i}) - \gamma^t(i)(a_i - \underline{A_i}) \\ -\grave{\gamma}^t(i)(\overline{A_i} - a_i) - \delta^t(j)(b_j - \underline{B_j}) - \grave{\delta}^t(j)(\overline{B_j} - b_j) \\ -\vartheta^t(i)(c_I - \underline{C_I}) - \grave{\vartheta}^t(i)(\overline{C_I} - c_I) - \theta^t(i)(d_J - \underline{D_J}) \\ -\grave{\theta}^t(i)(\overline{D_J} - d_J) - \sigma^t(j)(e_K - \underline{E_K}) - \grave{\sigma}^t(j)(\overline{E_K} - e_K) \\ -\pi^t(j)(f_L - \underline{F_L}) - \grave{\pi}^t(j)(\overline{F_L} - f_L) \end{aligned} \quad (46)$$

for each player $i \in T_1 = \{1,2,...,m\}$, and

$$\begin{aligned} L_j[x,u,\upsilon,a,b,c,d,e,f,\lambda,\mu,\rho,\beta,\gamma,\delta,\vartheta,\theta,\sigma,\pi] &= F_j(x,u,\upsilon,a) \\ -\lambda^t(i)g(x,u,\upsilon,c) - \mu^t(i)h(x,u,\upsilon,d) - \rho^t(j)\varphi(x,u,\upsilon,e) \\ -\beta^t(j)\psi(x,u,\upsilon,f) - \gamma^t(i)(a_i - \underline{A_i}) - \gamma^t(i)(a_i - \underline{A_i}) \\ -\grave{\gamma}^t(i)(\overline{A_i} - a_i) - \delta^t(j)(b_j - \underline{B_j}) - \grave{\delta}^t(j)(\overline{B_j} - b_j) \\ -\vartheta^t(i)(c_I - \underline{C_I}) - \grave{\vartheta}^t(i)(\overline{C_I} - c_I) - \theta^t(i)(d_J - \underline{D_J}) \\ -\grave{\theta}^t(i)(\overline{D_J} - d_J) - \sigma^t(j)(e_K - \underline{E_K}) - \grave{\sigma}^t(j)(\overline{E_K} - e_K) \\ -\pi^t(j)(f_L - \underline{F_L}) - \grave{\pi}^t(j)(\overline{F_L} - f_L) \end{aligned} \quad (47)$$

for each player $j \in T_2 = \{m+1,...,r\}$.
While, the partial derivatives of L_j are evaluated using the two sets of multipliers $\rho(j),\ \beta(j)\ \forall\ j \in \tau \subset \{m+1,...,r\}$ and $\bar{\rho}(j), \bar{\beta}(j)\ \forall\ j \notin \tau$ as per the minimization or maximization criteria for the player j.

3 Parametric Analysis of NMMFHCSGs

This section deals with some basic notions in parametric NMMFHCSGs problem (4). These notions are: (i) The set of feasible parameters, (ii) The solvability set of feasible parameters, (iii) The stability set of the first kind, and (iv) The stability set of the second kind.
Let us consider $\alpha-$NMMHCSG programming problem (8) where the functions $F_i(\cdot) : \Re^n \times \Re^{2m} \times \Re^s \times \Re^r \to \Re^m$, $g_I(\cdot) : \Re^{3n} \times \Re^s \to \Re^{n_1}$, and $\varphi_K(\cdot) : \Re^n \times \Re^{2\ell} \times \Re^s \to \Re^{n_2}$ are convex functions and assumed to be of class C^1. The Function $F_j(\cdot) : \Re^n \times \Re^m \times \Re^s \times \Re^{2r} \to \Re^r$ is convex function in $v_j, j \in \tau \subset \{m+1,...,r\}$ and is concave function in $v_j, j \notin \tau$ assumed to be of class C^1, and the two functions $h_J(\cdot) : \Re^n \times \Re^{2q} \times \Re^s \to \Re^{q_1}$, $\psi_L(\cdot) : \Re^n \times \Re^{2\varsigma} \times \Re^s \to \Re^{q_2}$ are concave and also assumed to be of class C^1 with $|\ \partial g(x,u,\upsilon,b)/\partial x\ | \neq 0$ in a ball about a solution point (x,u,υ,b) and $|\ \partial\varphi(x,u,\upsilon,e)/\partial x\ | \neq 0$ in a ball about a solution point (x,u,υ,e), $\Re^{n_1} \times \Re^{n_2} \to \Re^n$ and $\Re^{q_1} \times \Re^{q_2} \to \Re^q$.
For this problem, let we define the following notions:

3.1 The Set of Feasible Parameters

The set of feasible parameters S for $\alpha-$NMMHCSG problem (8) is defined by

$$S=\left\{q\in\Re^{2(2n+q+\ell+\varsigma+s)}\mid \Omega(q)\neq\emptyset\right\} \quad (48)$$

where $q=(c,d,e,f)$ and $\Omega(q)$ is defined in (8).

3.2 The Solvability Set of Feasible Parameters

The solvability set of Feasible Parameters S_1 for $\alpha-$NMMHCSG problem (8) is defined by

$$S_1(P)=\left\{(p,q)\in\Re^{2(r+2n+q+\ell+\varsigma+2s)}\left|\begin{array}{l}\text{The solution of}\\ \text{problem (8) exist}\end{array}\right.\right\} \quad (49)$$

where $p=(a,b)$ and $q=(c,d,e,f)$.

