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New Basis for Solving Fractional Order Models on Semi-Infinite Intervals

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Abstract: In this note, we apply a numerical method to solve viscoelastic models involving fractional derivatives. Our method generalizes rational Legendre collocation scheme. It uses new functions named “fractional rational Legendre functions” as the basis. The new basis convergence more rapidly than rational Legendre functions in solving fractional differential equations. Numerical results show the efficiency and performance of the new basis in comparison with rational Legendre functions.

Keywords: Fractional calculus; Rational approximation; Fractional order viscoelasticity; Fractional rational Legendre

1 Introduction

The concept of differentiation and integration to noninteger order is not certainly new. It is as old as conventional calculus, but has not been very popular in science and engineering for years. It has gained its considerable popularity and importance in science and engineering, over the last decades. Nowadays, utilization of fractional calculus for modeling natural phenomena has grown and it is finding increasing use in many areas of science and engineering from the nano to the macro scale. Fractional calculus models, represent a relatively simple way to describe dynamics in complex systems. It has been widely applied in different science fields such as dynamical systems theory, controller tuning, legged robots, redundant robots, heat diffusion, digital circuit synthesis, viscoelastic materials, electricity, mechanics, chaos and fractals [1,2].

By growing the applications of the fractional calculus, it is required to extend methods for solving fractional differential equations. Since fractional differentiation and integration is a non-local property, obtaining solutions of such equations comes with a high cost. In most problems, it is impossible to have exact solutions, so approximate techniques play an important role in finding the solution of these equations. Although numerous schemes are introduced to solve fractional differential equations, none of them is a reliable method for solving all fractional differential equations. In other words, our information about the structure of the problem is the key to choosing the method. In this paper, we have introduced a new basis to solve fractional differential equations which occur on real positive line. The basis is named fractional rational Legendre functions (FRLFs). These functions are constructed from rational Legendre functions (RLFs) and improve the rate of convergence. To show the efficiency of this new basis, collocation scheme is employed to solve fractional viscoelasticity model.

While the first section includes some basic definitions and necessary formulas of fractional calculus, fractional viscoelastic model is considered in Section 2 and its governing equations are given there. In the beginning of Section 3, the formulation of RLFs, FRLFs together with some basic properties of them are presented. After that, the implementation of collocation method based on RLFs and FRLFs is presented and a brief argument about convergence is discussed. In Section 4 the results of applying the proposed methods for solving fractional viscoelastic equation are summarized. Eventually, the summary and conclusions are outlined in Section 5.

1.1 Preliminary definitions

In this section some notations of the fractional calculus that will be used throughout the paper, are presented. The base function in fractional calculus is the Euler’s Gamma

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function, which generalizes the factorial function, and allows to take non-integer values.

Definition 1(Euler's Gamma function). *The Gamma function for all complex z is defined as [1,2,3,4]:*

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} \mathrm{d}t, \tag{1}$$

which converges in the right half of the complex plane, $Re(z) > 0$.

Mittag-Leffler is another important function in fractional calculus. In fractional calculus, this function plays a similar role of the exponential function in integer order calculus. In other words, is an alternative for the exponential function in fractional calculus.

Definition 2(Mittag-Leffler function). *For* $\alpha, \beta > 0$, *the two parameter Mittag-Leffler function is defined as [1,2,3]:*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}. \tag{2}$$

Taking $\beta = 1$ *gives the one parameter Mittag-Leffler function,* $E_\alpha(z)$.

There are several definitions of fractional derivative. In this paper, because of its benefits for initial value problems, the Caputo's definition of fractional derivative is used. The fractional derivative in the Caputo sense is defined based on the Riemann-Liouville fractional integral.

