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Analysis and Modeling of Fractional Order Model for Hepatitis B at Different Stages

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Abstract: Fractional operator is used to construct the framework of complex hepatitis B by using Caputo and Caputo Fabrizio fractional order derivative. Examination the uniqueness and stability to test the viability of the fractional order model with the proposed numerical plan as well as analyzes qualitatively. Union of different parts behind iterative approach on account of Fabrizio offers a bounded solution that accomplished required outcomes. The fractional system of differential equations which has four parts , susceptible individuals $A(t)$, acute infected $B(t)$, $C(t)$ is chronic hepatitis and $I(t)$ represents individuals who have retrieve after the infection with a life time freedom. At the end, the impact of the framework parameter on the spread of the ailment are begun to analyze using the numerical simulations.

Keywords: Hepatitis B, Fabrizio sense, fractional order, stability analysis

1 Introduction

Mathematics always plays a key role and has major effects with physical sciences for the development of social sciences including lives. The well growing and major fields for the improvement of mathematician subjects are one of biological sciences in different areas. Mathematical analysis and modeling are used to recognize biological system which becomes more qualitative and quantitative [1]. Modeling always manipulate, represents and communicate the real life objects day by day which is much more interesting in humans. Observer may provide the identical item which shows single perspectives of any object i.e. there's no unique observer to approach the reality about any physical phenomena. Every observer gathered different types of facts and provide reliable hypothesis which may become the part of records. This logical procedure is called abduction which is not always infallible, even with admirable systematic unknown. Modeling is the method of manufacturing models model is the depiction of structures and the functioning of some value structure. The model can be reconfigured and it is usually impossible or costly to play with in the structure it represents [2]. In a community, Epidemic part of study kept an important role to understand the effect of infectious disease. We may analyze model building, identify parameter estimation, test mathematical model with sensitivity with different kinds of parameters including numerical simulations in mathematical modeling. Research of this special type always help to understand the disease spread as well as to control their parameters in the world wide community [3-6]. Hepatitis B is the life threatening liver disease are faced by developing countries [7]. A major worldwide health problem is Hepatitis B virus and due to infection of acute and chronic hepatitis B, every year 600000 people die in which many deaths happen due to liver cancer of chronic infection [8].

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Fractional calculus always play a key role to describe and understand the dynamical behavior for the different kind of physical phenomena in recent decades [9]. The fractional order integral as well as derivatives provide an important development to understand the impact of diseases in recent years [10-12]. Precise opinion is provide that the paper to provided that an instinctual understanding for different type of fractional derivatives. The diverse from consistent geometrical figures in its non-integer ascending coefficient [4]. Our main purpose is to create relationship between the geometry features and fractional operators which are related to the physical origins and hereditary belongings . Also observation are made in numerous usual situation and which are the distinguishing feature of fractional operators [13].

Laplace with adomian technique is the beneficial operator in numerous kinds of organical science. Laplace with adomian polynomials helps to understand the behavior of actual phenomina as well as provvide the breakdown of non-linear terms in system of differential equations which is much more powerful than general ADM technique [14-16]. The Caputo and Fabrizio presented a innovative derivative with fractional operator to develops the exponential kernel. The advanced CF fractional operator also has been recycled magnificently in mathematical modeling [18]. Also fractional AdamsBashforth method via the CF derivative was accessible [17].

In this paper, to suggest a fractional order for acute and chorionic hepatitis B. To find the system of fractional differential equations, the Caputo and Caputo Fabrizio fractional derivative operator of order is used. The result for time fractional model has been acquired by using (LADM)and Sumudu transform. The legitimacy of the scheme is accessible by stability analysis. Also the numerical simulations are derived using Matlab to see the detail behavior of the disease.

2 Materials and method

For the fractional order system, we used a mathematical model outlined in [18]. Thus, model is represented in two major ways of fractional order

2.1 Caputo sense

The extension of the fractional order in the form of a nonlinear system can be written as

$$D^{\eta_1}A(t) = b - \rho A(t)C(t) - (\xi_0 + \psi)A(t) \quad (1)$$

$$D^{\eta_2}B(t) = \rho A(t)C(t) - (\xi_1 + \gamma + \sigma_1)B(t) \quad (2)$$

$$D^{\eta_3}C(t) = \gamma B(t) - (\xi_0 + \xi_1 + \sigma_2)C(t) \quad (3)$$

$$D^{\eta_4}I(t) = \sigma_1 B(t) + \sigma_2 C(t) + \psi A(t) - \xi_0 I(t) \quad (4)$$

The fractional system of differential equations which has four parts, susceptible individuals $A(t)$, acute infected is $B(t)$, $C(t)$ is chronic hepatitis and $I(t)$ shows those individuals who have retrieve after the infection with a life time freedom. Here M represents the total population, it can follows as $A+B+C+I = M$.

3 Qualitative analysis

We obtain the disease-free equilibrium point by replacing the left hand equivalent to zero for qualitative research equilibrium points for the $(1-4)$. $E = (\frac{b}{a}, 0, 0, \frac{\psi b}{\xi_0 a})$, where $l = \xi_0 + \psi$ and $E^0 = (A^0,\ B^0,\ C^0,\ D^0)$ is a fractional process endemic equilibrium, where

$$A^0 = \frac{1}{\rho\gamma} ji$$

$$B^0 = \frac{1}{\rho\gamma}(li)[R_0 - 1]$$

$$C^0 = \frac{1}{\rho}(l)[R_0 - 1]$$

$$I^0 = \frac{1}{\xi_0}[(\frac{\sigma_1}{\rho\gamma}(l)(i) + \frac{\sigma_2}{\rho}(l)[R_0 - 1]) + \frac{\psi}{\rho\gamma}(j)(i)]$$

where $j = \xi_0 + \gamma + \sigma_1$ and $i = \xi_0 + \sigma_1 + \sigma_2$

The major condition for the model of fractional order is R_0 and is obtain below

$$R_0 = \frac{\rho\gamma b}{l ji}$$

4 The Laplace-Adomian decomposition method

By busing Laplace transform, we have

$$\mathscr{L}\{D_t^{\eta_1}A(t)\} = b\mathscr{L}\{1\} - \rho\mathscr{L}\{A(t)C(t)\} - (\xi_0+\psi)\mathscr{L}\{A(t)\} \quad (5)$$