3.3 The Stability Set of the First Kind

Suppose that $P^*=(p^*,q^*)\in S_1(P)$ with a corresponding $\alpha-$Nash Min-Max solution $\omega^*=(u^*,\upsilon^*,a^*,b^*,c^*,d^*,e^*,f^*)$ to the problem (8) then the stability set of the first kind corresponding to ω^* denoted by S_2 is defined by

$$S_2(\omega^*)=\{P\in S_1(P)\mid \omega^* \text{ is solution to problem (8)}\} \quad (50)$$

The following algorithm can be constructed for determining the stability set of the first kind $S_2(\omega^*)$ based on **Theorem 2.1.** section 2.

Algorithm 3.1.

Step 1: Starting with any $P^*\in S_1(P)$ and then using a suitable algorithm to solve $\alpha-$NMMFHCSGs problem (8), we obtain an $\alpha-$Nash Min-Max solution $\omega^*=(u^*,\upsilon^*,a^*,b^*,c^*,d^**,e^**,f^*)$.

Step 2: Substituting in the system of equations (9) - (45) that stated in **Theorem 2.1.** section 2, we obtain N as follows:

$$N=\left\{\frac{\partial F_i(x^*,u^*,\upsilon^*,a^*)}{\partial x}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I(x^*,u^*,\upsilon^*,c^*)}{\partial x}\right)\right]\right.$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J(x^*,u^*,\upsilon^*,d^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial \varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\beta_L(i)\left(\frac{\partial \psi_L(x^*,u^*,\upsilon^*,f^*)}{\partial x}/\right)\right]=0,$$
$$\frac{\partial F_i(x^*,u^*,\upsilon^*,a^*)}{\partial u_i}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I(x^*,u^*,\upsilon^*,c^*)}{\partial u_i}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J(x^*,u^*,\upsilon^*,d^*)}{\partial u_i}\right)\right]$$
$$-\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial \varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial u_i}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\beta_L(i)\left(\frac{\partial \psi_L(x^*,u^*,\upsilon^*,f^*)}{\partial u_i}\right)\right]=0,$$
$$\frac{\partial F_j(x^*,u^*,\upsilon^*,b^*)}{\partial x}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I(x^*,u^*,\upsilon^*,c^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J(x^*,u^*,\upsilon^*,d^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial \varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\beta_L(i)\left(\frac{\partial \psi_L(x^*,u^*,\upsilon^*,f^*)}{\partial x}/\right)\right]=0,\ j\in\tau$$
$$\frac{\partial F_j(x^*,u^*,\upsilon^*,b^*)}{\partial x}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I(x^*,u^*,\upsilon^*,c^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J(x^*,u^*,\upsilon^*,d^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{K=1}^{\ell}\bar{\rho}_K(j)\left(\frac{\partial \varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial x}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\bar{\beta}_L(i)\left(\frac{\partial \psi_L(x^*,u^*,\upsilon^*,f^*)}{\partial x}\right)\right]=0,\ j\notin\tau$$
$$\frac{\partial F_j(x^*,u^*,\upsilon^*,b^*)}{\partial v_j}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I(x^*,u^*,\upsilon^*,c^*)}{\partial v_j}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J(x^*,u^*,\upsilon^*,d^*)}{\partial v_j}\right)\right]$$
$$-\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial \varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial v_j}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\beta_L(i)\left(\frac{\partial \psi_L(x^*,u^*,\upsilon^*,f^*)}{\partial v_j}\right)\right]=0,\ j\in\tau$$
$$\frac{\partial F_j(x^*,u^*,\upsilon^*,b^*)}{\partial v_j}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I(x^*,u^*,\upsilon^*,c^*)}{\partial v_j}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J(x^*,u^*,\upsilon^*,d^*)}{\partial v_j}\right)\right]$$
$$-\left[\sum_{K=1}^{\ell}\bar{\rho}_K(j)\left(\frac{\partial \varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial v_j}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\bar{\beta}_L(j)\left(\frac{\partial \psi_L(x^*,u^*,\upsilon^*,f^*v)}{\partial v_j}\right)\right]=0,\ j\notin\tau$$
$$\left[\frac{\partial F_i(x^*,u^*,\upsilon^*,a^*)}{\partial a}\right]-\gamma_i(i)+\hat{\gamma}_i(i)=0,$$
$$\left[\frac{\partial F_j(x^*,u^*,\upsilon^*,b^*)}{\partial b}\right]-\delta_j(j)+\hat{\delta}_j(j)=0,$$
$$\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I(x^*,u^*,\upsilon^*,c^*)}{\partial c}\right)\right]-\vartheta_I(i)+\hat{\vartheta}_I(i)=0,$$