Definition 3(Riemann–Liouville fractional integral). *Let* $\alpha \in \mathbb{R}^+$. *The Riemann–Liouville fractional integral of order* α *for* $0 < t < a$ *is defined as [1,3,4]:*

$$J_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau. \tag{3}$$

Definition 4(Fractional derivative in the Caputo sense). *Let* $\alpha \in \mathbb{R}^+$. *The Caputo's fractional derivative of order* α *for* $t > 0$ *is defined as [1]:*

$$\begin{aligned} {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha f(t) &= J_0^{\lceil\alpha\rceil-\alpha} \mathrm{D}^{\lceil\alpha\rceil} f(t) \\ &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_0^t (t-\tau)^{\lceil\alpha\rceil-\alpha-1} \mathrm{D}^{\lceil\alpha\rceil} f(\tau) \mathrm{d}\tau, \end{aligned} \tag{4}$$

where $\mathrm{D}^{\lceil\alpha\rceil}$ *is the ordinary differential operator.*

2 Fractional viscoelastic model

Viscoelasticity describes a material behavior that is time-dependent, or retains memory of the material history. This is in contrast to an elastic material, which is time invariant. A viscoelastic material exhibit both elastic (often Hookean) and viscous (often Newtonian) behavior. The mechanics of these viscoelastic materials can be modeled as arrangements of lossless elastic springs and lossy viscous dampers (dashpots). In fact, many common models for viscoelasticity consider springs and dashpots in various configurations. The viscoelastic properties of a medium can be determined from multiple methods, such as stress–relaxation, creep, and dynamic testing. The stress–relaxation experiment is particularly useful because it contains a wide spectrum of storage and loss properties. For a distributed material the stress due to the elastic element is proportional to the strain and the stress due to the viscous element is proportional to the time rate of change of strain [5,6,7,8].

Polymers in general show a weak frequency dependence of their viscoelastic characteristics. This frequency dependence is difficult to describe with classical viscoelasticity. The classical models for polymer materials are based on constitutive equations with differential operators of integer order. A large number of derivative operators (or internal variables), resulting in many parameters, is required to obtain a reasonably accurate description of the observed viscoelastic characteristics.

Fractional order operators are very useful for modeling viscoelastic materials. The fractional derivative model significantly reduces the number of elements, and terms, needed to robustly model viscoelasticity. The model replaces the dashpot of the Maxwell element with a fractional spring-pot. The spring-pot interpolates between spring and dashpot behavior, giving the fractional model more flexibility. Mathematically, this is described as:

$$\sigma = \eta \frac{\mathrm{d}^\alpha \varepsilon}{\mathrm{d}t^\alpha} \tag{5}$$

where α is a rational number between 0 and 1, and η is a material parameter similar to a damping coefficient. The nature of the spring-pot element is clear; if $\alpha = 0$ then the element becomes a spring. If $\alpha = 1$ then the element becomes a dashpot. For any α between 0 and 1, the element has both spring and dashpot behavior [8].

An advantage of using fractional order operators in viscoelasticity is that a whole spectrum of viscoelastic mechanisms can be included in a single internal variable. The stress–relaxation spectrum for the fractional order model is continuous with the relaxation constant as the most probable relaxation time, while the order of the operator plays the role of a distribution parameter. Note that the spectrum is discrete for the classical model that is based on integer order derivatives. By a suitable choice of material parameters for the classical viscoelastic model it is observed both numerically and analytically that the classical model with a large number of internal variables (each representing a specific viscoelastic mechanism) converges to the fractional model with a single internal variable.

It has been proven that fractional viscoelastic models are thermodynamically consistent. Ever since fractional order models have been used to describe relaxation and creep behavior as well as damping properties. The fractional order viscoelastic model has successfully been fitted to experimental data over a broad frequency range for several polymers using only four parameters in the uniaxial. The fractional order viscoelastic model has also been successfully fitted to time domain rubber data at small strains [5,6,7].

By using the concept of internal variables the simplest uniaxial fractional viscoelastic model that can reproduce instantaneous and long time elastic responses, is formulated as follows [5]

$$\sigma(t) = E_1(\varepsilon(t) - \epsilon(t)) + E_2\varepsilon(t), \tag{6}$$

$$\epsilon^{(\alpha)}(t) + \frac{1}{\tau^\alpha}\epsilon(t) = \frac{1}{\tau^\alpha}\varepsilon(t), \qquad 0 < \alpha < 1, \tag{7}$$

where σ and ε represent the stress and (macroscopic) strain, respectively. ϵ is an internal variable of strain type representing a distribution of irreversible microstructural processes in the material. $E_1 > 0$ and $E_2 > 0$ are elastic stiffness and $\tau > 0$ is the relaxation constant. α is the order of fractional derivative, which is in the Caputo sense. Equation (7) comes with the following initial condition

$$\epsilon(0) = 0, \tag{8}$$

which means that the model predicts an initial response following Hooke's elastic law

$$\sigma(0) = (E_1 + E_2)\,\varepsilon(0) = E_{(0)}\varepsilon(0), \tag{9}$$

where $E_{(0)} = E_1 + E_2$ is the instantaneous stiffness of the model.

