Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100633

Approximating Properties of Generalized Dunkl Analogue of Szász Operators

M. Mursaleen, Taqseer Khan and Md Nasiruzzaman*

Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India

Received: 17 Jun. 2016, Revised: 17 Jul. 2016, Accepted: 20 Jul. 2016
Published online: 1 Nov. 2016

Abstract: In this paper we present a generalization of Szász operators using the Dunkl generalization of the exponential function. We investigate approximating properties for these operators using the Korovkin approximation theorem and the weighted Korovkin-type theorem. We obtain quantitative estimates by using the modulus of continuity and the rate of convergence for functions belonging to the Lipschitz class. Furthermore, we obtain the rate of convergence in terms of the classical, the second order, and the weighted modulus of continuity.

Keywords: Dunkl analogue; Korovkin theorem; Szász operator; modulus of continuity and weighted modulus of continuity.

1 Introduction

The approximation theory concerns with working out a function by functions which are easy to compute like algebraic polynomials. Weierstrass obtained the first potential result for the real valued continuous functions defined on the compact intervals $[a,b]$. This theorem has played a significant role in the development of functional analysis, theory of functions and several other branches of Mathematics. Several researchers have proved the celebrated Weierstrass' theorem using singular integrals. The most elegant proof of this theorem was given by S. N. Bernstein in 1912. Using probabilistic approach, Bernstein [6] constructed the following sequence of operators $B_n : C[0,1] \to C[0,1]$

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad x \in [0,1], \quad (1)$$

for $n \in \mathbb{N}$ and $f \in C[0,1]$.

In 1950, Szász [19] introduced the following sequence of linear positive operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad f \in C[0,\infty), \quad (2)$$

where $x \geq 0$ and f is a continuous and nondecreasing function on $[0,\infty)$. He obtained approximation results and rate of convergence for these operators.

For any $x \in [0,\infty)$, $n \in \mathbb{N}$, $\mu \geq 0$ and $f \in C[0,\infty)$, Sucu [18] defined the following sequence of positive linear operators

$$L_n(f;x) = \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \quad (3)$$

where the generalized exponential function is defined by

$$e_\mu(x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_\mu(k)}, \quad (4)$$

and the coefficients γ_μ are defined as follows for $k \in \mathbb{N}_0$ and $\mu > -\frac{1}{2}$

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)}$$

and

$$\gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma\left(k+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)}.$$

A recursion formula for γ_μ is given by

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k), \quad k = 0,1,2,\cdots,$$

* Corresponding author e-mail: mursaleenm@gmail.com

where

$$\theta_k = \begin{cases} 0 & \text{if } k \in 2\mathbb{N} \\ 1 & \text{if } k \in 2\mathbb{N}+1. \end{cases}$$

The operators in (3) is called as Dunkl analogue of the Szász operators. Sucu investigated their approximating properties and some direct results. Very recently İçöz and Çekim gave a Dunkl generalization of Kantrovich integral generalization of Szász operators in [9] and a Dunkl generalization of Szász operators via q-calculus in [10].

In the twentieth century, the quantum calculus began with the study of Jackson, but Euler and Jakobi had worked out this kind of calculus known as q calculus. q calculus plays an important role in natural sciences such as mathematics, physics and chemistry. It has many applications in number theory, combinatorics, orthogonal polynomials, hyper geometric functions, mechanics, the theory of relativity and quantum theory. Basic definitions and properties of quantum calculus can be found in the books [4,12]. In recent years, the topic of quantum calculus has attracted considerable attention of many mathematicians. The rapid improvement of quantum calculus has given rise to finding new generalization of Bernstein polynomials including q integers. The q analogue of Bernstein polynomials was firstly introduced and investigated its approximation properties by Lupaş in 1987 [15]. Then, another generalization of Bernstein polynomials based on q integers was presented by Phillips [17].

The remainder of the paper is organized as follows. In Section 2, Construction of operators and moment estimation is obtained. In Section 3, the convergence of the operators (5) is examined with the help of universal Korovkin-type theorem is given. In Section 4,5, The rate of convergence is established by means of a classical approach, the class of Lipschitz functions, the second order modulus of continuity, Peetre's K-functional. In last Section of this paper the conclusion is given.