$$\left[\sum_{J=1}^{q}\mu_J(i)\left(\frac{\partial h_J(x^*,u^*,\upsilon^*,d^*)}{\partial d}\right)\right]-\theta_J(i)+\grave{\theta}_J(i)=0,$$
$$\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial\varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial e}\right)\right]-\sigma_K(j)+\grave{\sigma}_K(j)=0,$$
$$\left[\sum_{L=1}^{\varsigma}\beta_L(j)\left(\frac{\partial\psi_L(x^*,u^*,\upsilon^*,f^*)}{\partial f}\right)\right]-\pi_L(j)+\grave{\pi}_L(j)=0,$$
$$\left[\sum_{K=1}^{\ell}\bar{\rho}_K(j)\left(\frac{\partial\varphi_K(x^*,u^*,\upsilon^*,e^*)}{\partial e}\right)\right]-\sigma_K(j)+\grave{\sigma}_K(j)=0,$$
$$\left.\left[\sum_{L=1}^{\varsigma}\bar{\beta}_L(j)\left(\frac{\partial\psi_L(x^*,u^*,\upsilon^*,f^*)}{\partial f}\right)\right]-\pi_L(j)+\grave{\pi}_L(j)=0\right\}$$

where
$\mu(i)\geq 0,\beta(j)\geq 0,\gamma(i)\geq 0,\grave{\gamma}(i)\geq 0,\delta(j)\geq 0,\grave{\delta}(j)\geq 0,$
$\vartheta(i)\geq 0,\grave{\vartheta}(i)\geq 0,\theta(i)\geq 0,\grave{\theta}(i)\geq 0,\sigma(j)\geq 0,\grave{\sigma}(j)\geq 0,$
$\pi(j)\geq 0,\grave{\pi}(j)\geq 0$ and $\bar{\beta}(j)\leq 0$.
Note that, this system denoted by N represents H equations in the M unknowns which are linear in $\lambda,\mu,\rho,\beta,\gamma,\grave{\gamma},\delta,\grave{\delta},\ \vartheta,\grave{\vartheta},\theta,\grave{\theta},\ \sigma,\grave{\sigma},\pi$ and $\grave{\pi}$. If $H=M$, we can obtain $\lambda,\mu,\rho,\beta,\ \gamma,\grave{\gamma},\delta,\grave{\delta},\vartheta,\grave{\vartheta},\theta,\ \grave{\theta},\sigma,\grave{\sigma},\pi$ and $\grave{\pi}$ explicitly. Otherwise, if $H<M$, then we obtain λ,ρ as a function of other multipliers.

Step 3: The stability set of the first kind is defined by

$$S_2(\omega^*)=\left\{P\in\Re^{2\left(\substack{m+r+3n+\\ q+\ell+\varsigma+2s}\right)}\ \middle|\ \begin{pmatrix}\lambda,\mu,\rho,\beta,\gamma,\\ \delta,\vartheta,\theta,\sigma,\pi\end{pmatrix}\in N\right\}$$

and $\omega^*=(u^*,\upsilon^*,a^*,b^*,c^*,d^{*}*,e^*,f^*)$ is the solution of problem (8).

3.4 The Stability Set of The Second Kind

Suppose that $P^*\in S_1(P)$ with a corresponding $\alpha-$Nash Min-Max solution $u^*=(u^*,\upsilon^*)=\left(u_i^*,\upsilon_j^*\right),i=1,2,\ldots,m$ and $j=m+1,\ldots r$ to problem (8), and $\chi(\kappa)$ denotes either the unique side of Ω which contains u^* defined by

$$\chi(\kappa)=\left\{(u,\upsilon)\in\Re^s\ \middle|\ \begin{array}{l} g_I[\zeta_1(u,\upsilon,c^*),u,\upsilon,c^*]=0\\ \forall I=\{1,2,\ldots,n\},\\ h_J[\zeta_1(u,\upsilon,c^*),u,\upsilon,d^*]=0\\ \forall J\in Z_1\subset\{1,2,\ldots,q\},\\ h_J[\zeta_1(u,\upsilon,c^*),u,\upsilon,d^*]>0\\ \forall J\notin Z_1,\\ \varphi_K[\zeta_2(u,\upsilon,e^*),u,\upsilon,e^*]=0\\ \forall K=\{1,2,\ldots,\ell\},\\ \psi_L[\zeta_2(u,\upsilon,e^*),u,\upsilon,f^*]=0\\ \forall L\in Z_2\subset\{1,2,\ldots,\varsigma\},\\ \psi_L[\zeta_2(u,\upsilon,e^*),u,\upsilon,f^*]>0\\ \forall L\notin Z_2,\\ \underline{C}\leq c^*\leq\overline{C},\ \underline{D}\leq d^*\leq\overline{D},\\ \underline{E}\leq e^*\leq\overline{E},\ \underline{F}\leq f^*\leq\overline{F}\end{array}\right\}$$