$$\mathscr{L}\{D_t^{\eta_2}B(t)\} = \rho\mathscr{L}\{A(t)C(t)\} - (\xi_1+\gamma+\sigma_1)\mathscr{L}\{B(t)\} \quad (6)$$

$$\mathscr{L}\{D_t^{\eta_3}C(t)\} = \gamma\mathscr{L}\{B(t)\} - (\xi_0+\xi_1+\sigma_2)\mathscr{L}\{C(t)\} \quad (7)$$

$$\mathscr{L}\{D_t^{\eta_4}I(t)\} = \sigma_1\mathscr{L}\{B(t)\} + \sigma_2\mathscr{L}\{C(t)\} + \psi\mathscr{L}\{A(t)\} - \xi_0\mathscr{L}\{I(t)\} \quad (8)$$

$$\mathscr{L}\{A\} = \frac{A(0)}{S} + \frac{b}{S^{\eta_1+1}} - \frac{\rho}{S^{\eta_1}}\mathscr{L}\{AC\} - \frac{(\xi_0+\psi)}{S^{\eta_1}}\mathscr{L}\{A\} \quad (9)$$

$$\mathscr{L}\{B\} = \frac{B(0)}{S} + \frac{\rho}{S^{\eta_2}}\mathscr{L}\{AC\} - \frac{(\xi_1+\gamma+\sigma_1)}{S^{\eta_2}}\mathscr{L}\{B\} \quad (10)$$

$$\mathscr{L}\{C\} = \frac{C(0)}{S} + \frac{\gamma}{S^{\eta_3}}\mathscr{L}\{B\} - \frac{(\xi_0+\xi_1+\sigma_2)}{S^{\eta_3}}\mathscr{L}\{C\} \quad (11)$$

$$\mathscr{L}\{I(t)\} = \frac{I(0)}{S} + \frac{\sigma_1}{S^{\eta_4}}\mathscr{L}\{B\} + \frac{\sigma_2}{S^{\eta_4}}\mathscr{L}\{C\} + \frac{\psi}{S^{\eta_4}}\mathscr{L}\{A\} - \frac{\xi_0}{S^{\eta_2}}\mathscr{L}\{I\} \quad (12)$$

including the initial conditions as

$$A(0) = k_1 = 100, B(0) = k_2 = 40, C(0) = k_3 = 20, I(0) = k_4 = 5 \quad (13)$$

Infinite series solutions ia as follows

$$A = \sum_{k=1}^{\infty} A_k, B = \sum_{k=1}^{\infty} (B)_k, C = \sum_{k=1}^{\infty} (C)_k, I = \sum_{k=1}^{\infty} I_k \quad (14)$$

We can write the nonlinearity AC as

$$AC = \sum_{k=1}^{\infty} U_k$$

where U_k is can be written as

$$U_k = \frac{1}{k!}\frac{d^k}{\lambda^k}[\sum_{j=0}^{k}\lambda^j A_j \sum_{j=0}^{k}\lambda^j (C)_j]|_{\lambda=0} \quad (15)$$

$$\mathscr{L}\{A_{k+1}\} = -\frac{\rho}{S^{\eta_1}}\mathscr{L}\{U_k\} - \frac{(\xi_0+\psi)}{S^{\eta_1}}\mathscr{L}\{A_k\}$$

$$\mathscr{L}\{(B)_{k+1}\} = \frac{\rho}{S^{\eta_2}}\mathscr{L}\{U_k\} - \frac{(\xi_1+\gamma+\sigma_1)}{S^{\eta_2}}\mathscr{L}\{(B)_k\}$$

$$\mathscr{L}\{(C)_{k+1}\} = \frac{\gamma}{S^{\eta_3}}\mathscr{L}\{(B)_k\} - \frac{(\xi_0+\xi_1+\sigma_2)}{S^{\eta_3}}\mathscr{L}\{(C)_k\}$$

$$\mathscr{L}\{I_{k+1}\} = \frac{\sigma_1}{S^{\eta_4}}\mathscr{L}\{(B)_k\} + \frac{\sigma_2}{S^{\eta_4}}\mathscr{L}\{(C)_k\} + \frac{\psi}{S^{\eta_4}}\mathscr{L}\{A_k\} - \frac{\xi_0}{S^{\eta_2}}\mathscr{L}\{I_k\} \quad (16)$$

The technique gives analytical solution as.

$$A(t) = 100 - 0.146\frac{t_1^{\eta}}{\eta_1!} + 0.84\frac{t^{2\eta_1}}{2\eta_1!} - 0.009\frac{t^{3\eta_1}}{3\eta_1!} + 0.72\frac{t^{\eta_1+\eta_3}}{(\eta_1+\eta_3)!} + 0.0028\frac{(\eta_1+\eta_3)!}{\eta_1!\eta_3!}t^{2\eta_2+\eta_3} \quad (17)$$

$$B(t) = 40 + 7.52\frac{t^{\eta_2}}{\eta_2!} - 0.4662\frac{t^{2\eta_2}}{(2\eta_2)!} - 2.48\times 10^{-3}\frac{t^{\eta_1+2\eta_2}}{(\eta_1+2\eta_2)!} - 1.46\frac{t^{\eta_1+\eta_2}}{(\eta_1+\eta_2)!} - 7.2\frac{t^{\eta_2+\eta_3}}{(\eta_2+\eta_3)!} - 0.0028\frac{(\eta_1+\eta_3)!}{\eta_1!\eta_3!}t^{2\eta_2+\eta_3}, \quad (18)$$

$$C(t) = 20 - 1.44\frac{t^{\eta_3}}{\eta_3!} + 0.13248\frac{t^{2\eta_3}}{2\eta_3!} + 0.0752\frac{t^{\eta_2+\eta_3}}{(\eta_2+\eta_3)!} + 0.0004\frac{t^{\eta_1+\eta_2+\eta_3}}{(\eta_1+\eta_2+\eta_3)!} \quad (19)$$

$$I(t) = 5 + 5.05\frac{t^{\eta_4}}{\eta_4!} - 0.1515\frac{t^{2\eta_4}}{2\eta_4!} + 0.008\frac{t^{\eta_1+\eta_4}}{(\eta_1+\eta_4)!} + 0.376\frac{t^{\eta_2+\eta_4}}{(\eta_2+\eta_4)!} \quad (20)$$

5 Caputo-Fabrizio sense

Use of Caputo Fabrizio derivative on the proposed model with a fractional order parameter of $\kappa \in (0,1]$.