3 Rational approach

Rational Chebyshev functions were introduced by Boyd [9] to solve problems on positive real line. This basis allows free use of the Fast Fourier Transform. Later, Guo et al. [10] developed Boyd's idea and introduced rational Legendre functions. The rational Chebyshev and rational Legendre functions have been successfully used in a wide range of applications [9,10,11,12,13,14,15,16,17,18].

In this section, at first, RLFs are introduced and some basic properties of them are explained. Then, we generalize RLFs and construct FRLFs which have more rapid convergence for solving fractional differential equations. At last, implementation of the collocation scheme and some points about convergence are given.

3.1 Rational Legendre functions

RLFs are obtained from Legendre polynomials, using the algebraic function $\frac{t-L}{t+L}$. The RLF of order n, which is denoted by $R_n(t)$, is the nth eigenfunction of the following singular Sturm-Liouville problem

$$\frac{1}{w(t)}\left(tR_n'(t)\right)' + n(n+1)R_n(t) = 0, \qquad t \geq 0,$$
$$n = 0,1,2,\ldots, \tag{10}$$

where $w(t)$ is the weight function as follows

$$w(t) = \frac{L}{(t+L)^2}. \tag{11}$$

L is a user-selective constant which is called "map parameter" and sets the length scale of the mapping. This parameter should be optimized by trial-and-error. The accuracy is usually quite insensitive to L so long as it is of the same order of magnitude as the optimum value. Strategies for optimizing L are given in [9,11].

RLFs are orthogonal on the semi-infinite interval with respect to $w(t)$ and satisfy the following orthogonality property

$$\int_0^\infty R_n(t)R_m(t)w(t)\mathrm{d}t = \begin{cases} 0 & n \neq m, \\ \frac{1}{2n+1} & n = m. \end{cases} \tag{12}$$

These functions can be obtained using the following recursive relation for $n \geq 1$

$$R_{n+1}(t) = \frac{2n+1}{n+1}R_1(t)R_n(t) - \frac{n}{n+1}R_{n-1}(t), \tag{13}$$

and initial functions

$$R_0(t) = 1, \qquad R_1(t) = \frac{t-L}{t+L}. \tag{14}$$

Using the previous relations, we have the following explicit representation for RLFs:

$$R_n(t) = \sum_{i=0}^{n} \frac{(-1)^i(n+i)!}{(i!)^2(n-i)!}\left(\frac{L}{t+L}\right)^i. \tag{15}$$

3.2 Fractional rational Legendre functions

RLFs are suitable basis for solving ordinary differential equations on semi-infinite intervals, but their convergence rate for solving fractional differential equations is slow. In present paper, we have introduced FRLFs to subdue this problem. While these new functions have all of the good features of RLFs, they converge more rapidly than RLFs in solving fractional differential equations.

Similar to RLFs, FRLFs, which we denote by $R_{n,\alpha}(t)$, are orthogonal functions on semi-infinite interval with respect to the weight function

$$w_\alpha(t) = \frac{\alpha}{L}\left(\frac{L}{t+L}\right)^{\alpha+1}. \tag{16}$$

Their orthogonality property doesn't differ from RLFs; i.e.

$$\int_0^{\infty} R_{n,\alpha}(t)R_{m,\alpha}(t)w_\alpha(t)\mathrm{d}t = \begin{cases} 0 & n \neq m, \\ \frac{1}{2n+1} & n = m. \end{cases} \quad (17)$$

Also, FRLFs are the eigenfunctions of the following singular Sturm-Liouville problem

$$\frac{1}{w_\alpha(t)}\left(\frac{1}{w_\alpha(t)}\left(\left(\frac{L}{t+L}\right)^{\alpha} - \left(\frac{L}{t+L}\right)^{2\alpha}\right)R'_{n,\alpha}(t)\right)' + n(n+1)R_{n,\alpha}(t) = 0, \quad t \geq 0, \quad n = 0,1,2,\ldots. \quad (18)$$

