Approximation by Bivariate Bernstein-Durrmeyer Operators on a Triangle

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Abstract: In the present paper, we obtain some approximation properties for the bivariate Bernstein-Durrmeyer operators on a triangle. We characterize the rate of convergence in terms of K-functional and the usual and second order modulus of continuity. We estimate the order of approximation by Voronovskaja type result and illustrate the convergence of these operators to a certain function through graphics using Mathematica algorithm. We also discuss the comparison of the convergence of the bivariate Bernstein-Durrmeyer operators and the bivariate Bernstein-Kantorovich operators to the function through illustrations using Mathematica. Lastly, we study the simultaneous approximation for first order partial derivatives and the shape preserving properties of these operators.

Keywords: Modulus of continuity, rate of convergence, simultaneous approximation, shape preserving properties

1 introduction

Let \( \psi(x, y) \) be a continuous function in a closed region \( R : 0 \leq x \leq 1, 0 \leq y \leq 1 \). Kingsley [7] introduced the Bernstein polynomials for functions of two variables as

\[
B_{m,n}(\psi; x, y) = \sum_{k=0}^{n} \sum_{l=0}^{m} \left( \frac{n}{k} \right) \left( \frac{m}{l} \right) \lambda_{n,k}(x) \lambda_{m,l}(y),
\]

where \( \lambda_{r,t}(x) = \binom{r}{t} x^r (1-x)^t, x \in [0, 1] \). He studied the simultaneous approximation for these operators. Butzer [3] also discussed the simultaneous approximation in a direct manner. In [8], Pop obtained the rate of convergence in terms of the modulus of continuity and established the Voronovskaja type asymptotic theorem for the operators \( B_{m,n}(\psi; x, y) \).

Stancu [10] defined another bivariate Bernstein operators on the triangle \( \triangle := S = \{(x, y) : x + y \leq 1, 0 \leq x, y \leq 1\} \) for functions \( f : S \rightarrow \mathbb{R} \), as

\[
M_n(f; x, y) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} b_{n,k,l}(x,y) f \left( \frac{k}{n}, \frac{l}{n} \right), \quad (x, y) \in S
\]

where \( b_{n,k,l}(x,y) = \binom{n}{k} \binom{n-k}{l} (1-x)^{n-k-l} (1-y)^{k+l} \). He derived the rate of convergence in terms of complete modulus of continuity for \( M_n(f; x, y) \).

Pop and Fărcășa [9] discussed the convergence and approximation properties of the Bernstein-Kantorovich type operators defined as

\[
\psi_n(f; x, y) = (n+1)^2 \sum_{k=0}^{n} \sum_{l=0}^{n-k} b_{n,k,l}(x,y) \int_{\frac{l}{n-1}}^{\frac{l+1}{n}} \int_{\frac{k}{n-1}}^{\frac{k+1}{n}} f(s,t)dsdt
\]

and the associated GBS operators on the triangle. In [1], Acar and Aral studied the approximation properties of two dimensional Bernstein-Stancu-Chlodowsky operators on a triangular domain with mobile boundaries, and gave shape preserving properties and also obtained weighted approximation properties of these operators.

Derriennic [6] studied multivariate Bernstein polynomials defined for integral functions on a triangle and proved the convergence of these operators and its derivative in \( L_p \) spaces. In 1992, Zhou [11] defined the two-dimensional Bernstein-Durrmeyer operators \( \mathcal{Y}_n : f \rightarrow \mathcal{Y}_n(f; \ldots) \) with \( f \in C(S) \) (the space of all continuous functions on \( S \)),

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endowed with the norm \( \| f \| = \sup_{(x,y) \in S} |f(x,y)| \), as
\[
\mathcal{V}_n(f;x,y) = (n+1)(n+2) \sum_{k=0}^{n-k} \sum_{l=0}^{k} b_{n,k,l}(x,y) \times \int_0^1 \int_0^{1-t} b_{n,k,l}(s,t)f(s,t)dsdt
\]
and obtained the rate of convergence in terms of the \( K \)-functional and the smoothness of the functions in \( L_p \) spaces. Deo and Bhardwaj [5] also studied some direct theorems and established an inverse theorem for the operators \( \mathcal{V}_n \), on \( S \).

The aim of this paper is to study the approximation properties of bivariate Bernstein-Durrmeyer operators \( \mathcal{V}_n \) on the triangle \( S \). We obtain the rate of convergence by means of \( K \)-functional, usual and second order modulus of continuity and establish the asymptotic formula to find the order of approximation for the operators \( \mathcal{V}_n \) in continuous function spaces. We demonstrate the convergence of the operators \( \mathcal{V}_n \) to a certain function and the comparison of the convergence with the bivariate Bernstein-Kantorovich operators to the function using Mathematica. We also study the simultaneous approximation for first order partial derivatives and shape preserving properties of these operators.