Then the stability set of the second kind of problem (8) for each player corresponding to $\chi(\kappa)$ denoted by $\Pi(\kappa)$ is defined by

$$\Pi(\kappa)=\left\{P\in S_1(P)\ \middle|\ m_{Opt.}(p)\cap\chi(\kappa)\neq\emptyset\right\}$$

where

$$m_{Opt.}(p)=\left\{u^*\in\Re^s\ \middle|\ \begin{array}{l} F_i[\zeta_1(u^*,c^*),u^*,a^*]\\ \qquad=\text{Min}\ (F_i[\zeta_1(u,c),u,a])\\ \forall i=1,2,\ldots,m,\\ F_j[\zeta_2(u^*,e^*),u^*,b^*]\\ \qquad=\text{Min}\ (F_j[\zeta_2(u,e),u,b])\\ \forall j=m+1,\ldots,r,\\ \underline{A}\leq a\leq\overline{A},\ \underline{B}\leq b\leq\overline{B},\\ \underline{C}\leq c\leq\overline{C},\ \underline{E}\leq e\leq\overline{E}\end{array}\right\}$$

and

$$\chi(\kappa)=\left\{(u,\upsilon)\in\Re^s\ \middle|\ \begin{array}{l} g_I[\zeta_1(u,\upsilon,c),u,\upsilon,c]=0\\ \forall I=\{1,2,\ldots,n\},\\ h_J[\zeta_1(u,\upsilon,c),u,\upsilon,d^*]=0\\ \forall J\in Z_1\subset\{1,2,\ldots,q\},\\ h_J[\zeta_1(u,\upsilon,c),u,\upsilon,d]>0\\ \forall J\notin Z_1,\\ \varphi_K[\zeta_2(u,\upsilon,e),u,\upsilon,e]=0\\ \forall K=\{1,2,\ldots,\ell\},\\ \psi_L[\zeta_2(u,\upsilon,e),u,\upsilon,f]=0\\ \forall L\in Z_2\subset\{1,2,\ldots,\varsigma\},\\ \psi_L[\zeta_2(u,\upsilon,e),u,\upsilon,f]>0\\ \forall L\notin Z_2,\\ \underline{C}\leq c\leq\overline{C},\ \underline{D}\leq d\leq\overline{D},\\ \underline{E}\leq e\leq\overline{E},\ \underline{F}\leq f\leq\overline{F}\end{array}\right\}$$

Now the following algorithm can be used to determine the stability set of the second kind as follows

Algorithm 3.2.

Step 1: Choosing $P^*\in S_1(P)$ and find the corresponding $\alpha-$ Nash Min-Max solution (u^*,υ^*) for the problem (8).
Step 2: Determine the side $\chi(\kappa)$ for which $(u^*,\upsilon^*)\in\chi(\kappa)$. This determination is carried by substituting in the constraints by (u^*,υ^*) to find the active constraints and inactive one.
Step 3: Compute the stability set of the second kind $\Pi(\kappa)$ by solving the following Kuhn-Tucker conditions

$$\frac{\partial F_i[u,\upsilon,a]}{\partial u_i}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I[u,\upsilon,c]}{\partial u_i}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J[u,\upsilon,d]}{\partial u_i}\right)\right]-\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial \varphi_K[u,\upsilon,e]}{\partial u_i}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\beta_L(i)\left(\frac{\partial \psi_L[\zeta_2(u,\upsilon,e),u,\upsilon,f]}{\partial u_i}\right)\right]=0,\quad i=1,2,...,m$$

$$\frac{\partial F_j[u,\upsilon,b]}{\partial v_j}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I[u,\upsilon,c]}{\partial v_j}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J[u,\upsilon,d]}{\partial v_j}\right)\right]-\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial \varphi_K[u,\upsilon,e]}{\partial v_j}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\beta_L(i)\left(\frac{\partial \psi_L[u,\upsilon,f]}{\partial v_j}\right)\right]=0,\quad j\in\tau\subset\{m+1,...,r\}$$

$$\frac{\partial F_j[u,\upsilon,b]}{\partial v_j}-\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I[u,\upsilon,c]}{\partial v_j}\right)\right]$$
$$-\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J[u,\upsilon,d]}{\partial v_j}\right)\right]-\left[\sum_{K=1}^{\ell}\bar{\rho}_K(j)\left(\frac{\partial \varphi_K[u,\upsilon,e]}{\partial v_j}\right)\right]$$
$$-\left[\sum_{L=1}^{\varsigma}\bar{\beta}_L(i)\left(\frac{\partial \psi_L[u,\upsilon,f]}{\partial v_j}\right)\right]=0,\quad j\notin\tau\subset\{m+1,...,r\}$$