FRLFs satisfy the following recursive relation for $n \geq 1$

$$R_{n+1,\alpha}(t) = \frac{2n+1}{n+1}R_{1,\alpha}R_{n,\alpha}(t) - \frac{n}{n+1}R_{n-1,\alpha}(t), \quad (19)$$

and initial relations

$$R_{0,\alpha}(t) = 1, \qquad R_{1,\alpha}(t) = 1 - 2\left(\frac{L}{t+L}\right)^{\alpha}, \quad (20)$$

which gives an explicit form for FRLFs

$$R_{n,\alpha}(t) = \sum_{i=0}^{n} \frac{(-1)^i(n+i)!}{(i!)^2(n-i)!}\left(\frac{L}{t+L}\right)^{\alpha i}. \quad (21)$$

3.3 Function approximation

Here, we present collocation scheme for solving differential equations using RLFs and FRLFs. The first step for solving differential equations, using collocation method, is to choose appropriate trial functions. Since we are going to solve fractional differential equations on semi-infinite interval, we use FRLFs as the basis, although we employ RLFs, for considering the rate of convergence.

By choosing FRLFs as the basis, the solution of the differential equation is approximated as

$$u_N(\alpha,t) = \sum_{n=0}^{N} c_n R_{n,\alpha}(t) + c_{N+1}\frac{L^{\lceil\alpha\rceil+1}}{\Gamma(\alpha+1)}\frac{t^{\alpha}}{(t+L)^{\lceil\alpha\rceil+1}}, \quad (22)$$

where c_ns are unknown coefficients that should be found. The last term in the right hand side is included, since for all smooth functions, the fractional derivative in the Caputo sense, at origin, is zero. So, we add this term to surmount difficulties.

In the next step, the unknown function in the differential equation and its initial condition are replaced by the approximating function, $u_N(\alpha,t)$, and the residual function is constructed. Since the residual function is identically equal to zero for the exact solution, the challenge is to choose the series coefficients, c_n, so that the residual function is minimized. In the collocation approach it is required that the residual function be satisfied exactly at a set of collocation points, t_i:

$$res(t_i) = 0. \quad (23)$$

These equations together with the initial conditions form a system of algebraic equations which should be solved to give the expansion coefficients c_ns.

3.4 Convergence

The order of the convergence of spectral methods is based on the behavior of the series coefficients, c_n. These quantities, have the property of decaying to zero with increasing n at the same qualitative rate, usually exponentially. The smoother the function, the more rapidly its spectral coefficients converge. It is shown that representing the coefficients on a log-linear or log-log graph gives the order of convergence. More details are given in [11].

Definition 5(Algebraic index of convergence). *The algebraic index of convergence, k, is the largest number for which*

$$\lim_{n\to\infty} |c_n| n^k < \infty \quad (24)$$

where the c_n are the coefficients of the series. In other words, k is the algebraic index of convergence, if for $n >> 1$

$$c_n \sim \mathrm{O}\left(n^{-k}\right). \quad (25)$$

4 Numerical Results

In this section, collocation scheme based on RLFs and FRLFs is used to solve fractional viscoelasticity model introduced by equationas (6), (7) and (8). The obtained solutions for both basis are compared and the algebraic indexes of convergence for them are computed.

Using appropriate replacements, Equation (7) is converted to the following non-dimensional equation [5]