2 Construction of operators and moment estimation

For $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$ and $\mu \geq 0$ we introduce the Dunkl generalization of the Szász operators as follows:
We replace x by $r_n(x) = x - \frac{1}{2n}$, $n \in \mathbb{N}$, in (3) to obtain the following sequence of operators

$$L_n^*(f;x) = \frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \quad (5)$$

and call them as generalized Dunkl analogue of Szász operators. It is easily seen that the operators $L_n^*(f;x)$ are positive and linear.

In this paper we present a generalization of Szász operators using the Dunkl generalization of the exponential function. We investigate approximating properties for these operators using the Korovkin approximation theorem and the weighted Korovkin-type theorem. We obtain quantitative estimates by using the modulus of continuity and the rate of convergence for functions belonging to the Lipschitz class. Furthermore, we obtain the rate of convergence in terms of the classical, the second order, and the weighted modulus of continuity.

We prove the following lemmas.

Lemma 1. *Let $L_n^*(f;x)$ be the operators given by (5). Then we have the following identities:*

1. $L_n^*(1;x) = 1$,
2. $L_n^*(t;x) = r_n(x) = x - \frac{1}{2n}$,
3. $L_n^*(t^2;x) = x^2 + 2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\frac{x}{n} - \frac{1}{n^2}\left(\frac{1}{4} + \mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)$,
4. $L_n^*(t^3;x) \leq x^3 + \left(3 - 2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x^2}{n} + \left(3 + 4\mu^2 + (4+2\mu)\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\frac{x}{n^2}\right) + \frac{1}{8n^3}$,
5. $L_n^*(t^4;x) \leq x^4 + \left(6 + 4\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x^3}{n}\left(3 + 12\mu^2 - 12\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x^2}{n^2} + \left(17 + 6\mu^2(2+n^2) + (42\mu + 8\mu^3)\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x}{n^3} + \left(3 + 2\mu^2(4+2\mu)\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x^2}{n}$

Proof. Using the definition of the generalized exponential function, (1) is obvious.

$$\begin{aligned} L_n^*(t;x) &= \frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left(\frac{k+2\mu\theta_k}{n}\right) \\ &= \frac{1}{ne_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(k+2\mu\theta_k) \\ &= \frac{1}{ne_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{(k+2\mu\theta_k)\gamma_\mu(k-1)}(k+2\mu\theta_k) \\ &= \frac{1}{ne_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^{k+1}}{\gamma_\mu(k)} \\ &= r_n(x) \\ &= x - \frac{1}{2}, \end{aligned}$$

which proves (2).

$$L_n^*(t^2;x) = \frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left(\frac{k+2\mu\theta_k}{n}\right)^2$$
$$= \frac{1}{n^2 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(k+2\mu\theta_k)^2$$
$$= \frac{1}{n^2 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k-1)}(k+2\mu\theta_k)$$
$$= \frac{1}{n^2 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^{k+1}}{\gamma_\mu(k)}(k+1+2\mu\theta_{k+1}),$$

using the relation

$$\theta_{k+1} = \theta_k + (-1)^k,$$

$$= \frac{r_n(x)}{ne_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(k+2\mu\theta_k)$$
$$+\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)} + \frac{r_n(x)}{ne_\mu(nr_n(x))}2\mu\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}$$
$$= (x-\frac{1}{2n})^2 + \frac{1}{n}(x-\frac{1}{2n}) + \frac{2\mu}{n}(x-\frac{1}{2n})\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}$$
$$= x^2 + 2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\frac{x}{n} - \frac{1}{n^2}\left(\frac{1}{4}+\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right),$$

and this proves (3).

$$L_n^*(t^3;x) = \frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left(\frac{k+2\mu\theta_k}{n}\right)^3$$
$$= \frac{r_n(x)}{n^2 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(k+2\mu\theta_k)^2$$
$$+ \frac{r_n(x)}{n^2 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(1+2\mu(-1)^k)^2$$
$$+ \frac{2r_n(x)}{n^2 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(k+2\mu\theta_k)(1+2\mu(-1)^k)$$

and after some simple computations we arrive at the desired inequality.

$$L_n^*(t^4;x) = \frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left(\frac{k+2\mu\theta_k}{n}\right)^4$$
$$= \frac{r_n(x)}{n^3 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(k+2\mu\theta_k)^3$$
$$+ \frac{r_n(x)}{n^3 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}(1+2\mu(-1)^k)^3$$
$$+ \frac{3r_n(x)}{n^3 e_\mu(nr_n(x))}\sum_{k=0}^{\infty}(k+2\mu\theta_k)$$
$$\times\ (1+2\mu(-1)^k)(k+2\mu\theta_k+1+2\mu(-1)^k).$$

Some intermediary calculations yield the required inequality.