2 Preliminary results

**Lemma 1.** For \( e_{ij} = s^i t^j, (i,j) \in \mathbb{N}^0 \times \mathbb{N}^0, \mathbb{N}^0 = \mathbb{N} \cup \{0\} \), we have

(i) \( \mathcal{V}_n(e_{00};x,y) = 1; \)

(ii) \( \mathcal{V}_n(e_{10};x,y) = \frac{1+nx}{n+3}; \)

(iii) \( \mathcal{V}_n(e_{01};x,y) = \frac{n+3}{n+1}; \)

(iv) \( \mathcal{V}_n(e_{20};x,y) = \frac{n(n-1)x^2 + 4nx + 2}{(n+1)(n+4)}; \)

(v) \( \mathcal{V}_n(e_{02};x,y) = \frac{n(n-1)y^2 + 4ny + 2}{(n+1)(n+4)}; \)

(vi) \( \mathcal{V}_n(e_{11};x,y) = \frac{nxy + n(x+y) + 1}{(n+1)(n+4)}; \)

(vii) \( \mathcal{V}_n(e_{40};x,y) = \frac{n^3x^3(16 - 6x) + n^2x^2(72 - 48x + 11x^2) + nx(96 - 72x + 32x^2 - 6x^3) + 4x^3}{(n+1)(n+4)(n+5)(n+6)}; \)

(viii) \( \mathcal{V}_n(e_{04};x,y) = \frac{n^3y^3(16 - 6y) + n^2y^2(72 - 48y + 11y^2) + ny(96 - 72y + 32y^2 - 6y^3) + 4y^3}{(n+1)(n+4)(n+5)(n+6)}; \)

The moments (i) – (vi) are given in [5]. The proof of (vii) and (viii) can be obtained by a simple computation. Hence the details are omitted.

**Lemma 2.** [5] For \( h_{ij} = (s-x)^i(t-y)^j, (i,j) \in \mathbb{N}^0 \times \mathbb{N}^0 \), we have

(i) \( \mathcal{V}_n(h_{00};x,y) = 1; \)

(ii) \( \mathcal{V}_n(h_{10};x,y) = \frac{1-3x}{n+3}; \)

(iii) \( \mathcal{V}_n(h_{01};x,y) = \frac{1-3y}{n+3}; \)

(iv) \( \mathcal{V}_n(h_{20};x,y) = \frac{2((6-n)x^2 + (n-4)x + 1)}{(n+3)(n+4)}; \)

(v) \( \mathcal{V}_n(h_{02};x,y) = \frac{2((6-n)y^2 + (n-4)y + 1)}{(n+3)(n+4)}; \)

**Lemma 3.** For the bivariate operators \( \mathcal{V}_n(f;x,y) \), we have

(i) \( \lim_{n \to \infty} n\mathcal{V}_n((s-x);x,y) = 1 - 3x; \)

(ii) \( \lim_{n \to \infty} n\mathcal{V}_n((t-y);x,y) = 1 - 3y; \)

(iii) \( \lim_{n \to \infty} n\mathcal{V}_n((s-x)^2;x,y) = 2x(1-x); \)

(iv) \( \lim_{n \to \infty} n\mathcal{V}_n((t-y)^2;x,y) = 2y(1-y); \)

(v) \( \lim_{n \to \infty} n\mathcal{V}_n((s-x)(t-y);x,y) = -2xy; \)

(vi) \( \lim_{n \to \infty} n^2\mathcal{V}_n((s-x)^4;x,y) = 12x^2(x-1)^2; \)

(vii) \( \lim_{n \to \infty} n^2\mathcal{V}_n((t-y)^4;x,y) = 12y^2(y-1)^2. \)

**Proof.** The proof of this lemma easily follows. Hence we omit the details.

**Lemma 4.** For every \( x \in [0,1] \) and \( n \in \mathbb{N} \), we have

\( \mathcal{V}_n((s-x)^2;x,y) + \left(\frac{1+nx}{n+3} - x\right)^2 < \frac{3}{n+3} \left(\phi^2(x) + \frac{1+9x^2}{n+3}\right) \)

where \( \phi(x) = \sqrt{x(1-x)}. \)

**Proof.** From Lemma 2, we have

\( \mathcal{V}_n((s-x)^2;x,y) + \left(\frac{1+nx}{n+3} - x\right)^2 < \frac{2((6-n)x^2 + (n-4)x + 1) + (1-3x)^2}{(n+3)^2} \)

\( = \frac{(21-2n)x^2 + (2n-14)x + 3}{(n+3)^2} \)

\( = \frac{(2n-14)(1-x) + 7x^2 + 3}{(n+3)^2} \)

\( \leq \frac{1}{n+3} \left(\phi^2(x) + \frac{27x^2 + 20x + 3}{n+3}\right) \)

\( \leq \frac{3}{n+3} \left(\phi^2(x) + \frac{1+9x^2}{n+3}\right). \)

3 Main results

**Basic convergence theorem**

**Theorem 1** [12] Let \( \mathcal{V}_n : C(S) \to C(\mathbb{R}), n \in \mathbb{N} \), be linear positive operators. If

\( \lim_{n \to \infty} \mathcal{V}_n(e_{ij}) = e_{ij}, (i,j) \in \{(0,0),(1,0),(0,1)\} \)
and
\[ \lim_{n \to \infty} V_n(e_{20} + e_{02}) = e_{20} + e_{02} \]
uniformly in \( S \), then the sequence \( V_n(f) \) converges to \( f \) uniformly in \( S \), for any \( f \in C(S) \).