Then the Sumudu Transform is applied.

$$\begin{aligned}
X(\kappa)\frac{\{A(t)\}-A(0)}{1-\kappa+\kappa\mu} &= S[b-\rho AC-(\xi_0+\psi)A] \\
X(\kappa)\frac{\{B(t)\}-B(0)}{1-\kappa+\kappa\mu} &= S[\rho AC-(\xi_1+\gamma+\psi_1)B] \\
X(\kappa)\frac{\{C(t)\}-C(0)}{1-\kappa+\kappa\mu} &= S[\gamma B-(\xi_0+\xi_1+\sigma_2)C] \\
X(\kappa)\frac{\{I(t)\}-I(0)}{1-\kappa+\kappa\mu} &= S[\sigma_1 B+\sigma_2 C+\psi A-\xi_0 I]
\end{aligned} \tag{21}$$

We get the Inverse Sumudu transformation rearranged and implemented

$$\begin{aligned}
A_{m+1}(t) &= A_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b-\rho AC_m-(\xi_0+\psi)A_m]] \\
B_{m+1}(t) &= B_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\rho AC_m-(\xi_1+\gamma+\psi_1)B_m]] \\
C_{m+1}(t) &= C_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\gamma B_m-(\xi_0+\xi_1+\sigma_2)C_m]] \\
I_{m+1}(t) &= D_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\sigma_1 B_m+\sigma_2 C_m+\psi A_m-\xi_0 I_m]]
\end{aligned} \tag{22}$$

The solution will be given by

$$A(t)=\lim_{m\to\infty}A_m(t), B(t)=\lim_{m\to\infty}B_m(t), C(t)=\lim_{m\to\infty}C_m(t), I(t)=\lim_{m\to\infty}I_m(t)$$

5.1 Stability and fixed point theorem analysis as application

Let us suppose $(U_1,\|.\|)$ as a Banach space and Q as a self-map of U_1. Let $g_{m+1}=y(Q,g_m)$ be particular recursive procedure. Our aim is to prove that Q is picard Q-stable.

Theorem 1. Let $(U_1,\|.\|)$ be a Banach space and Q be a self-map of U_1 satisfying $\|Q_x-Q_g\|\leq C\|x-Q_x\|+c\|x-g\|$ for all x,g$\in N_1$ where $0\leq C$,$0\leq c<1$ .Suppose Q is a Q-stable picard. Suppose the equation below is recursive

$$\begin{aligned}
A_{m+1}(t) &= A_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b-\rho AC_m-(\xi_0+\psi)A_m]] \\
B_{m+1}(t) &= B_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\rho AC_m-(\xi_1+\gamma+\sigma_1)B_m]] \\
C_{m+1}(t) &= C_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\gamma B_m-(\xi_0+\xi_1+\sigma_2)C_m]] \\
I_{m+1}(t) &= I_m(0)+S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\sigma_1 B_m+\sigma_2 C_m+\psi A_m-\xi_0 I_m]]
\end{aligned}$$

where $\frac{1-\kappa+\kappa\mu}{X(\kappa)}$ is the multiplier of the fractional word.

Theorem 2. Suppose self-map Q is defined as

$$Q(A_{m+1}(t)) = A_{m+1}(t) = A_m(0) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b - \rho AC_m - (\xi_0+\psi)A_m]]$$
$$Q(B_{m+1}(t)) = B_{m+1}(t) = B_(m)(0) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\rho AC_m - (\xi_1+\gamma+\sigma_1)B_m]]$$
$$Q(C_{m+1}(t)) = C_{m+1}(t) = C_(m)(0) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\gamma B_m - (\xi_0+\xi_1+\sigma_2)C_m]]$$
$$Q(I_{m+1}(t)) = I_{m+1}(t) = I_m(0) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\sigma_1 B_m + \sigma_2 C_m + \psi A_m - \xi_0 I_m]]$$

is Q-stable in $L^1(a,b)$ if

$$\begin{gathered}\{1-(\xi_0+\psi)x(s) - \rho Vy(s) - \rho Wz(s)\} < 1\\ \{1-(\xi_1+\gamma+\sigma_1)x(s) - \rho Vy(s) + Wz(s)\} < 1\\ \{1+\gamma x(s) - (\xi_0+\xi_1+\sigma_1)y(s)\} < 1\\ \{1+\sigma_1 x(s) + \sigma_2 y(s) + \psi z(s) - \xi_0 P(s)\} < 1\end{gathered} \tag{23}$$

Proof: Here, we evaluate Q has fixed point as follows $\forall(n,m)\varepsilon N\times N$

$$\begin{aligned}Q(A_m(t)) - Q(A_n(t)) &= A_m(t) - A_n(t) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b - \rho A_m C_m - (\xi_0+\psi)A_m]]\\ &\quad - S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b - \rho A_n C_n - (\xi_0+\psi)A_n]]\\ Q(B_m(t)) - Q(B_n(t)) &= B_m(t) - B_n(t) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\rho A_m C_(m) - (\xi_1+\gamma+\sigma_1)B_m]]\\ &\quad - S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\rho A_n C_(n) - (\xi_1+\gamma+\sigma_1)B_n]]\\ Q(C_m(t)) - Q(C_n(t)) &= C_m(t) - C_n(t) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\gamma B_m - (\xi_0+\xi_1+\sigma_2)B_m]]\\ &\quad - S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\gamma B_n - (\xi_0+\xi_1+\sigma_2)B_n]]\\ Q(I_m(t)) - Q(I_n(t)) &= I_m(t) - D_n(t) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\sigma_1 B_m + \sigma_2 C_m + \psi S_m - \xi_0 I_m]]\\ &\quad - S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[\sigma_1 C_n + \sigma_2 C_n + \psi S_n - \xi_0 I_n]]\end{aligned} \tag{24}$$