Lemma 2. *Let the operators $L_n^*(f;x)$ be given by (5). Then*

1. $L_n^*(t-x;x) = -\frac{1}{2n}$,
2. $L_n^*((t-x)^2;x) = \left(1+2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x}{n} - \frac{1}{n^2}\left(\frac{1}{4}+\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)$,
3. $L_n^*((t-x)^4;x) \le \left(6+24\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x^3}{n} + \left(3-4\mu^2-20\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x^2}{n^2} + \left(3+4\mu^2+(4+2\mu)\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x^2}{n} + \left(17+6\mu^2(2+\mu^2)+(42\mu+8\mu^3)\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x}{n^3}$.

Proof. In view of

$L_n^*(t-x;x) = L_n^*(t;x) - L_n^*(1;x)$,

(1) is obvious.

$L_n^*((t-x)^2;x)$

$$= L_n^*(t^2;x) - 2xL_n^*(t;x) + x^2 L_n^*(1;x)$$
$$= x^2 + 2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\frac{x}{n} - \frac{1}{n^2}\left(\frac{1}{4}+\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)$$
$$-\ 2x(x-\frac{1}{2}) + x^2$$
$$= 2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\frac{x}{n} - \frac{1}{n^2}\left(\frac{1}{4}+\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right) + \frac{x}{n}$$
$$= \left(1+2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x}{n} - \frac{1}{n^2}\left(\frac{1}{4}+\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right),$$

this ends the proof of (2).

$L_n^*((t-x)^4;x)$

$$= L_n^*(t^4;x) - 4xL_n^*(t^3;x) + 6x^2L_n^*(t^2;x) - 4x^3L_n^*(t;x) + x^4L_n^*(1;x)$$

and using the Lemma 1, the desired inequality is obtained.

3 Korovkin's type results

We prove the following results by using Korovkin's theorem which states as follows (see [1], [2], [3], [8], [11], [14], [16]):

Theorem A. Let (T_n) be a sequence of positive linear operators from $C[0,1]$ into $C[0,1]$. Then $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[0,1]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0$, where $f_i(x) = x^i$ for $i = 0,1,2$.

Theorem 1. *Let $L_n^*(f;x)$ be the operators given by (5) and let*

$$H := \{f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty\},$$

then for $f \in H$, we have

$$\lim_{n\to\infty} L_n^*(f;x) = f$$

uniformly on each compact subset of $[0,\infty)$.

*Proof.*The proof is based on Theorem A regarding the convergence of a sequence of linear positive operators, so it is enough to prove the conditions

$$\lim_{n\to\infty} L_n^*(t^j;x) = x^j, \quad j=0,1,2, \quad \{\text{as } n\to\infty\}$$

uniformly on each compact subset of $[0,1]$. Using Lemma 1 it is easily seen that the above assertions are supplied. Hence the theorem is proved.

To study the weighted approximation, we recall the weighted spaces of functions defined on $\mathbb{R}^+ = [0,\infty)$ as follows:

$$P_\rho(\mathbb{R}^+) = \{f : |f(x)| \le M_f \rho(x)\},$$
$$Q_\rho(\mathbb{R}^+) = \{f : f\in P_\rho(\mathbb{R}^+)\cap C[0,\infty)\},$$
$$Q_\rho^k(\mathbb{R}^+) = \left\{f : f\in Q_\rho(\mathbb{R}^+) \ \text{ and } \ \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = k (k \ \text{ is a constant})\right\},$$

where

$$\rho(x) = 1+x^2$$

is a weight function and M_f is a constant depending only on f. Note that $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm

$$\| f \|_\rho = \sup_{x\ge 0}\frac{|f(x)|}{\rho(x)}.$$

Theorem 2.*Let $L_n^*(.\ ;\ .)$ be the operators given by (5). Then for any function $f\in Q_\rho^k(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty} \| L_n^*(f;x) - f \|_\rho = 0.$$