**Estimates of rate of convergence**

For \( f \in C(S) \), the complete modulus of continuity for the bivariate case is defined as follows:

\[ \omega(f; \delta_1, \delta_2) = \sup \{ |f(s,t) - f(x,y)| : |s-x| \leq \delta_1, |t-y| \leq \delta_2 \}, \]

where \( \delta_1, \delta_2 > 0 \). Taking into account that on triangle \( S \), we have

\[ |f(s,t) - f(x,y)| \leq \omega(|s-x|, |t-y|) \leq \omega(f; \delta_1, \delta_2) \]

whenever \( |s-x| \leq \delta_1, |t-y| \leq \delta_2, \delta_1 > 0, \delta_2 > 0 \) and

\[ \omega(f; \lambda_1 \delta_1, \lambda_2 \delta_2) = (1 + \lambda_1 + \lambda_2) \omega(f; \delta_1, \delta_2), \lambda_1 > 0, \lambda_2 > 0. \]

Further, the partial moduli of continuity with respect to \( x \) and \( y \) is defined as

\[ \omega_1(f; \delta) = \sup \{ |f(x_1,y) - f(x_2,y)| : y \in [0,1] \text{ and } \delta \geq 0 \}, \]

\[ \omega_2(f; \delta) = \sup \{ |f(x,y_1) - f(x,y_2)| : x \in [0,1] \text{ and } \delta \geq 0 \}. \]

It is clear that they satisfy the properties of the usual modulus of continuity. The details of the modulus of continuity for the bivariate case can be found in [2].

In what follows, \( \delta_0(x) = \sqrt{\gamma_n((s-x)^2;x,y)}, \)
\( \delta_0(y) = \sqrt{\gamma_n((t-y)^2;y,x)}. \)

**Theorem 2** Let \( f \) be continuous on \( S \), then we have

\[ |V_n(f;x,y) - f(x,y)| \leq 3 \omega(f; \delta_0(x), \delta_0(y)). \]

**Proof.** Applying Lemma 2 and the Cauchy-Schwarz inequality, the proof of this theorem is straightforward. Hence the details are omitted.

**Theorem 3** Let \( f \in C(S) \). Then, we have the following inequality

\[ |V_n(f(s,t);x,y) - f(s,t)| \leq 2(\omega_1(f; \delta_0(x)) + \omega_2(f; \delta_0(y))). \]

**Proof.** The definition of partial moduli of continuity and using Cauchy-Schwarz inequality, proof of this theorem easily follows.

**Local approximation**

For \( f \in C(S) \), let \( C^2(S) = \{ f \in C(S) : f^{(i,j)} \in C(S), 0 \leq i + j \leq 2 \} \), where \( f^{(i,j)} \) is \((i,j)\)th-order partial derivative with respect to \( x,y \) of \( f \), endowed with the norm

\[ ||f||_{C^2(S)} = ||f|| + \sum_{i=1}^{2} \left( \| \frac{\partial f}{\partial x^i} \| + \| \frac{\partial f}{\partial y^j} \| \right). \]

The Peetre’s \( K \)–functional of the function \( f \in C(S) \) is given by

\[ \mathcal{K}(f; \delta) = \inf_{g \in C^2(S)} \{ ||f - g|| + \delta ||g||_{C^2(S)} : \delta > 0 \}. \]

It is also known that the following inequality

\[ \mathcal{K}(f; \delta) \leq M_1 \{ \delta_0^2(f; \sqrt{\delta}) + \min(1, \delta) ||f|| \}, \]

holds for all \( \delta > 0 \) ([4], page 192). The constant \( M_1 \) is independent of \( \delta \) and \( f \) and \( \delta_0^2(f; \sqrt{\delta}) \) is the second order modulus of continuity.

Now, we find the order of approximation of the sequence \( V_n(f;x,y) \) to the function \( f(x,y) \in C(S) \) by Peetre’s \( K \)-functional.