In eq (38), we take norm on both sides

$$\begin{aligned}||Q(A_m(t)) - Q(A_n(t))|| &= ||A_m(t) - A_n(t) + S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b - \rho A_m C_m - (\xi_0+\psi)A_m]]\\ &\quad - S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b - \rho A_n C_n - (\xi_0+\psi)A_n]]||\end{aligned} \tag{25}$$

by applying triangular inequality, we have

$$\begin{aligned}||Q(A_m(t)) - Q(A_n(t))|| &\le ||A_m(t) - A_n(t)|| + ||S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b - \rho A_m C_m - (\xi_0+\psi)A_m]]\\ &\quad - S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}S[b - \rho A_n C_n - (\xi_0+\psi)A_n]]||\end{aligned} \tag{26}$$

Despite additional analysis

$$\begin{aligned}||Q(A_m(t)) - Q(A_n(t))|| &\le ||A_m(t) - A_n(t)|| + ||S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}|| - (\xi_0+\psi)(A_m - A_n)||\\ &\quad + || - \rho A_m(C_m - C_n)|| + || - \rho C_n(A_m - A_n)||]\end{aligned} \tag{27}$$

$$||A_m - An|| \cong ||C_m - C_n||$$

We have equation (31) to replace

$$||Q(A_m(t)) - Q(A_n(t))|| \le ||A_m(t) - A_n(t)|| + ||S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}|| - (\xi_0+\psi)(A_m - A_n)|| + || - \rho A_m(A_m - A_n)|| + || - \rho C_n(A_m - A_n)||] \quad (28)$$

Since A_n, A_m, C_m are convergent and bounded sequence, so we need to find three different positive constants V, V_1, and M for all t

$$||A_n|| < V, ||A_m|| < V_1, ||C_(m)|| < W \quad (29)$$

Now consider equation (32) and (33) , we get

$$||Q(A_m(t)) - Q(A_n(t))|| \le \{1 - (\xi_0+\psi)x(s) - \rho V y(s) - \rho W z(s)\}||A_m - A_n|| \quad (30)$$

where $S^{-1}[\frac{(1-\kappa+\kappa\mu)}{X(\kappa)}]$ are the function of x, y, and z In the same way, we will obtain

$$||Q(B_m(t)) - Q(B_n(t))|| \le \{1 - (\xi_1+\gamma+v_1)x(s) - \rho V y(s) + W z(s)\}||B_m - B_n|| \quad (31)$$

$$||Q(C_m(t)) - Q(C_n(t))|| \le \{1 + \gamma x(s) - (\xi_0+\xi_1+\sigma_1)y(s)\}||C_m - C_n|| \quad (32)$$

$$||Q(I_m(t)) - Q(I_n(t))|| \le \{1 + \sigma_1 x(s) + \sigma_2 y(s) + \psi z(s) - \xi_0 P(s)\}||I_m - I_n|| \quad (33)$$

where

$$\{1 - (\xi_0+\psi)x(s) - \rho V y(s) - \rho W z(s)\} < 1$$
$$\{1 - (\xi_1+\gamma+\sigma_1)x(s) - \rho V y(s) + W z(s)\} < 1$$
$$\{1 + \gamma x(s) - (\xi_0+\xi_1+\sigma_1)y(s)\} < 1$$
$$\{1 + \sigma_1 x(s) + \sigma_2 y(s) + \psi z(s) - \xi_0 P(s)\} < 1$$

Hence the non linear Q-self mapping has a fixed point and satisfied the required result. Let (33)- (36) hold and therefore using

$$C = (0,0,0,0)$$

$$C = \begin{cases} \{1 - (\xi_0+\psi)x(s) - \rho V y(s) - \rho W z(s)\} \\ \{1 - (\xi_1+\gamma+\sigma_1)x(s) - \rho V y(s) + W z(s)\} \\ \{1 + \gamma x(s) - (\xi_0+\xi_1+\sigma_1)y(s)\} \\ \{1 + \sigma_1 x(s) + \sigma_2 y(s) + \psi z(s) - \xi_0 P(s)\} \end{cases}$$

It shows that the given condition satisfied the Theorem (6.1) and (6.2) for the nonlinear mapping Q, so Q is picard Q-stable.

5.2 *Uniqueness of the special solution*

Here $X = M(a,b) \times (0,T)$ is the Hilbert space and specified as $y : (a,b) \times (0,T) \longrightarrow$ R, such that $\int\int wydwdy < \infty$. The operator will be used as

$$Q(A,B,C,I) = \begin{cases} b - \rho AC - (\xi_0+\psi)A \\ \rho AC - (\xi_1+\gamma+\sigma_1)B \\ \gamma B - (\xi_0+\xi_1+\sigma_2)C \\ \sigma_1 B + \sigma_2 C + \psi A - \xi_0 I \end{cases}$$

The main purpose is to show the inner product of
$Q((U_{11}-U_{12}, U_{21}-U_{22}, U_{31}-U_{32}, U_{41}-U_{42})(z_1,z_2,z_3,z_4))$

$$\begin{aligned} Q((U_{11}-U_{12}, U_{21}-U_{22}, U_{31}-U_{32}, U_{41}-U_{42})(z_1,z_2,z_3,z_4)) = \\ (-\rho(U_{11}-U_{12})(U_{31}-U_{32}) - (\xi_0+v)(U_{11}-U_{12}), z_1) \\ (\rho(U_{11}-U_{12})(U_{31}-U_{32}) - (\xi_1+\gamma+\sigma_1)(U_{21}-U_{22}), z_2) \\ (\gamma(U_{21}-U_{22}) - (\xi_0+\xi_1+\sigma_2)(U_{31}-U_{32}), z_3) \\ (\sigma_1(U_{21}-U_{22}) + \sigma_2(U_{31}-U_{32}) + \psi(U_{11}-U_{12}) - \xi_0(U_{41}-U_{42}), z_4) \end{aligned} \quad (34)$$