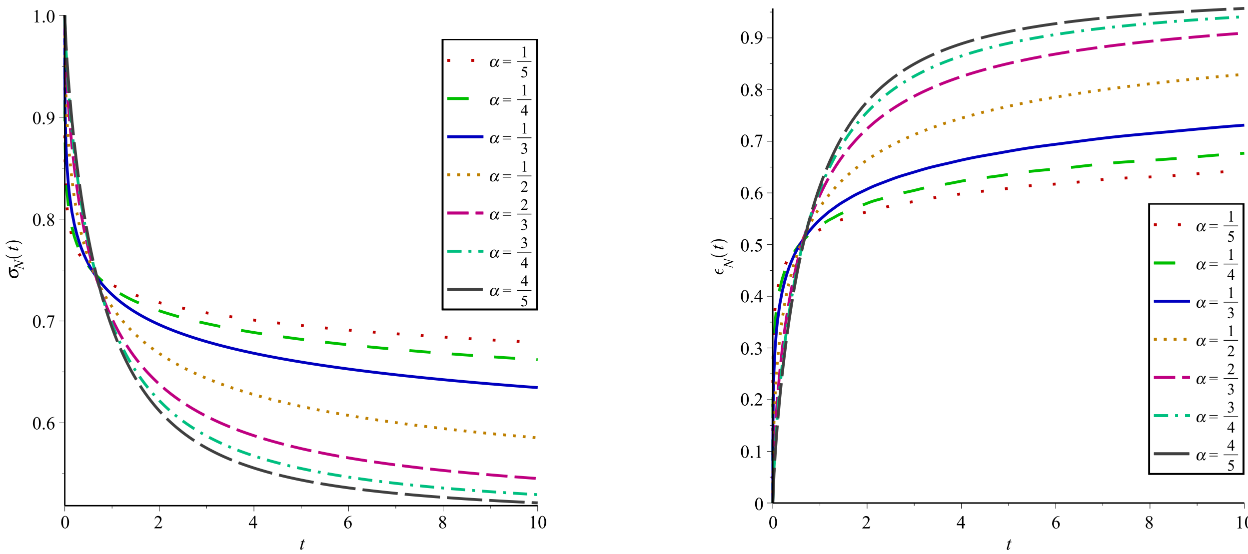
$$\epsilon^{(\alpha)}(t) + \epsilon(t) = \varepsilon(t). \quad (26)$$

By choosing the stress response due to the following step strain

$$\varepsilon(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases} \quad (27)$$

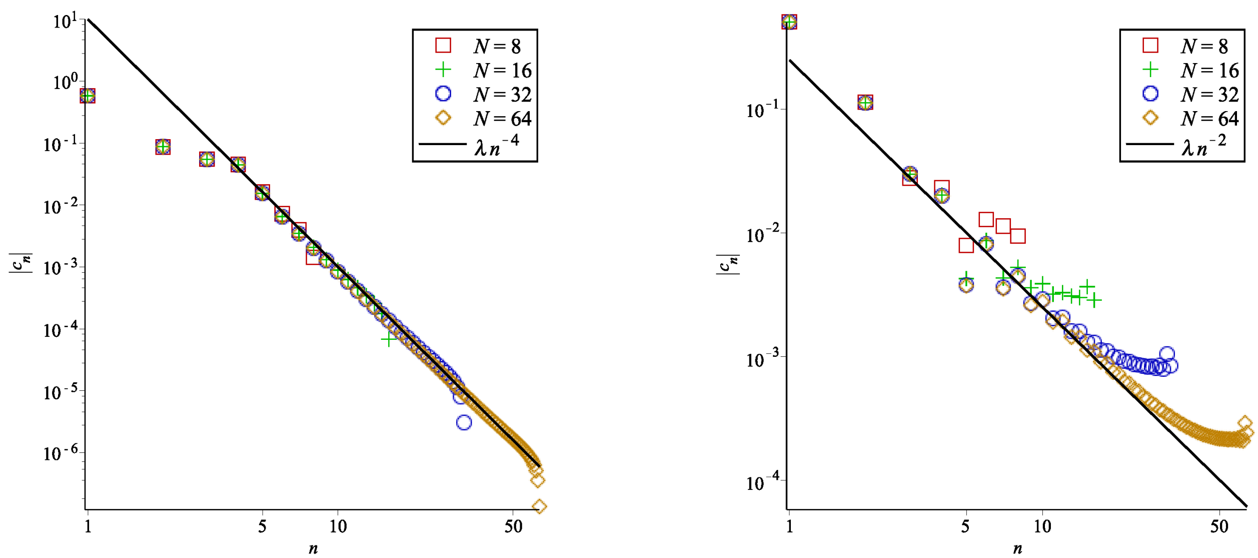
the exact solution of the model is as follows

$$\sigma(t) = E_1 E_\alpha\left(-t^{\alpha}\right) + E_2. \quad (28)$$



(a) Stress **(b)** Strain

Fig. 1: The stress and strain functions of the model for differnt values of α.



(a) FRLF **(b)** RLF

Fig. 2: Spectral coefficients of the solution of the fractional viscoelasticity model using FRLFs and RLFs for $\alpha = \frac{1}{2}$ and different values of N.