*Proof.*Application of (1) and (2) of Lemma 1 implies that

$$\| L_n^*(1;x) - 1 \|_\rho = 0$$

and

$$\| L_n^*(t;x) - x \|_\rho = 0$$

respectively. Using (3) of Lemma 1 and the definition of norm, the following is obtained

$$\sup_{x\in[0,\infty)}\frac{|L_n^*(t^2;x)-x^2|}{1+x^2} \le \frac{((8n+1)\mu+1)}{4n^2}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}.$$

From where we easily find that

$$\lim_{n\to\infty}\|L_n^*(t^2;x)-x^2\|_\rho = 0.$$

This completes the proof by the weighted Korovkin-type theorems proved by Gadzhiev [8].

4 Rate of Convergence

In what follows we calculate the rate of convergence of the operators (5) by means of modulus of continuity and Lipschitz type maximal functions.

Let $f\in C[0,\infty]$. The modulus of continuity of f, denoted by $\omega(f,\delta)$, gives the maximum oscillation of f in any interval of length not exceeding $\delta>0$ and is given by

$$\omega(f,\delta) = \sup_{|y-x|\le\delta} |f(y)-f(x)|, \quad x,y\in[0,\infty). \tag{6}$$

It is known that $\lim_{\delta\to 0+}\omega(f,\delta)=0$ for $f\in C[0,\infty)$ and for any $\delta>0$ one has

$$|f(y)-f(x)| \le \left(\frac{|y-x|}{\delta}+1\right)\omega(f,\delta). \tag{7}$$

We give the rate of convergence of the operators $T_{n,q}^*(f;x)$ defined in (5) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$. Let $f\in C[0,\infty)$, $M>0$ and $0<\nu\le 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \{f : |f(\zeta_1)-f(\zeta_2)| \le M|\zeta_1-\zeta_2|^\nu \quad (\zeta_1,\zeta_2\in[0,\infty))\} \tag{8}$$

Theorem 3.*Let $L_n^*(.\ ;\ .)$ be the operators defined by (5). Then for each $f\in Lip_M(\nu)$ satisfying (8), we have*

$$|L_n^*(f;x)-f(x)| \le M(\lambda_n(x))^{\frac{\nu}{2}},$$

where

$$\lambda_n(x) = L_n^*\left((t-x)^2;x\right).$$

*Proof.*By using (8) and the linearity property, the following is obtained

$$\begin{aligned} |L_n^*(f;x)-f(x)| &= |L_n^*(f(t)-f(x);x)| \\ &\le L_n^*(|f(t)-f(x)|;x) \\ &\le ML_n^*(|t-x|^\nu;x). \end{aligned}$$

Therefore, by Lemma 1 and the Hölder's inequality, we obtain

$$\begin{aligned} &|L_n^*(f;x)-f(x)| \\ &\le M\frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^\infty\left(\frac{(nr_n(x))^k}{\gamma_\mu(k)}\right)^{\frac{2-\nu}{2}} \\ &\times\left(\frac{(nr_n(x))^k}{\gamma_\mu(k)}\right)^{\frac{\nu}{2}}\left|\frac{k+2\mu\theta_k}{n}-x\right|^\nu \\ &\le M\frac{1}{e_\mu(nr_n(x))}\left(\sum_{k=0}^\infty\frac{(nr_n(x))^k}{\gamma_\mu(k)}\right)^{\frac{2-\nu}{2}} \\ &\times\left(\sum_{k=0}^\infty\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left|\frac{k+2\mu\theta_k}{n}-x\right|^2\right)^{\frac{\nu}{2}} \\ &= M\left(L_n^*(t-x)^2;x\right)^{\frac{\nu}{2}}, \end{aligned}$$

which completes the proof.