**Theorem 4** For the function \( f \in C(S) \), the following inequality

\[ |V_n(f;x,y) - f(x,y)| < 4 \mathcal{K}(f; J_n(x,y)) \]

\[ + \omega \left( f, \sqrt{\frac{1 - 3\delta y}{n + 3}} \right) \left( \frac{1 - 3\delta y}{n + 3} \right)^2 \]

\[ \leq M \left\{ \delta_0^2 \left( f, \sqrt{J_n(x,y)} \right) + \min(1, J_n(x,y)) ||f||_{C^2(S)} \right\} \]

\[ + \omega \left( f, \sqrt{\frac{1 - 3\delta y}{n + 3}} \right) \left( \frac{1 - 3\delta y}{n + 3} \right)^2 \]

holds. The constant \( M > 0 \) is independent of \( f \) and \( J_n(x,y) \), where

\[ J_n(x,y) = \frac{1}{n + 3} \left( \phi(x) \left( \frac{1 + 9\delta^2}{n + 3} \right) + \frac{1}{n + 3} \left( \phi^2(x) + \frac{1 + 9\delta^2}{n + 3} \right) \right) \]

and \( \phi(x) = \sqrt{x(1-x)} \).

**Proof.** We define the auxiliary operators as follows:

\[ T_n(f;x,y) \]

\[ = V_n(f;x,y) - f \left( \frac{1 + nx}{n + 3}, \frac{1 + ny}{n + 3} \right) + f(x,y). \]

Then, from Lemma 2, we have

\[ T_n(1;x,y) = 1, \quad T_n((s - x);x,y) = 0 \quad \text{and} \quad T_n((t - y);x,y) = 0. \]

Let \( g \in C^2(S) \) and \((s,t) \in S \). Using the Taylor’s theorem,
we have  
\[ g(s,t) - g(x,y) = \frac{\partial g(x,y)}{\partial x}(s-x) + \int_x^s (s-\alpha) \frac{\partial^2 g(\alpha,y)}{\partial \alpha^2} d\alpha \]
\[ + \frac{\partial g(x,y)}{\partial y}(t-y) + \int_y^t (t-\beta) \frac{\partial^2 g(x,\beta)}{\partial \beta^2} d\beta. \]

Operating by $\mathcal{T}_n$ on the equation (3), we get
\[ \mathcal{T}_n(g(x,y)) - g(x,y) = \mathcal{T}_n\left( \int_x^s (s-\alpha) \frac{\partial^2 g(\alpha,y)}{\partial \alpha^2} d\alpha \right) \]
\[ + \int_y^t (t-\beta) \frac{\partial^2 g(x,\beta)}{\partial \beta^2} d\beta. \]

Hence,
\[ |\mathcal{T}_n(g(x,y)) - g(x,y)| \leq \mathcal{D}_n\left( \int_x^s (s-\alpha) \frac{\partial^2 g(\alpha,y)}{\partial \alpha^2} d\alpha \right) \]
\[ + \int_y^t (t-\beta) \frac{\partial^2 g(x,\beta)}{\partial \beta^2} d\beta. \]

Now, for every $g \in C^2(S)$ and from equation (4), we get
\[ |\mathcal{T}_n(f;x,y) - f(x,y)| \]
\[ \leq |\mathcal{T}_n(g(x,y)) - g(x,y)| + |g(x,y) - f(x,y)| \]
\[ + f\left( \frac{1+nx}{n+3}, \frac{1+ny}{n+3} \right) \]
\[ \leq \mathcal{D}_n\left( \int_x^s (s-\alpha) \frac{\partial^2 g(\alpha,y)}{\partial \alpha^2} d\alpha \right) \]
\[ + \int_y^t (t-\beta) \frac{\partial^2 g(x,\beta)}{\partial \beta^2} d\beta + \mathcal{D}_n\left( f\left( \frac{1+nx}{n+3}, \frac{1+ny}{n+3} \right) \right). \]

Taking the infimum on the right hand side over all $g \in C^2(S)$ and using (1), we obtain
\[ |\mathcal{T}_n(f;x,y) - f(x,y)| \leq 4\mathcal{K}(f;J_n(x,y)) \]
\[ + \mathcal{D}_n\left( f; \sqrt{J_n(x,y)} \right) \]
\[ + \min\{1,J_n(x,y)\} ||f||_{C^2(S)} \]
\[ + \mathcal{D}_n\left( f; \sqrt{J_n(x,y)} \right) \]
\[ \leq M\left\{ \mathcal{O}\left( f; \sqrt{J_n(x,y)} \right) \right\}. \]

where $M = 4M_1$. Hence, the proof is completed.

**Theorem 5** Let $f \in C^1(S)$ and $(x,y) \in S$. Then, we have
\[ |\mathcal{T}_n(f;x,y) - f(x,y)| \leq ||f'|| \delta_n(x) + ||f'|| \delta_n(y). \]

**Proof** Let $(x,y) \in S$ be a fixed point. Then, we may write
\[ f(s,t) - f(x,y) = \int_x^s f_u'(u,t)du + \int_y^t f_v'(x,v)dv. \]

Now, applying $\mathcal{T}_n(.,x,y)$ on both sides of the above equation,
\[ |\mathcal{T}_n(f(s,t;x,y) - f(x,y)| \leq |\mathcal{T}_n\left( \int_x^s f_u'(u,t)du; x,y \right) \]
\[ + |\mathcal{T}_n\left( \int_y^t f_v'(x,v)dv; x,y \right). \]