$$\frac{\partial F_i[u,\upsilon,a]}{\partial a}-\gamma_i(i)+\grave{\gamma}_i(i)=0,$$
$$\frac{\partial F_j[u,\upsilon,b]}{\partial b}-\delta_j(j)+\grave{\delta}_j(j)=0,$$
$$\left[\sum_{I=1}^{n}\lambda_I(i)\left(\frac{\partial g_I[u,\upsilon,c]}{\partial c}\right)\right]-\vartheta_I(i)+\grave{\vartheta}_I(i)=0,$$
$$\left[\sum_{J=1}^{q}\mu_I(i)\left(\frac{\partial h_J[u,\upsilon,d]}{\partial d}\right)\right]-\theta_J(i)+\grave{\theta}_J(i)=0,$$
$$\left[\sum_{K=1}^{\ell}\rho_K(j)\left(\frac{\partial \varphi_K[u,\upsilon,e]}{\partial e}\right)\right]-\sigma_K(j)+\grave{\sigma}_K(j)=0,$$
$$\left[\sum_{L=1}^{\varsigma}\beta_L(i)\left(\frac{\partial \psi_L[u,\upsilon,f]}{\partial f}\right)\right]-\pi_L(j)+\grave{\pi}_L(j)=0,$$
$$\left[\sum_{K=1}^{\ell}\bar{\rho}_K(j)\left(\frac{\partial \varphi_K[u,\upsilon,e]}{\partial e}\right)\right]-\sigma_K(j)+\grave{\sigma}_K(j)=0,$$
$$\left[\sum_{L=1}^{\varsigma}\bar{\beta}_L(i)\left(\frac{\partial \psi_L[u,\upsilon,f]}{\partial f}\right)\right]-\pi_L(j)+\grave{\pi}_L(j)=0,$$
$$g_I[u,\upsilon,c]=0,\ \ h_J[u,\upsilon,d]\geq 0,$$
$$\varphi_K[u,\upsilon,e]=0,\ \ \psi_L[u,\upsilon,f]\geq 0,$$
$$(a_i-\underline{A_i})\geq 0,\ \ (\overline{A_i}-a_i)\geq 0,$$
$$\left(b_j-\underline{B_j}\right)\geq 0,\ \ (\overline{B_j}-b_j)\geq 0,$$
$$(c_I-\underline{C_I})\geq 0,\ \ (\overline{C_I}-c_I)\geq 0,$$
$$(d_J-\underline{D_J})\geq 0,\ \ (\overline{D_J}-d_J)\geq 0,$$
$$(e_K-\underline{E_K})\geq 0,\ \ (\overline{E_K}-e_K)\geq 0,$$
$$(f_L-\underline{F_L})\geq 0,\ \ (\overline{F_L}-f_L)\geq 0$$

where
$\mu(i)\geq 0,\beta(j)\geq 0,\gamma(i)\geq 0,\grave{\gamma}(i)\geq 0,\delta(j)\geq 0,\grave{\delta}(j)\geq 0,\vartheta(i)\geq 0,\grave{\vartheta}(i)\geq 0,\theta(i)\geq 0,\grave{\theta}(i)\geq 0,\sigma(j)\geq 0,\grave{\sigma}(j)\geq 0,\pi(j)\geq 0,\grave{\pi}(j)\geq 0$ and $\bar{\beta}(j)\leq 0$.

Note that this system is linear in $\lambda,\mu,\rho,\beta,\gamma,\grave{\gamma},\delta,\grave{\delta},\vartheta,\grave{\vartheta},\theta,\grave{\theta},\sigma,\grave{\sigma},\pi$ and $\grave{\pi}$ and non-linear in u,υ,a,b,c,d,e and f. This system can be solved if $F_i(\cdot),i=1,2,...,m$, $F_j(\cdot),j=m+1,...,r,g_I(\cdot),I=\{1,2,...,n\},\varphi_K(\cdot),K=\{1,2,...,\ell\}$ are linear or quadratic functions, $h_J(\cdot)$, $J=\{1,2,...,q\}$ and $\psi_L(\cdot)$, $L=\{1,2,...,\varsigma\}$ are linear or nonlinear functions.

4 Solution Procedure for Solving NMMFHCSGs [Problem (4)]

Solution procedure for solving NMMFHCSGs problem (4) can be summarized in the following steps

Step 1: Ask the decision maker to specify $\alpha-$cut where $(0\leq\alpha\leq 1)$.

Step 2: Employing **Definition 2.1.** and **Definition 2.2.** section 2, for a certain (α), Nash Min-Max fuzzy hybrid continuous static games easily converted to $\alpha-$NMMHCSG problem (8).

Step 3: State the necessary optimal conditions that stated in **Theorem 2.1.** section 2.

Step 4: Solve the system of equations (9) - (45) by using any computer package, i.e. **[gamultiobj]** built-in MATLAB tool, to get $\alpha-$Nash Min-Max solution point to problem (8).

Step 5: Using **Algorithm 3.1.** and **Algorithm 3.2.**, section 3, to determine the stability set of the first and second kind corresponding to $\alpha-$Nash Min-Max solution point from **Step 4**, respectively.

5 Numerical Example

Let us consider the following Nash Min-Max hybrid continuous static game with fuzzy parameters. The game consists of four players whose cost functions are

$$F_1=(u_1-\tilde{a}_1)^2-8u_2+5u_1$$
$$F_2=(u_2+\tilde{a}_2)^2+2u_1$$
$$F_3=(6+u_4)-(u_3+10)+\tilde{a}_3u_3^2$$
$$F_4=(u_3+10)-(u_4+6)+\tilde{a}_3u_4^2$$

respectively. Assuming that the first two players plaiyng independently using NES with control vectors u_1 and u_2, respectively. The remaining players playing under a secure concept using **MMS** with control vectors u_3 and u_4, where

$$4-u_1-u_2\geq 0,\ 6-2u_1-u_2\geq 0,\ -10\leq u_3\leq 4,$$
$$-6\leq u_4\leq 4,\ u_1\geq 0,\ \text{and}\ \ u_2\geq 0$$