Theorem 4.*Let $\tilde{C}[0,\infty)$ denote the space of uniformly continuous functions defined on $[0,\infty)$ and $f\in\tilde{C}[0,\infty)\cap H$, then the following holds*

$$\begin{aligned} &|L_n^*(f;x)-f(x)| \\ &\le\left\{1+\left(x\left(1+2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\right)^{\frac{1}{2}}\right\}\omega(f,\frac{1}{\sqrt{n}}). \end{aligned}$$

*Proof.*Making use of Lemma 1, the definition of modulus of continuity and the Cauchy-Schwarz inequality, we obtain the following

$$| L_n^*(f;x) - f(x) |$$

$$\leq \frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left|f\left(\frac{k+2\mu\theta_k}{n}\right)-f(x)\right|$$

$$\leq \left(1+\frac{1}{\delta}\frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left|\frac{k+2\mu\theta_k}{n}-x\right|\right)\omega(f,\delta)$$

$$\leq \left\{1+\frac{1}{\delta}\left(\frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left(\frac{k+2\mu\theta_k}{n}-x\right)^2\right)^{\frac{1}{2}}\right\}$$

$$\times\ \omega(f,\delta)$$

$$\leq \left\{\frac{1}{\delta}\left(\frac{x}{n}\left(1+2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)-\frac{1}{n^2}\left(\frac{1}{4}\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\right)^{\frac{1}{2}}\right\}$$

$$+\ \omega(f,\delta)$$

$$\leq \left\{1+\frac{1}{\delta}\left(\frac{x}{n}\left(1+2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\right)^{\frac{1}{2}}\right\}\omega(f,\delta).$$

We rearrange the resulting terms and arrive at the desired inequality.

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $\mathbb{R}^+ = [0,\infty)$ and

$$C_B^2(\mathbb{R}^+) = \{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\} \tag{9}$$

with the norm

$$\| g \|_{C_B^2(\mathbb{R}^+)} = \| g \|_{C_B(\mathbb{R}^+)} + \| g' \|_{C_B(\mathbb{R}^+)} + \| g'' \|_{C_B(\mathbb{R}^+)}\ . \tag{10}$$

Also,

$$\| g \|_{C_B(\mathbb{R}^+)} = \sup_{x\in\mathbb{R}^+} | g(x) |\ . \tag{11}$$

Lemma 3.*Let $L_n^*(.\ ;\ .)$ be the operator defined by (5). Then for any $g \in C_B^2(\mathbb{R}^+)$, we have*

$$| L_n^*(g;x) - g(x) | \leq \left(1+\frac{\lambda_n(x)}{2}\right) \| g \|_{C_B^2(\mathbb{R}^+)},$$

where

$$\lambda_n(x) = |L_n^*\left((t-x)^2;x\right)|$$

*Proof.*Let $g \in C_B^2(\mathbb{R}^+)$, then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi)\frac{(t-x)^2}{2}, \quad \psi \in (x,t).$$

Operating by L_n^* on both sides, we have

$$L_n^*(g,x) - g(x) = g'(x)L_n^*((t-x);x) + \frac{g''(\psi)}{2}L_n^*\left((t-x)^2;x\right).$$

Combining this with the Lemma 2, the following is obtained

$$| L_n^*(g,x) - g(x) |$$

$$\leq \frac{1}{2n} \| g' \|_{C_B(\mathbb{R}^+)} + \frac{\| g'' \|_{C_B(\mathbb{R}^+)}}{2}$$

$$\times \left|\left(1+2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x}{n}-\frac{1}{n^2}\left(\frac{1}{4}+\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\right|$$

$$\leq \| g \|_{C_B^2(\mathbb{R}^+)} + \frac{\| g \|_{C_B^2(\mathbb{R}^+)}}{2}$$

$$\times \left|\left(1+2\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\frac{x}{n}-\frac{1}{n^2}\left(\frac{1}{4}+\mu\frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\right|$$

$$= \| g \|_{C_B^2(\mathbb{R}^+)} + \frac{\| g \|_{C_B^2(\mathbb{R}^+)}}{2}\lambda_n(x)$$

$$= \| g \|_{C_B^2(\mathbb{R}^+)}\left(1+\frac{\lambda_n(x)}{2}\right),$$

where

$$\lambda_n(x) = |L_n^*\left((t-x)^2;x\right)|.$$

This completes the proof of the lemma.