By using the inequalities,
\[ |\int_x^s f_u'(u,t)du| \leq ||f'|| ||s-x|| \]

and
\[ |\int_y^t f_v'(x,v)dv| \leq ||f'|| ||t-y||, \]

\[ |\mathcal{T}_n(f;x,y)| \leq |\mathcal{T}_n(f;x,y)| + \left| f\left( \frac{1+nx}{n+3}, \frac{1+ny}{n+3} \right) \right| \]
\[ + |f(x,y)| \leq 3||f||_{C^2(S)}. \]
we get
\[ |\mathcal{Y}_n(f(s,t);x,y) - f(x,y)| \leq \|f'_n\| \mathcal{Y}_n((s-x);x,y) + \|f''_n\| \mathcal{Y}_n((t-y);x,y). \]

Now, by applying Cauchy-Schwarz inequality, we obtain
\[ |\mathcal{Y}_n(f(s,t);x,y) - f(x,y)| \leq \|f'_n\| (\mathcal{Y}_n((s-x)^2;x,y))^{1/2} + \|f''_n\| (\mathcal{Y}_n((t-y)^2;x,y))^{1/2} \]
\[ = \|f'_n\| \delta_n(x) + \|f''_n\| \delta_n(y). \]

This completes the proof.

**Voronovskaja type theorem**

**Theorem 6** Let \( f \in C^2(S) \). Then, we have
\[ \lim_{n \to \infty} n (\mathcal{Y}_n(f;x,y) - f(x,y)) = f'_n(x,y)(1 - 3x) + f''_n(x,y)(1 - 3y) + f''_n(x,y)x(1 - x) \]
\[ - 2f''_n(x,y)xy + f''_n(x,y)y(1 - y), \]
uniformly in \((x,y) \in S\).

**Proof:** Let \((x,y) \in S\). By the Taylor’s theorem, we have
\[ f(s,t) = f(x,y) + f'_n(x,y)(s-x) + f'_n(x,y)(t-y) \]
\[ + \frac{1}{2} (f''_n(x,y)(s-x)^2 + 2f''_n(x,y)(s-x)(t-y) \]
\[ + f''_n(x,y)(t-y)^2) \]
\[ + \eta(s,t;x,y)\{(s-x)^2 + (t-y)^2\}, \tag{5} \]
where \( \eta(s,t;x,y) \to 0, \) as \((s,t) \to (x,y)\).

Operating \( \mathcal{Y}_n(f;x,y) \) on both sides of (5), we get
\[ \mathcal{Y}_n(f;x,y) = f(x,y) + f'_n(x,y) \mathcal{Y}_n((s-x);x,y) \]
\[ + f''_n(x,y) \mathcal{Y}_n((t-y);x,y) \]
\[ + \frac{1}{2} (f''_n(x,y) \mathcal{Y}_n((s-x)^2;x,y) \]
\[ + 2f''_n(x,y) \mathcal{Y}_n((s-x)(t-y);x,y) \]
\[ + f''_n(x,y) \mathcal{Y}_n((t-y)^2;x,y)) \]
\[ + \mathcal{Y}_n(\eta(s,t;x,y)\{(s-x)^2 + (t-y)^2\};x,y) \tag{6} \]

Now, by applying Cauchy-Schwarz inequality to the last term of (6), we have
\[ \mathcal{Y}_n(\eta(s,t;x,y)\{(s-x)^2 + (t-y)^2\};x,y) \]
\[ \leq (\mathcal{Y}_n(\eta^2(s,t;x,y);x,y))^{1/2} \sqrt{\mathcal{Y}_n((s-x)^4;x,y)} \]
\[ + \sqrt{\mathcal{Y}_n((t-y)^4;x,y)}. \]

Since \( \eta(s,t;x,y) \in C(S) \) and \( \eta(s,t;x,y) \to 0, \) as \((s,t) \to (x,y), \) applying Theorem 1
\[ \lim_{n \to \infty} \mathcal{Y}_n(\eta^2(s,t;x,y);x,y) = 0 \]

This completes the proof.

**Numerical Examples**

Let us consider
\[ f : S \to \mathbb{R}, f(x,y) = x^2 - \sqrt{7}(1 - x - y)^2 - 10xy. \]

The convergence of bivariate Bernstein-Durrmeyer operators \( \mathcal{Y}_n(f;x,y) \) to the function \( f \) is illustrated in Examples 1 and 2.

**Example 1.** For \( n = 20, 50 \) the convergence of the operators \( \mathcal{Y}_n(f;x,y) \) (blue) to the function \( f(x,y) \) (red) is demonstrated in figures 1 and 2 respectively. We notice that the error in the approximation of the function by the operators becomes smaller as \( n \) increases.