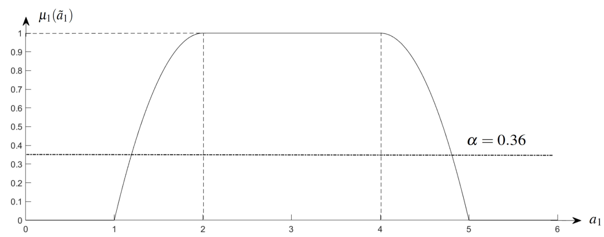
with the membership functions:

$$\mu_i(a_i)=\begin{cases}0, & -\infty\le a_i\le P_{i1}\\ 1-\left(\dfrac{a_i-P_{i2}}{P_{i1}-P_{i2}}\right)^2, & P_{i1}\le a_i\le P_{i2}\\ 1, & P_{i2}\le a_i\le P_{i3}\\ 1-\left(\dfrac{a_i-P_{i3}}{P_{i4}-P_{i3}}\right)^2, & P_{i3}\le a_i\le P_{i4}\\ 0, & P_{i4}\le a_i\le\infty\end{cases}$$

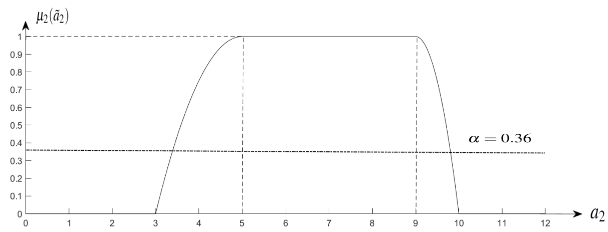
and

$$\begin{array}{llll} P_{11}=1, & P_{12}=2, & P_{13}=4, & P_{14}=5,\\ P_{21}=3, & P_{22}=5, & P_{23}=9, & P_{24}=10,\\ P_{31}=0, & P_{32}=0.1, & P_{33}=0.3, & P_{34}=0.5\end{array}$$

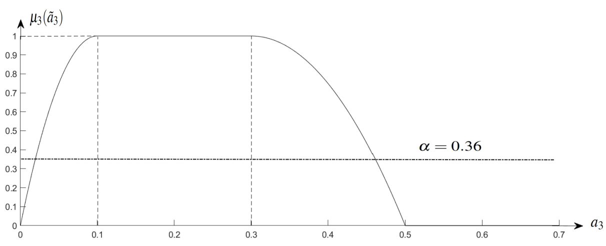
See **Fig.** 2. Taking $\alpha = 0.36$, then $1.2 \le a_1 \le 4.8$, $3.4 \le a_2 \le 9.8$, $0.02 \le a_3 \le 0.46$.
According to the **Definition 2.1.** and , **Definition 2.2.**, section 2, for $\alpha = 0.36$, NMMFHCSG converted to



Membership function for the fuzzy number $\tilde{a}_1 = [1,2,4,5]^T$



Membership function for the fuzzy number $\tilde{a}_2 = [3,5,9,10]^T$



Membership function for the fuzzy number $\tilde{a}_3 = [0,0.1,0.3,0.5]^T$

Fig. 2: Membership function for the fuzzy numbers $\tilde{a}_1, \tilde{a}_2$ and $\tilde{a}_3$

$\alpha-$NMMHCSG as the following NLPP

$$\begin{aligned} \textbf{Min } F_1 &= (u_1-a_1)^2 - 8u_2 + 5u_1,\\ F_2 &= (u_2+a_2)^2 + 2u_1,\\ F_3 &= (6+u_4) - (u_3+10) + a_3u_3^2, \text{for } P_3 \text{ Min-Max},\\ \text{or } F_4 &= (u_3+10) - (u_4+6) + a_3u_4^2, \text{for } P_4 \text{ Min-Max} \end{aligned}$$

Subject to

$$\begin{gathered} 4-u_1-u_2\ge 0,\ 6-2u_1-u_2\ge 0,\ u_1\ge 0,\ u_2\ge 0,\\ -10\le u_3\le 4,\ -6\le u_4\le 4,\ 1.2\le a_1\le 4.8,\\ 3.4\le a_2\le 9.8,\ 0.02\le a_3\le 0.46 \end{gathered}$$

and

$$\begin{aligned} \textbf{Max } F_4 &= (u_3+10) - (u_4+6) + a_3u_4^2, \text{for } P_3 \text{ Min-Max},\\ \text{or } F_3 &= (6+u_4) - (u_3+10) + a_3u_3^2, \text{for } P_4 \text{ Min-Max} \end{aligned}$$

Subject to

$$\begin{gathered} 4-u_1-u_2\ge 0,\ 6-2u_1-u_2\ge 0,\ u_1\ge 0,\ u_2\ge 0,\\ -10\le u_3\le 4,\ -6\le u_4\le 4,\ 1.2\le a_1\le 4.8,\\ 3.4\le a_2\le 9.8,\ 0.02\le a_3\le 0.46 \end{gathered}$$