Theorem 5.*For every $f \in C_B[0,\infty)$ and $x \in [0,\infty)$, there holds the following*

$$| L_n^*(g;x) - g(x) |$$

$$\leq\ 2\tilde{M}\left\{\omega_2\left(f;\left(\frac{\lambda_n(x)}{4}\right)^{\frac{1}{2}}\right)+min\left(1,\frac{\lambda_n(x)}{4}\right)\| f \|_{C_B(\mathbb{R}^+)}\right\},$$

where $\tilde{M}$ia a positive constant independent of n and ω_2 is the second modulus of continuity of a function f defined by

$$\omega_2(f;\delta) = \sup_{0<t\leq\delta}\|f(.+2t)-2f(.+t)+f(.)\|_{C_B[0,\infty)}. \tag{12}$$

*Proof.*Let $g \in C_B^2[0,\infty)$. Then using triangle inequality and the Lemma 3, we obtain the following

$$| L_n^*(f;x) - f(x) |$$

$$\leq | L_n^*(f-g;x) | + | L_n^*(g;x) - g(x) | + | f(x) - g(x) |$$

$$\leq 2\|f-g\|_{C_B[0,\infty)} + \frac{\lambda_n(x)}{2}\|g\|_{C_B^2[0,\infty)}.$$

Using the definition of Peetre's K-functional, we get

$$| L_n^*(f;x) - f(x) | \leq 2K\left(f,\frac{\lambda_n(x)}{4}\right).$$

Using the relation between the second modulus of continuity and the Peetre's K-functional, we arrive at the desired estimate.

*Remark.*For the operators $L_n(f;x)(.\ ;\ .)$ defined by (3) we may write that, for every $f \in C_B[0,\infty)$, $x \geq 0$ and $n \in \mathbb{N}$

$$|L_n(f;x)(f;x) - f(x)| \leq 2\omega\left(f;\lambda_{n,x}^*\right), \tag{13}$$

where

$$\lambda_{n,x}^* = \sqrt{L_n(f;x)((t-x)^2;x)} \tag{14}$$

$$= \sqrt{\left(1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n}}.$$

Now we claim that the error estimation in Theorem 4 is better than that of (14) provided $f \in C_B[0,\infty)$ and $x \geq \frac{1}{2n}, \quad n \in \mathbb{N}$. Indeed, for $x \geq \frac{1}{2n}, \quad \mu \geq 0$ and $n \in \mathbb{N}$, it guarantees that

$$L_n^*((t-x)^2;x) \leq L_n((t-x)^2;x), \tag{15}$$

where $L_n^*((t-x)^2;x)$ and $L_n((t-x)^2;x)$ are defined in Theorem 4 and in (14). Again if we put $r_n(x) = x$, then the result in [18] by (14) is obtained.

5 Order of convergence

We shall determine the order of convergence of the function $f \in Q_\rho^k(\mathbb{R}^+)$. For $f \in Q_\rho^k(\mathbb{R}^+)$, the weighted modulus of continuity, introduced by Atakut and Ispir [5], is defined by

$$\Omega(f;\delta) = \sup_{0\leq x<\infty,\ |h|\leq\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

The importance of this type of modulus of continuity is due to its following properties. For $f \in Q_\rho^k(\mathbb{R}^+)$,

$$\lim_{\delta\to 0}\Omega(f;\delta) = 0,$$

$$|f(t)-f(x)|$$

$$\leq 2\left(1+\tfrac{1}{\delta}|t-x|\right)(1+\delta^2)(1+x^2)(1+(t-x)^2)\Omega(f;\delta);\ 0\leq x,t<\infty. \tag{16}$$

Details of this modulus of continuity can be found in [5]. We prove the following theorem.

Theorem 6. *Let $f \in Q_\rho^k(\mathbb{R}^+)$. Then there holds*

$$\sup_{0\leq x<\infty} \frac{|L_n^*(f;x)-f(x)|}{(1+x^2)^3} \leq M_\mu\left(1+\frac{1}{n}\right)\omega\left(f;\frac{1}{n^{\frac{1}{2}}}\right),$$

where M_μ is constant independent of n.