To obtain the approximate solution using collocation scheme, the residual function is constructed by replacing $\epsilon(t)$ in Equation (26) with $\epsilon_N(t) = u_N(\alpha,t)$ defined in (22). So, we have

$$res(t) = {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}\,\epsilon_N(t) + \epsilon_N(t) - \varepsilon(t). \qquad (29)$$

In the collocation technique, once the basis set has been chosen, the optimal sets for nodal points are the Gaussian points for each basis [11]. Since the equation is defined on real positive line, fractional rational Legendre-Gauss-Lobatto nodes are used as collocation points. These nodes are the roots of

$R_{N+1,\alpha}(t) - R_{N-1,\alpha}(t)$. By evaluating the residual function at these nodes, we have the following system of algebraic equations

$$\begin{aligned} &res(t_i) = 0, \quad i = 1,2,\ldots,N+1, \\ &\epsilon_N(0) = 0. \end{aligned} \tag{30}$$

The approximate solution, $\epsilon_N(t)$, is determined after solving this system for unknown coefficients c_n. We computed these coefficients using `fsolve` command of the *Maple* software. The approximate stress, $\sigma_N(t)$, is computed using (6) and $\epsilon_N(t)$.

The graphs of stress and strain functions for different values of α, obtained by FRLFs, are shown in Figures 1a and 1b. Since the obtained graphs for RLFs and exact solutions coincide with these graphs, they are omitted to prevent iteration.

Maximum errors between exact and approximate solutions, $\|\sigma - \sigma_N\|_\infty$, obtained by FRLFs and RLFs are compared in Table 1. Considering this table we observe that although using RLFs give solutions which are in good approximation with exact solution, but applying FRLFs give better solutions. Comparing obtained errors with errors reported in [5], it is found that collocation scheme together with FRLFs or RLFs give more accurate solutions than discontinuous Galerkin method.

Table 1: Maximum errorrs between approximation and exact solutions of stress function.

α	N	FRLFs	RLFs
$\frac{1}{2}$	8	2.10×10^{-3}	1.10×10^{-2}
	16	3.02×10^{-4}	5.74×10^{-3}
	32	4.10×10^{-5}	2.94×10^{-3}
	64	5.36×10^{-6}	1.49×10^{-3}
$\frac{2}{3}$	8	3.82×10^{-5}	1.90×10^{-3}
	16	4.96×10^{-5}	8.97×10^{-4}
	32	1.35×10^{-5}	3.82×10^{-4}
	64	3.68×10^{-6}	1.56×10^{-4}
$\frac{3}{4}$	8	9.14×10^{-5}	5.13×10^{-4}
	16	1.62×10^{-5}	2.59×10^{-4}
	32	8.17×10^{-6}	1.07×10^{-4}
	64	8.04×10^{-6}	4.08×10^{-5}
$\frac{4}{5}$	8	2.50×10^{-4}	1.03×10^{-4}
	16	2.29×10^{-5}	8.49×10^{-5}
	32	1.93×10^{-5}	3.90×10^{-5}
	64	1.93×10^{-5}	1.95×10^{-5}

Log-Log graphs of the absolute values of spectral coefficients, $|c_n|$, versus n obtained by FRLFs and RLFs for $\alpha = \frac{1}{2}$ and different values of N, are given in Figure 2. As we anticipated, the coefficients tend to zero by increasing n. The rate of tending to zero, specifies the order of convergence. As fast as the coefficients decay, the approximate solution converges to the exact solution. Figure 2 illustrates that the spectral coefficients obtained by FRLFs are approximately proportional to $\mathrm{O}\left(n^{-4}\right)$, while the spectral coefficients obtained by RLFs are proportional to less than $\mathrm{O}\left(n^{-2}\right)$. This fact is true for other values of α. From these graphs and Definition 5, it is found out that the collocation scheme with FRLFs has a fourth-order convergence, whereas this scheme with RLFs has a convergence rate less than two. The value λ in these graphs is used to show a constant coefficient.

5 Conclusion

Fractional rational Legendre functions and rational Legendre functions are employed to compute approximate solutions of fractional order viscoelasticity model. FRLFs are introduced for the first time in this paper. They enhance the convergence rate of RLFs approximately twice. The results exhibit the high accuracy of the proposed methods. This property and simple implementation of the methods demonstrate their reliability for solving other fractional problems which lie in semi-infinite intervals.

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