According to **Theorem 2.1.** section 2, the first two players playing independently, then the Lagrangian function for this team can be defined by using equation (46) as follows

$$\begin{aligned} L_1 = \ & (u_1-a_1)^2 - 8u_2 + 5u_1 - \mu_1(1)(4-u_1-u_2)\\ & -\mu_2(1)(6-2u_1-u_2) - \mu_3(1)u_1 - \mu_4(1)u_2 - \mu_5(1)\\ & (u_3+10) - \mu_6(1)(4-u_3) - \mu_7(1)(u_4+6) - \mu_8(1)(4-u_4)\\ & -\mu_9(1)(a_1-1.2) - \mu_{10}(1)(4.8-a_1) - \mu_{11}(1)(a_2-3.4)\\ & -\mu_{12}(1)(9.8-a_2) - \mu_{13}(1)(a_3-0.02) - \mu_{14}(1)(0.46-a_3), \end{aligned}$$

$$\begin{aligned} L_2 = \ & (u_2+a_2)^2 + 2u_1 - \mu_1(2)(4-u_1-u_2)\\ & -\mu_2(2)(6-2u_1-u_2) - \mu_3(2)u_1 - \mu_4(2)u_2 - \mu_5(2)\\ & (u_3+10) - \mu_6(2)(4-u_3) - \mu_7(2)(u_4+6) - \mu_8(2)(4-u_4)\\ & -\mu_9(2)(a_1-1.2) - \mu_{10}(2)(4.8-a_1) - \mu_{11}(2)(a_2-3.4)\\ & -\mu_{12}(2)(9.8-a_2) - \mu_{13}(2)(a_3-0.02) - \mu_{14}(2)(0.46-a_3) \end{aligned}$$

On the other hand, the remaining players P_3 and P_4 playing under Min-Max solution then

1.For determining the Min-Max solution for player P_3 , as shown in (47), define

$$\begin{aligned} L_3 = \ & (6+u_4) - (u_3+10) + a_3u_3^2 - \mu_1(3)(4-u_1-u_2)\\ & -\mu_2(3)(6-2u_1-u_2) - \mu_3(3)u_1 - \mu_4(3)u_2 - \mu_5(3)\\ & (u_3+10) - \mu_6(3)(4-u_3) - \mu_7(3)(u_4+6) - \mu_8(3)\\ & (4-u_4) - \mu_9(3)(a_1-1.2) - \mu_{10}(3)(4.8-a_1)\\ & -\mu_{11}(3)(a_2-3.4) - \mu_{12}(3)(9.8-a_2)\\ & -\mu_{13}(3)(a_3-0.02) - \mu_{14}(3)(0.46-a_3) \end{aligned}$$

For internal solution to the system of equations that produced from applying the necessary conditions (9) – (45) that stated in **Theorem 2.1.** the $\alpha-$Nash Min-Max solution for player P_3 is

$$(u_1^*, u_2^*, u_3^*, u_4^*, a_1^*, a_2^*, a_3^*) = (0,0,4,4,1.2,3.4,0.02)$$

Keeping the solution $(0,0,4,4,1.2,3.4,0.02)$ we get $\mu_i(l) = 0 \; \forall \; i = 1,...,14$ and $l = 1,2,3$ except $\mu_3(1) = \mu_9(1) = \mu_4(2) = \mu_{11}(2) = \mu_6(3) = \mu_8(3) = \mu_{13}(3) \neq 0$. Then Kuhn-Tucker conditions gives $\mu_3(1) = 2.6, \mu_9(1) = 2.4, \mu_1(2) = 0, \mu_4(2) = 6.8, \mu_{11}(2) = 6.8, \mu_6(3) = 0.84, \mu_8(3) = -1, \mu_{13}(3) = 16$.
If $b_1 \leq a_1^* \leq b_2, c_1 \leq a_2^* \leq c_2, d_1 \leq a_3^* \leq d_2$, then

$$b_1 = 1.2, b_2 \geq 1.2, c_1 = 3.4, c_2 \geq 3.4, d_1 = 0.02, d_2 \geq 0.02$$

i.e., the stability set of the first kind for player P_3 corresponding to the solution $w_3^* = (0,0,4,4,1.2, 3.4,0.02)$ is

$$S_2(w_3^*) = \left\{ (p,q,r) \left| \begin{array}{l} p_{31} \leq p_{32} = 12 - 9p_{31} \leq p_{33} \leq p_{34}, \\ \quad 0 \leq p_{31} \leq 1.2, \\ q_{31} \leq q_{32} = 34 - 9q_{31} \leq q_{33} \leq q_{34}, \\ \quad 0 \leq q_{31} \leq 3.4, \\ r_{31} \leq r_{32} = 0.2 - 9r_{31} \leq r_{33} \leq r_{34}, \\ \quad 0 \leq r_{31} \leq 0.02 \end{array} \right. \right\}$$