Assessing the first condition in the system without loss of all generality

$$(-\rho(U_{11}-U_{12})(U_{31}-U_{32})-(\xi_0+\psi)(U_{11}-U_{12}),z_1)$$
$$(-\rho(U_{11}-U_{12})(U_{31}-U_{32}),z_1)+((-(\xi_0+\psi)(U_{11}-U_{12}),z_1)) \quad (35)$$

we can suppose that

$$(U_{11}-U_{12})\cong(U_{21}-U_{22})\cong(U_{31}-U_{32})\cong(U_{41}-U_{42}) \quad (36)$$

Equation (40) will be

$$(-\rho(U_{11}-U_{12})^2-(\xi_0+\psi)(U_{11}-U_{12}),z_1)$$

We obtain on the basis of the relation between the norm and the inner product as obtain,

$$\begin{aligned}&(-\rho(U_{11}-U_{12})^2-(\xi_0+\psi)(U_{11}-U_{12}),z_1)\\ &\cong(-\rho(U_{11}-U_{12})^2,z_1)+(-(\xi_0+\psi)(U_{11}-U_{12}),z_1)\\ &\leq\rho||(U_{11}-U_{12})^2||||z_1||+(\xi_0+\psi)||(U_{11}-U_{12})||||z_1||\\ &=(\rho\overline{z_1}+\xi_0+\psi)||U_{11}-U_{12}||||z_1||\end{aligned} \quad (37)$$

Repeating a similar way, from system formula 2nd 3rd and 4th, we obtain

$$\begin{aligned}&(\rho(U_{11}-U_{12})(U_{31}-U_{32})-(\xi_1+\gamma+\sigma_1)(U_{21}-U_{22}),z_2)\\ &\leq(\rho\overline{z_2}+\xi_1+\gamma+\sigma_1)||U_{21}-U_{22}||,||z_2||\\ &.(\gamma(U_{21}-U_{22})-(\xi_0+\xi_1+\sigma_2)(U_{31}-U_{32}),z_3)\\ &\leq(\gamma+\xi_0+\xi_1+\sigma_2)||(U_{31}-U_{32})||||z_3||\\ &.(\xi_1(U_{21}-U_{22})+\xi_2(U_{31}-U_{32})+\xi(U_{11}-U_{12})-\xi_0(U_{41}-U_{42}),z_4)\\ &\leq(\sigma_1+\sigma_2+\psi-\xi_0)||U_{41}-U_{42}||||z_4||\end{aligned} \quad (38)$$

Substitute equation (41) and (42) in (38), we get

$$P\Big((U_{11}-U_{12},U_{21}-U_{22},U_{31}-U_{32},U_{41}-U_{42}),(Z_1,Z_2,Z_3,Z_4)\Big)$$
$$\leq\begin{cases}(\rho\overline{w_1}+\xi_0+\psi)||U_{11}-U_{12}||||z_1||,\\ (\rho\overline{z_2}+\xi_1+\gamma+\sigma_1)||U_{21}-U_{22}||||z_2||,\\ (\gamma+\xi_0+\xi_1+\sigma_2)||U_{31}-U_{32}||||z_3||,\\ (\sigma_1+\sigma_2+\psi+\xi_0)||U_{41}-U_{42}||||z_4||,\end{cases}$$

But both solutions converge for a sufficiently large value of Vi with $i=1,2,3,4$, Using $\exists$ topology for very small + ve parameters kv_1,kv_2,kv_3,kv_4, such that

$$||A-U_{11}||,||A-U_{12}||<\frac{kv_1}{3(\rho\overline{z_1}+\xi_0+\psi)||U_{11}-U_{12}||||z_1||}$$
$$||B-U_{21}||,||B-U_{22}||<\frac{kv_2}{3(\rho\overline{z_2}+\xi_1+\gamma+\sigma_1)||U_{21}-U_{22}||||z_2||}$$
$$||C-U_{31}||,||C-U_{32}||<\frac{kv_3}{3(\gamma+\xi_0+\xi_1+\sigma_2)||U_{31}-U_{32}||||z_3||}$$
$$||D-U_{41}||,||D-U_{42}||<\frac{kv_4}{3(\sigma_1+\sigma_2+\psi+\xi_0)||U_{41}-U_{42}||||z_4||}$$

By using equation (43) on the right side of the triangular inequality

$$V=max(v_1,v_2,v_3,v_4),K=max(kv_1,kv_2,kv_3,kv_4)$$

$$(\rho\overline{z_1}+\xi_0+\psi)||U_{11}-U_{12}||||z_1||$$
$$(\rho\overline{z_2}+\xi_1+\gamma+\sigma_1)||U_{21}-U_{22}||||z_2||$$
$$(\gamma+\xi_0+\xi_1+\sigma_2)||U_{31}-U_{32}||||z_3||$$
$$(\sigma_1+\sigma_2+\psi+\xi_0)||U_{41}-U_{42}||||z_4||$$

$$\begin{cases}(\rho\overline{z_1}+\xi_0+\psi)||U_{11}-U_{12}||||z_1||\\(\rho\overline{z_2}+\xi_1+\gamma+\sigma_1)||U_{21}-U_{22}||||z_2||\\(\gamma+\xi_0+\xi_1+\sigma_2)||U_{31}-U_{32}||||z_3||\\(\sigma_1+\sigma_2+\psi+\xi_0)||U_{41}-U_{42}||||z_4||\end{cases} < \begin{cases}k,\\k,\\k,\\k.\end{cases}$$