Proof. Making use of Lemma 1 and the expression (16), the following is obtained

$$|L_n^*(f;x)-f(x)|$$

$$\leq \frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left|f\left(\frac{k+2\mu\theta_k}{n}\right)-f(x)\right|$$

$$\leq 2(1+\delta^2)(1+x^2)\omega(f;\delta)\frac{1}{e_\mu(nr_n(x))}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}$$

$$\times\left(1+\frac{\left|\frac{k+2\mu\theta_k}{n}-x\right|}{\delta}\right)\left(1+\left(\frac{k+2\mu\theta_k}{n}-x\right)^2\right)$$

$$= 2(1+\delta^2)(1+x^2)\omega(f;\delta)\frac{1}{e_\mu(nr_n(x))}\left\{\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\right.$$

$$+\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left(\frac{k+2\mu\theta_k}{n}-x\right)^2$$

$$+\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left|\frac{k+2\mu\theta_k}{n}-x\right|$$

$$\left.+\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{(nr_n(x))^k}{\gamma_\mu(k)}\left|\frac{k+2\mu\theta_k}{n}-x\right|\left(\frac{k+2\mu\theta_k}{n}-x\right)^2\right\}.$$

Applying Cauchy-Schwarz inequality in the above, we obtain

$$|L_n^*(f;x)-f(x)|$$

$$\leq (1+\delta^2)(1+x^2)\omega(f;\delta)$$

$$\times\left\{1+L_n^*((t-x)^2;x)+\frac{1}{\delta}(L_n^*((t-x)^2;x))^{\frac{1}{2}}\right.$$

$$\left.+\frac{1}{\delta}(L_n^*((t-x)^2;x))\times(L_n^*((t-x)^4;x)))^{\frac{1}{2}}\right\}.$$

In the light of Lemma 2, we have the following estimates

$$L_n^*((t-x)^2;x) \leq \frac{1+2\mu}{n}x,$$

$$L_n^*((t-x)^4;x) \leq \frac{14+68\mu+16\mu^2+8\mu^3+6\mu^4}{n}(x+x^2+x^3).$$

Combining these with (17), we obtain the following

$$|L_n^*(f;x)-f(x)|$$

$$\leq (1+\delta^2)(1+x^2)\omega(f;\delta)\left\{1+(1+2\mu)x+\frac{1}{\delta}\frac{1}{\sqrt{n}}\sqrt{(1+2\mu)x}\right.$$

$$\left.+\frac{1}{\delta}\frac{1}{\sqrt{n}}\sqrt{(x^2+x^3+x^4)(14+68\mu+16\mu^2+8\mu^3+6\mu^4)}\right\}.$$

On choosing $\delta = \frac{1}{n^{\frac{1}{2}}}$, the theorem follows.

6 Applications

Korovkin-type theorems are very useful tools for determining whether a given sequence of positive linear operators acting on some function space is an approximation process. These theorems exhibit that the approximation (or the convergence) property holds on the whole space provided it holds on a test subsets of functions. The custom of calling these kinds of results "Korovkin-type theorems" refers to P. P. Korovkin [14] who discovered such a property for the functions $1, x$ and x^2 in the space $C[0,1]$ of all continuous functions on the real interval $[0,1]$ as well as for the functions $1, \cos$ and sin in the space of all continuous 2π -periodic functions on the real line (see [1], [2], [16]). Later on, several mathematicians have undertaken the program of extending Korovkin's theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces and so on. Recently, such theorems are generalized or extended by replacing the ordinary convergence by several other more general summabilyty methods, like statistical convergence, A-statistical convergence, statistical A-summabilty, weighted statistical summability etc. and using different set of test functions. This theory has fruitful connections with real analysis, functional analysis, harmonic analysis, measure theory and probability theory, summability theory and partial differential equations.

Conclusion

In this paper, we have modified the Dunkl analogue of Szász operators and defined a Dunkl generalization of these modified operators. This type of modifications enables better error estimation on the interval $[1/2;1)$ rather than the classical Dunkl Szász operators [18]. We obtained some approximation results via well known Korovkin's type theorem. We have also calculated the rate of convergence of operators by means of modulus of continuity and Lipschitz type maximal functions.

References

[1] F. Altomare, M. Campiti, Korovkin type approximation theory and its applications, *de Gruyter Stud. Math.*, 17, Berlin, 1994.

[2] F. Altomare, *Korovkin-type theorems and approximation by positive linear operators*, Surv. Approx. Theor., 5 (2010), 92–164.