To determine the stability set of the second kind for player P_3, let we define the following unique side of constraints that defined by

$$\chi = \left\{ (u_1,u_2,u_3,u_4) \left| \begin{array}{l} 4 - u_1 - u_2 > 0, \; 6 - 2u_1 - u_2 > 0, \\ u_1 = 0, \; u_2 = 0, \; u_3 = 4, \; u_4 = 4, \\ a_1 = 1.2, \; a_2 = 3.4, \; a_3 = 0.02 \end{array} \right. \right\}$$

then the stability set of the second kind for player P_3 as

$$\Pi = \left\{ \begin{array}{l} \mu_i(l) = 0 \; \forall \; i = 1,...,14 \text{ and } l = 1,2,3 \text{ except} \\ \mu_3(1) = 2.6, \mu_9(1) = 2.4, \mu_1(2) = 0, \mu_4(2) = 6.8, \\ \mu_{11}(2) = 6.8, \mu_6(3) = 0.84, \mu_8(3) = -1, \mu_{13}(3) = 16, \\ p_{31} \leq p_{32} = 12 - 9p_{31} \leq p_{33} \leq p_{34}, 0 \leq p_{31} \leq 1.2, \\ q_{31} \leq q_{32} = 34 - 9q_{31} \leq q_{33} \leq q_{34}, 0 \leq q_{31} \leq 3.4, \\ r_{31} \leq r_{32} = 0.2 - 9r_{31} \leq r_{33} \leq r_{34}, 0 \leq r_{31} \leq 0.02 \end{array} \right\}$$

2.To determine the $\alpha-$Nash Min-Max solution for player P_4, let we define

$$\begin{aligned} L_4 = & (u_3+10) - (u_4+6) + a_3 u_4^2 - \mu_1(4)(4 - u_1 - u_2) \\ & - \mu_2(4)(6 - 2u_1 - u_2) - \mu_3(4)u_1 - \mu_4(4)u_2 - \mu_5(4) \\ & (u_3+10) - \mu_6(4)(4 - u_3) - \mu_7(4)(u_4+6) - \mu_8(4) \\ & (4 - u_4) - \mu_9(4)(a_1 - 1.2) - \mu_{10}(4)(4.8 - a_1) \\ & - \mu_{11}(4)(a_2 - 3.4) - \mu_{12}(4)(9.8 - a_2) \\ & - \mu_{13}(4)(a_3 - 0.02) - \mu_{14}(4)(0.46 - a_3) \end{aligned}$$

Applying the necessary conditions (9) – (45) that stated in **Theorem 2.1.** we get $\alpha-$Nash Min-Max solution for player P_4 as follows

$$(u_1^*, u_2^*, u_3^*, u_4^*, a_1^*, a_2^*, a_3^*) = (0,0,4,1.09,1.2,3.4,0.46)$$

Keeping the solution as $w_4^* = (0,0,4,1.09,1.2,3.4,0.46)$, then the stability set of the first kind for player P_4 is

$$S_2(w_4^*) = \left\{ (p,q,r) \left| \begin{array}{l} p_{41} \leq p_{42} = 12 - 9p_{41} \leq p_{43} \leq p_{44}, \\ \quad 0 \leq p_{41} \leq 1.2, \\ q_{41} \leq q_{42} = 34 - 9q_{41} \leq q_{43} \leq q_{44}, \\ \quad 0 \leq q_{41} \leq 3.4, \\ r_{41} \leq r_{42} \leq r_{43} = 4.6 - 9r_{41} \leq r_{44}, \\ \quad 0 \leq r_{42} \leq 0.46 \end{array} \right. \right\}$$

To determine the stability set of the second kind for player P_4, let we define the following unique side of constraints by

$$\chi(\kappa) = \left\{ (u_1,u_2,u_3,u_4) \left| \begin{array}{l} 4 - u_1 - u_2 > 0, \; 6 - 2u_1 - u_2 > 0, \\ u_1 = 0, \; u_2 = 0, \; u_3 = 4, \; u_4 = 1.09, \\ a_1 = 1.2, \; a_2 = 3.4, \; a_3 = 0.46 \end{array} \right. \right\}$$

then the stability set of the second kind for player P_4 is given by

$$\Pi(\kappa) = \left\{ \begin{array}{l} \mu_i(l) = 0 \; \forall \; i = 1,...,14 \text{ and } l = 1,2,3 \text{ except} \\ \mu_9(1) = 2.4, \; \mu_4(2) = 6.8, \\ \mu_{11}(2) = 6.8, \; \mu_{13}(3) = 16, \\ p_{41} \leq p_{42} = 12 - 9p_{41} \leq p_{43} \leq p_{44}, 0 \leq p_{41} \leq 1.2, \\ q_{41} \leq q_{42} = 34 - 9q_{41} \leq q_{43} \leq q_{44}, 0 \leq q_{41} \leq 3.4, \\ r_{41} \leq r_{42} = 0.2 - 9r_{41} \leq r_{43} \leq r_{44}, 0 \leq r_{41} \leq 0.46 \end{array} \right\}$$

6 Conclusion and Recommendations

This paper deals with fuzzy hybrid continuous static games with multiple players having fuzzy parameters in both cost functions and constraints playing independently and others playing with the secure concept. All fuzzy parameters are characterized by fuzzy numbers. The solution to such types of games depends on converting the fuzziness problem to a deterministic one. Through the using of $\alpha-$level set, the fuzzy hybrid continuous static games has been converted to a non-fuzzy $\alpha-$HCSG problem. Nash Min-Max solution for solving such type of $\alpha-$HCSG is presented. The stability set of the first and second kind have been determined. Finally, the solution aspect has been illustrated through an illustrative numerical example.

Recommendations for future work include, solving industrial optimization problems that have the same formulation of the proposed fuzzy hybrid game. Furthermore, the development of the procedure to obtain a general structure of Nash Min-Max fuzzy hybrid continuous static game.

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Conflict of interest

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Authors' contributions

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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