As k is an extremely small positive parameter,therefore based on topological idea,we have

$$\begin{cases}(\rho\overline{z_1}+\xi_0+\psi)||U_{11}-U_{12}||||z_1||\\(\rho\overline{z_2}+\xi_1+\gamma+\sigma_1)||U_{21}-U_{22}||||z_2||\\(\gamma+\xi_0+\xi_1+\sigma_2)||U_{31}-U_{32}||||z_3||\\(\sigma_1+\sigma_2+\psi+\xi_0)||U_{41}-U_{42}||||z_4||\end{cases} < \begin{cases}0,\\0,\\0,\\0.\end{cases}$$

but it is obvious that

$$(\rho\overline{z_1}+\xi_0+\psi))\neq 0,(\rho\overline{z_2}+\xi_1+\gamma+\sigma_1)\neq 0$$
$$(\gamma+\xi_0+\xi_1+\sigma_2)\neq 0,(\sigma_1+\sigma_2+\psi+\xi_0)\neq 0$$

So that, we have
$||U_{11}-U_{12}||=0,||U_{21}-U_{22}||=0$ and $||U_{31}-U_{32}||=0,||U_{41}-U_{42}||=0$
which gives result that
$U_{11}=U_{12},U_{21}=U_{22}$ and $U_{31}=U_{32},U_{41}=U_{42}$. This proof the results.

6 Results and discussion

We investigate the fractional order model's qualitatively and furthermore check the mathematical model's non-negative arrangement. Susceptible, acute, chronic and recovered populace are delineated in Figures 1-4 for various estimations of η. In figures 5-8, Comparison of Caputo and Caputo Fabrizio has been discussed. We observed that Caputo Fabrizio fractional derivative gives more reliable result than Caputo. Another exceptional angle is to utilized generally low initial values, for a small time frame. If The initial values to the information are thought to be high over a huge time frame with the goal that populace probably won't be negative. Numerical impacts can be assessed at different fractional values utilizing tables and graphs.

Table 1: $A(t)$ at η_i

t	$\eta_i=1.01$	$\eta_i=0.951$	$\eta_i=0.903$	$\eta_i=0.849$
0	100.001	99.99	100.2	100.11
0.19	97.1112	99.8105	86.48523	96.2344
0.411	94.288	93.9107	93.5244	93.1283
0.621	91.5211	91.1527	90.7864	90.4233
0.811	88.82071	88.50671	88.2062	87.9211
0.99	86.1826	85.95683	85.75352	85.5741

Table 2: $B(t)$ at η_i

t	$\eta_i=1.01$	$\eta_i=0.951$	$\eta_i=0.903$	$\eta_i=0.849$
39.99	39.99	40.11	39.99	40.21
0.21	41.32153	41.42882	41.53641	41.6317
0.39	42.27781	42.33782	42.38121	42.40484
0.62	42.86910	42.83121	42.76593	42.67092
0.83	42.09523	42.93891	42.75152	42.53273
1.01	42.95612	42.67821	42.37191	42.03933

Table 3: $C(t)$ at η_i

t	$\eta_i = 1.01$	$\eta_i = 0.951$	$\eta_i = 0.903$	$\eta_i = 0.849$
19.99	19.988	20.03	19.98	20.03
0.19	19.71622	19.68682	19.65511	19.6212
0.41	19.44066	19.40459	19.36745	19.32955
0.61	19.17354	19.13873	19.10422	19.07029
0.801	18.914863	18.88602	18.85851	18.83291
1.001	18.66451	18.64492	18.62762	18.61282

Table 4: $I(t)$ at η_i

t	$\eta_i = 1.01$	$\eta_i = 0.951$	$\eta_i = 0.903$	$\eta_i = 0.849$
0	4.99	5.001	5.003	5.004
0.21	6.00711	6.11349	6.22901	6.35419
0.39	7.00973	7.14624	7.28812	7.43532
0.55	8.00939	8.14946	8.29088	8.43288
0.79	9.00763	9.13512	9.25994	9.38154
0.99	10.00611	10.10919	10.20648	10.29733

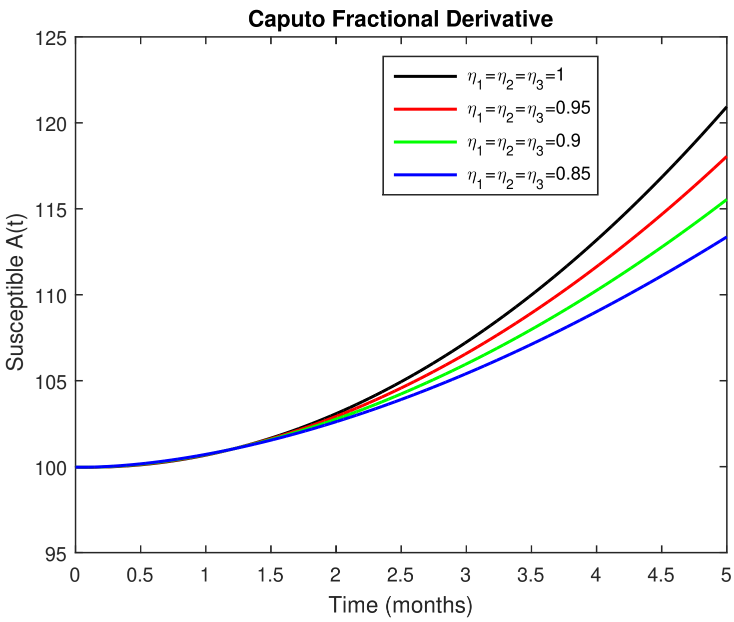

Figure 1. Dynamical behavior of susceptible $A(t)$ graphically at η_i where $i = 1,2,3,4$.

7 Conclusion

Hepatitis B is one of the network's significant disease influencing healthy living. It ought to likewise be noticed in the event that we don't utilize the powerful parameters to control the diseases, later on it will likewise influence human life.