**Example 2.** For \( n = 20, 50 \) the comparison of the convergence of bivariate Bernstein-Durrmeyer operators \( \mathcal{Y}_n(f;x,y) \) (blue) and bivariate Bernstein-Kantorovich operators \( \mathcal{B}_n(f;x,y) \) (green) to the function \( f(x,y) = x^2 - \sqrt{7}(1 - x - y)^2 - 10xy \) (red) is illustrated in figures 3 and 4 respectively. It is observed that the error in the approximation of \( f \) by the operators \( \mathcal{Y}_n \) is smaller than the operators \( \mathcal{B}_n. \)
In this section we study the simultaneous approximation property of the operators \( \mathcal{V}_n(\cdot;\cdot,\cdot) \).

**Theorem 7** Let \( f \in C^1(S) \). Then for every \((x,y) \in S^0 \) (the interior of \( S \)),

\[
\lim_{n \to \infty} \left( \frac{\partial}{\partial \omega} \mathcal{V}_n(f;\omega,x) \right)_{\omega=x} = \frac{\partial f}{\partial x}(x,y),
\]

and

\[
\lim_{n \to \infty} \left( \frac{\partial}{\partial v} \mathcal{V}_n(f;x,v) \right)_{v=y} = \frac{\partial f}{\partial y}(x,y).
\]

**Proof.** We shall prove only (8) because the proof of (9) is similar.

By the Taylor formula for \( f \in C^1(S) \), we have

\[
f(s,t) = f(x,y) + f_x(x,y)(s-x) + f_y(x,y)(t-y) + \psi(s,t;x,y)\sqrt{(s-x)^2 + (t-y)^2}
\]

for \((s,t) \in S\),

where \( \psi(s,t;x,y) \equiv \psi(\cdot,\cdot) \in C(S) \) and \( \psi(x,y) = 0 \).

Operating \( \mathcal{V}_n(\cdot;\cdot) \) to the above inequality and then by using Lemma 1, we get

\[
\frac{\partial}{\partial \omega} \mathcal{V}_n(f(s,t);\omega,y)_{\omega=x} = f(x,y)\left( \frac{\partial}{\partial \omega} \mathcal{V}_n(1;\omega,y)_{\omega=x} + f_x(x,y)\frac{\partial}{\partial \omega} \mathcal{V}_n(s-x;\omega,y)_{\omega=x} + f_y(x,y)\frac{\partial}{\partial \omega} \mathcal{V}_n(t-y;\omega,y)_{\omega=x} + \left( \frac{\partial}{\partial \omega} \mathcal{V}_n(\psi(s,t;x,y)\sqrt{(s-x)^2 + (t-y)^2};\omega,y)_{\omega=x} = f_x(x,y)\left( \frac{n}{n+3} \right) + E, (say).
\]

Hence, it is sufficient to prove that \( E \to 0 \), for every \((x,y) \in S^0 \), as \( n \to \infty \).

\[
E = (n+1)(n+2) \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \frac{\partial}{\partial \omega} b_{n,k,l}(\omega,y)_{\omega=x} \int_0^1 \int_0^{1-t} \psi(s,t;x,y)\sqrt{(s-x)^2 + (t-y)^2}dsdt
\]

\[
= (n+1)(n+2) \sum_{k=0}^{n-1} \frac{\partial}{\partial \omega} \psi(s,t;x,y)\sqrt{(s-x)^2 + (t-y)^2}dsdt
\]

\[
= (n+1)(n+2) \frac{\partial}{\partial \omega} \psi(s,t;x,y)\sqrt{(s-x)^2 + (t-y)^2}dsdt
\]

\[
= E_1 + E_2, (say).
\]
First, we estimate $E_1$. Applying Cauchy-Schwarz inequality, we have

\[
E_1 \leq \frac{(1-y)}{x(1-x-y)^2} \left( \sum_{k=0}^{n-k} \sum_{i=0}^{n-k} b_{n,k,i}(x,y)(k-nx)^2 \right)^{1/2} \times 
\left( (n+1)(n+2) \sum_{k=0}^{n-k} b_{n,k,i}(x,y) \right) \times \int_0^1 \int_0^{1-t} \psi^2(s,t;x,y)((s-x)^2 + (t-y)^2) \, ds \, dt \right)^{1/2} \leq \frac{n(1-y)}{x(1-x-y)^2} \left( \sum_{k=0}^{n-k} \sum_{i=0}^{n-k} b_{n,k,i}(x,y) \left( \frac{k}{n} - x \right)^2 \right)^{1/2} \times \left\{ \mathcal{Y}_n \left( \psi^2(s,t;x,y)((s-x)^2 + (t-y)^2);x,y \right) \right\}^{1/2} \leq \frac{n(1-y)}{x(1-x-y)^2} \left( M_n((s-x)^2;x,y) \right)^{1/4} \times \left\{ \mathcal{Y}_n(\psi^4(s,t;x,y);x,y) \right\}^{1/4} \times \left\{ \mathcal{Y}_n((t-y)^4;x,y) \right\}^{1/2} \right)^{1/2}.
\]

By making use of ([5], Lemma (2.5)), for every $(x,y) \in S^0$, we have $M_n((s-x)^2;x,y) = O \left( \frac{1}{n} \right)$, as $n \to \infty$. Thus, we get

\[
|E_1| \leq M(x,y) \{ \mathcal{Y}_n(\psi^4(s,t;x,y);x,y) \}^{1/4},
\]

in view of Lemma 3 ((vi) and (viii)).