[3] M. Altınok, M. Küçükaslan, A-statistical supremum-infimum and A-statistical convergence, *Azerbaijan Journal of Mathematics*, 4(2) **(**2014), 31-42.

[4] A. Aral, V. Gupta, R.P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer, New York, 2013.

[5] Ç. Atakut, N. Ispir, Approximation by modified Szász-Mirakjan operators on weighted spaces, *Proc. Indian Acad. Sci. Math. Sci.* 112 (2002), 571-578.

[6] S.N. Bernstein, Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités, *Commun. Soc. Math. Kharkow*, 2(13) (2012), 1-2.

[7] A.D. Gadjiev, The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P.P. Korovkin, *Sov. Math. Dokl.*, 15(5) (1974), 1433-1436.

[8] A.D. Gadjiev, Simultaneous statistical approximation of analytic functions and their derivatives by *k*-positive linear operators, *Azerbaijan Journal of Mathematics*, 1(1) (2011), 57-66.

[9] G. İçöz, B. Çekim, Stancu type generalization of Dunkl analogue of Szász-Kantorovich operators, *Math. Meth. Appl. Sci.*, (2015) DOI: 10.1002/mma.3602.

[10] G. İçöz, B. Çekim, Dunkl generalization of Szász operators via *q*-calculus, *Jour. Ineq. Appl.*, 2015 (2015): 284.

[11] N. Ispir, Approximation by modified complex Szasz-Mirakjan operators, *Azerbaijan Journal of Mathematics*, 3(2) (2013), 95-107.

[12] V. Kac, P. Cheung, *Quantum Calculus*, *Springer*, New York, 2002.

[13] A. Karaci, İ. Büyükyazici, M. Aktümen, Recognition of human speech using *q*-Bernstein polynomials, *Int. J. Computer Appl.*, 2(5) (2010), 22-28.

[14] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Co., Delhi,1960.

[15] A. Lupaş, A *q*-analogue of the Bernstein operator, In Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, Cluj-Napoca, 9 (1987), 85–92.

[16] M. Mursaleen, *Applied Summability Methods*, Springer Breifs, 2014.

[17] G.M. Phillips, Bernstein polynomials based on the *q*-integers, Ann. Numer. Math., 4 (1997) 511–518.

[18] S. Sucu, Dunkl analogue of Szász Operators, *Appl. Math. Comput.*, 244 (2014), 42-48.

[19] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Res. Natl. Bur. Stand.*, 45 (1950), 239-245.

M. Mursaleen is a full Professor and Chairman, Department of Mathematics, Aligarh Muslim University, India. His research interests are in the areas of pure and applied mathematics including Approximation Theory, Summability Theory, Operator Theory, Fixed Point Theory, Differential and Integral Equations. He has published more than 260 research papers in reputed international journals and six books. He is member of several scientific committees, advisory boards as well as member of editorial board of a number of scientific journals. He has visited a number of foreign universities/ institutions as a visiting scientist/ visiting professor. He has a high number of citations. Recently, he has been awarded the Outstanding Researcher of the Year 2014 of Aligarh Muslim University.

Taqseer Khan Taqseer Khan is a Ph.D. scholar at the dept. of Mathematics, Aligarh Muslim University (AMU), Aligarh, India. He obtained B.Sc. and M.Sc. (Gold Medalist) degrees in Mathematics from the Aligarh Muslim University (AMU), Aligarh, India. After that he earned MS degree in Mathematics from the Central European University (CEU), Budapest, Hungary. His research work is supported by the UGC, MHRD, Gov. of India. His research interests include approximation theory and sumability theory.

Md Nasiruzzaman has obtained his M.Sc. Degree in 2012, from the Indian Institute of Technology Guwahati, India. Recently he has completed Ph.D. under the supervision of Prof M. Mursaleen from Department of Mathematics AMU Aligarh, India. His research is presently centered on the Approximating Operators and Bicomplex Analysis and also his research interests are in the areas of pure and applied mathematics. He has qualified UGC-CSIR NET (June-2012) with All India Rank 57 and has also acknowledges the financial support of BSR Fellowship of the University Grants Commission, New Delhi, India. He has published a number of research papers in reputed international journals.