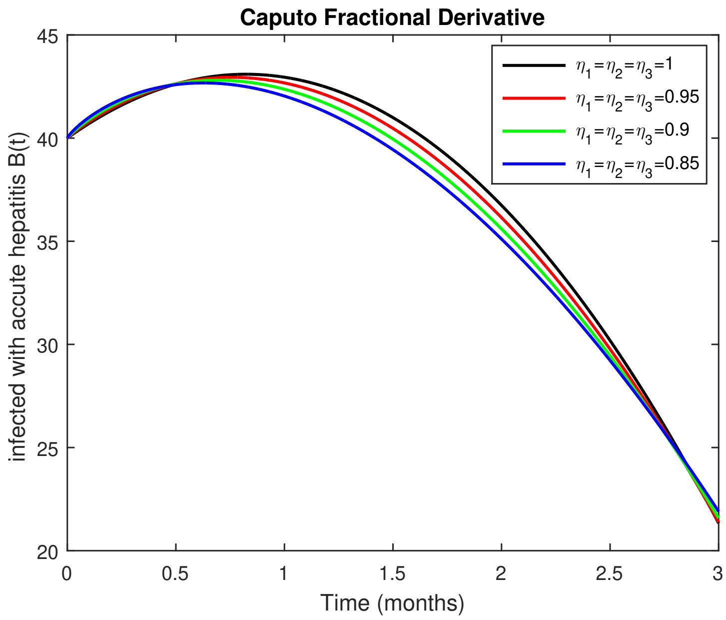

Figure 2.Dynamical behavior of exposed $B(t)$ graphically at η_i where $i = 1,2,3,4$.

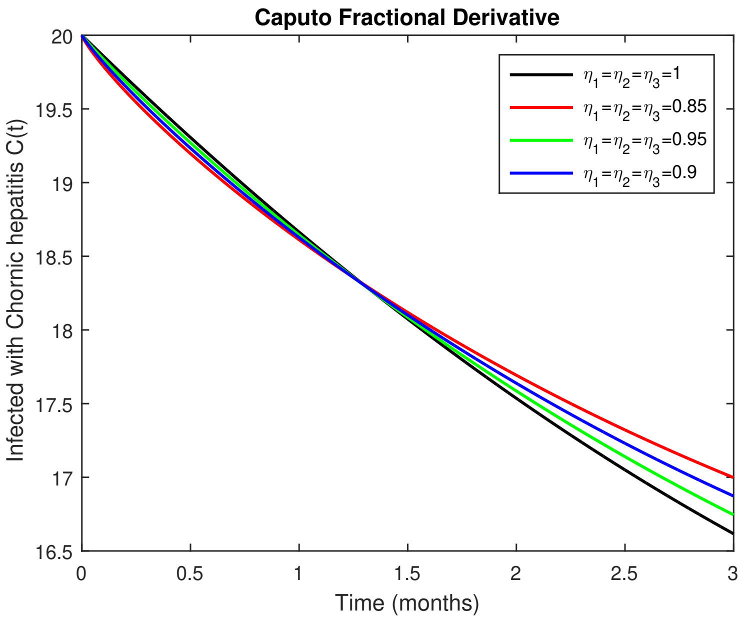

Figure 3.Dynamical behavior of infected $C(t)$ graphically at η_i where $i = 1,2,3,4$.

Numerical procedure and assessment are a progressively viable method for assessing, watching, resilience and figuring the effect of parameters in network control of the disease. We built up the scheme fractional order numerical model for Hepatitis B therefore in this paper using the sense of Caputo and the sense of Caputo Fabrizio. In this paper, we locate the model of hepatitis B shows the study of disease transmission A,B,C,I which portrays the populace for question. The technique for Laplace-Adomic Decomposition and Sumudu Transform is utilized to accomplish the series solution of the epidemic system of fractional A,B,C,D hepatitis B. L-ADM and Sumudu Transfom give progressively dependable outcomes, and the solution converges to diseases free equilibrium point since $R_0 < 1$. In order to obtain reliable results,

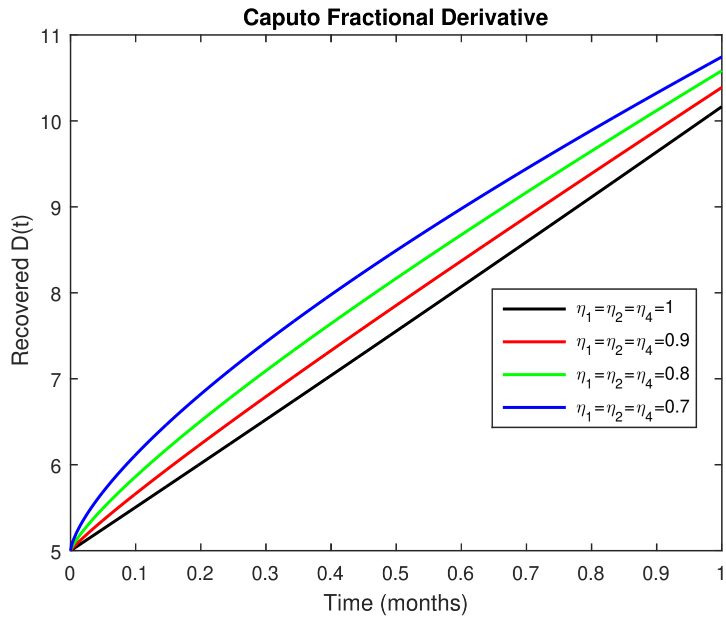

Figure 4.Dynamical behavior of recovered $I(t)$ graphically at η_i where $i = 1,2,3,4$.

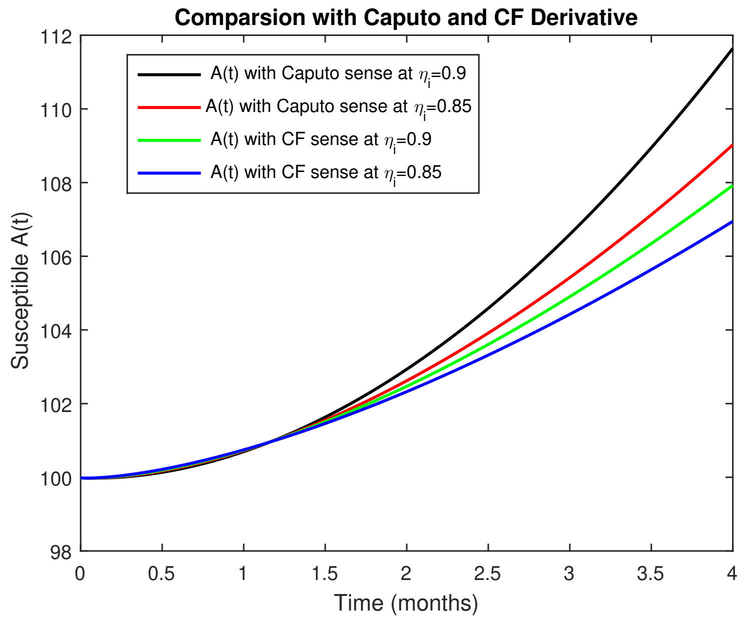

Figure 5.Comparison of $A(t)$ with Caputo and CF derivative graphically at η_i where $i = 1,2,3,4$.