From Theorem 1, for every $(x,y) \in S^0$, we obtain

\[
\lim_{n \to \infty} \mathcal{Y}_n(\psi^4(s,t;x,y);x,y) = \psi^4(x,y) = 0.
\]

To estimate $E_2$, proceeding in a manner similar to the estimate of $E_1$, for every $(x,y) \in S^0$, we get $E_2 \to 0$, as $n \to \infty$. Combining the estimates of $E_1$ and $E_2$, it follows that for every $(x,y) \in S^0$, $E \to 0$, as $n \to \infty$. Hence the proof is completed.

Similarly, we can prove the following theorem:

**Theorem 8** Let $f \in C^3(S)$. Then for every $(x,y) \in S^0$, we have

\[
\lim_{n \to \infty} \left\{ \left( \frac{\partial}{\partial \omega} \mathcal{Y}_n(f;\omega,y) \right) \right\} = \frac{\partial f}{\partial x}(x,y)
\]

\[
= -3f_x(x,y) + (2 - 5x)f_{xx}(x,y) + (1 - 5x)f_{xy}(x,y) + \chi(1-x)f_{xxx}(x,y) - 2xyf_{xxy}(x,y) + y(1-y)f_{xyy}(x,y)
\]

\[
and \lim_{n \to \infty} \left\{ \left( \frac{\partial}{\partial \omega} \mathcal{Y}_n(f;x,v) \right) \right\}_{\omega=\psi} - \frac{\partial f}{\partial y}(x,y)
\]

\[
= -3f_y(x,y) + (2 - 5y)f_{yy}(x,y) + (1 - 5y)f_{xy}(x,y) + y(1-y)f_{xxy}(x,y) + x(1-x)f_{xyy}(x,y).
\]

**5 Shape preserving properties**

In this section, we study convexity properties of the operators $\mathcal{Y}_n$ by proving that the operators $\mathcal{Y}_n$ is convex of order $(i,j)$ if $f(x,y)$ is convex of order $(i,j)$ for $0 < i + j \leq r$. We first recall the usual definition of convexity for bivariate functions.

For $f \in C(S), (x,y) \in S$ and $h \in \mathbb{R}$, $\triangle_h^{(i,j)}$ is defined by

\[
\triangle_h^{(1,0)}f(x,y) = f(x+h,y) - f(x,y),
\]

\[
\triangle_h^{(0,1)}f(x,y) = f(x,y+h) - f(x,y),
\]

\[
\triangle_h^{(1,1)}f(x,y) = f(x+h,y+h) + f(x,y) - f(x+h,y) - f(x,y+h),
\]

\[
\triangle_h^{(2,0)}f(x,y) = f(x+2h,y) - 2f(x+h,y) + f(x,y),
\]

\[
\triangle_h^{(0,2)}f(x,y) = f(x,y+2h) - 2f(x,y+h) + f(x,y),
\]

\[
\triangle_h^{(r,0)}f(x,y) = \sum_{i=0}^r (-1)^i \begin{pmatrix} r \\ i \end{pmatrix} f(x+(r-i)h,y),
\]

\[
\triangle_h^{(0,r)}f(x,y) = \sum_{i=0}^r (-1)^i \begin{pmatrix} r \\ i \end{pmatrix} f(x,y+(r-i)h).
\]

**Definition 1.** $f(x,y)$ is convex of order $(i,j)$, $i,j \in \mathbb{N}^0$, $0 < i + j \leq r$, if for $h \in \mathbb{R}$, $\triangle_h^{(i,j)}f \geq 0$.

**Remark.** Let $i, j \in \mathbb{N}^0$, $0 < i + j \leq r$. If $f \in C^{i+j}(S)$ and for all $(x,y) \in S$, $\frac{\partial^{(i+j)}}{\partial x^i \partial y^j} \psi_n(x,y) \geq 0$, then $f(x,y)$ is convex of order $(i,j)$.