the compression is rendered for different estimates of η, additional numerical simulation was performed to check the effect of the fractional parameter obtained on our solutions. The impact of the fractional parameter can be evaluated with tables and graphs. It is well-meaning to note that fractional derivatives show significant shifts and memory problems compared to ordinary derivatives. It should also be noted here that the fractional parameter for complex nonlinear differential conditions using fractional derivatives from Caputo and Caputo Fabrizio gives the more satisfactory results linked to the ideal model values but Caputo Fabrizio is to be preferred for reliable results.

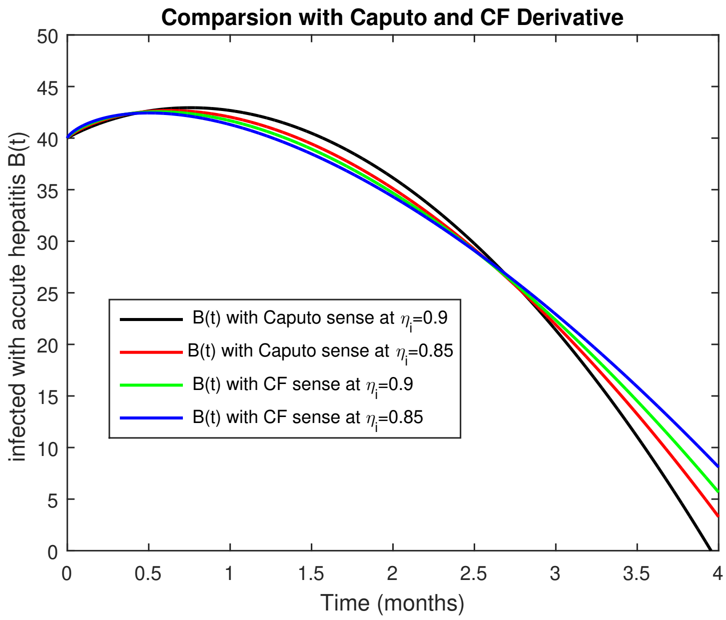

Figure 6.Comparison of $B(t)$ with Caputo and CF derivative graphically at η_i where $i = 1,2,3,4$.

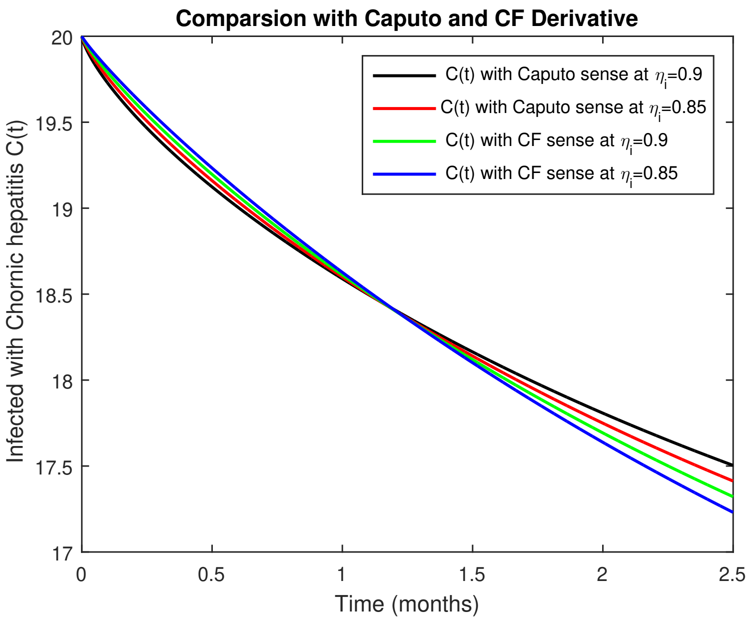

Figure 7. Comparison of $C(t)$ with Caputo anf CF derivative graphically at η_i where $i = 1,2,3,4$.

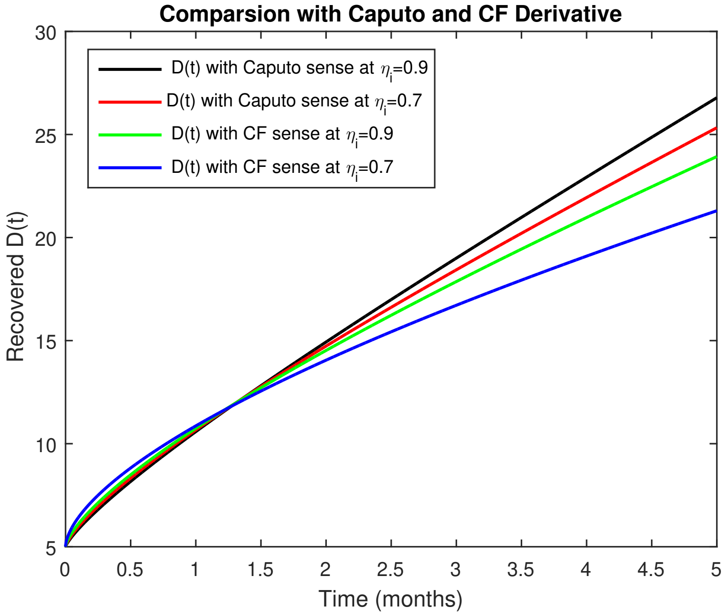

Figure 8. Comparison of $I(t)$ with Caputo and CF derivative graphically at η_i where $i = 1,2,3,4$.

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