**Lemma 5.** For $r = 0, 1, 2, \ldots$, $\frac{\partial^r}{\partial x^i \partial y^j} \mathcal{Y}_n(f;x,y)$ and $\frac{\partial^r}{\partial x^i \partial y^j} \mathcal{Y}_n(f;x,y)$ can be put in the form

\[
(a) \frac{\partial^r}{\partial x^i \partial y^j} \mathcal{Y}_n(f;x,y) = \frac{(n+2)!n!}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} \sum_{j=0}^{n-k} b_{n-r,k,l}(x,y) \times \int_0^1 \int_0^{1-t} b_{n+r+k+l,s,t} \frac{\partial^r}{\partial s^r} f(s,t) \, ds \, dt.
\]

\[
(b) \frac{\partial^r}{\partial x^i \partial y^j} \mathcal{Y}_n(f;x,y) = \frac{(n+2)!n!}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} \sum_{j=0}^{n-k} b_{n-r,k,l}(x,y) \times \int_0^1 \int_0^{1-t} b_{n+r+k+l,s,t} \frac{\partial^r}{\partial t^r} f(s,t) \, dt \, ds.
\]
Proof. (a) By Leibnitz theorem, we get

\[ \frac{\partial^r}{\partial x^r} \mathcal{J}_n(f;x,y) = (n+1)(n+2) \sum_{k=0}^{n-r} \binom{n-r}{k} \int_0^1 \int_0^{1-t} b_{n-k,j}(s,t) f(s,t) ds dt \times \frac{(-1)^{n-r} k! (n-k-l)! x^{n-k-l-r+j}}{(k-j)! (n-k-l-r+j)!} \times \left( \sum_{j=0}^{n-r-k} \binom{n-r-k}{j} \frac{t^j}{j!} \right) \times \frac{1}{(n-r)!(n+r)!} \sum_{k=0}^{n-r-k} \sum_{l=0}^{n-r-k-l} b_{n-r-k,l}(x,y) \times \int_0^1 \int_0^{1-t} b_{n-k+j,i}(s,t) f(s,t) ds dt. \]

From (10) and (11), we obtain

\[ \frac{\partial^r}{\partial x^r} \mathcal{J}_n(f;x,y) = \frac{(n+2)! n!}{(n-r)!(n+r)!} \sum_{k=0}^{n-r-k} \sum_{l=0}^{n-r-k-l} b_{n-r-k,l}(x,y) \times \int_0^1 \int_0^{1-t} b_{n-k+j,i}(s,t) f(s,t) ds dt. \]

The proof of (b) is similar to the proof of (a). Hence it is omitted.

Based on definition 1, Remark 5 and using Lemma 5, we give the following theorem:

**Theorem 9** Let \( f \in C^{i+j}(S) \) such that \( i, j \in \mathbb{N}_0 \) and \( 0 < i + j \leq r \). Then the following statement holds:

If \( f(x,y) \) is convex of order \((r,0)\) (resp. \((0,r)\)), then \( \mathcal{J}_n(f;x,y) \) is also convex of order \((r,0)\) (resp. \((0,r)\)).

**Algorithm:**

For the purpose of clarity we mention below the algorithm for one of the figures e.g. figure 4. The domain used in the graphics is \( \{(x,y) : x+y \leq 1, x, y \geq 0\} \).

\[
\text{Plot3D} \left[ \left\{ x^2 - \sqrt{7}(1-x-y)^2 - 10+x*y, 
\frac{50 \times 49 \times 2^4 + 50 \times x + 2}{(50 + 3) \times (50 + 4)} - \sqrt{7} \times \frac{1}{(50 + 3) \times (50 + 4)} \times [50 \times (50 - 1) \times (x+y)^2] - 2 \times 50 \times (50 + 1) \times (x+y) + (50^2 + 3 \times 50 + 2) - 10 \times (50 \times (50 - 1) \times x + y) + 50 \times (x+y + 1)] \right. \\
\left. \left/ (50 + 3) \times (50 + 4) \right. \right] \\
50^2 \times x^2 + 2 \times 50 \times x - 50 \times x^2 + \frac{1}{3} \\
\left/ (50 + 1)^2 \right. \right] \\
- \sqrt{7} \times \left( \frac{1}{(50 + 1)^2} + [50^2 \times x^2 + 50^2 \times y^2 + 2 \times 50 \times (x+y) - 50 \times (x^2 + y^2) + \frac{2}{3} - 2 \times x - 2 \times y + 4 \times 50 \times (50 - 1) \times x + y) + 2 \times (x+y) + 1] \right) \right/ + \left/ 2 \times (50 + 1)^2 \right. \\
\left/ 4 \times (50 + 1)^2 \right. \\
\{\{x,0,1\},\{y,0,1\}, \text{PlotStyle} \rightarrow \{\text{Red, Blue, Green}\}, \text{RegionFunction} \rightarrow \text{Function}[\{x,y,z\}, 0 \leq x+y \leq 1], \text{Mesh} \rightarrow \text{None}\} \right]
\]

**Conclusion:** The rate of convergence of the bivariate Bernstein-Durrmeyer type operators introduced by Zhou [11] is obtained in terms of the \( K \)–functional and moduli of continuity. We estimate the order of approximation by Voronovskaja type result and illustrate the convergence of these operators to a certain function through graphics using Mathematica algorithm. We also discuss the
comparision of the convergence of the bivariate Bernstein-Durrmeyer operators and the bivariate Bernstein-Kantorovich operators to the function through illustrations using Mathematica. Furthermore, we study the simultaneous approximation for first order partial derivatives and the shape preserving properties of these operators.

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References

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