A Generalized Class of Dual to Product-Cum-Dual to Ratio Type Estimators of Finite Population Mean In Sample Surveys

Housila P. Singh, Surya K. Pal* and Vishal Mehta.

School of Studied in Statistics, Vikram University, Ujjain, M. P., India.

Received: 13 Jul. 2015, Revised: 27 Aug. 2015, Accepted: 28 Aug. 2015.
Published online: 1 Jan. 2016.

Abstract: In this paper, we have suggested a general class of dual to product-cum-dual to ratio type estimators of finite population mean using an auxiliary variable \( x \) that is correlated with the variable \( y \) of interest. The proposed class of estimators includes several known estimators based on transformation in auxiliary variable \( x \). The bias and mean squared error (MSE) expressions of the proposed class of estimators have been obtained to the first degree of approximation. We have compared the proposed class of dual to product-cum-dual to ratio type estimators of finite population mean to various existing ratio, product, and ratio-cum-product type estimators and shown that the suggested class of estimators is better than other existing estimators under some realistic conditions.

Keywords: Auxiliary variable, Dual to product-cum-dual to ratio type estimator, Finite population mean, Simple random sampling, Bias, Mean squared error.

1 Introduction

It is well established fact that in sample surveys, auxiliary information is often used to improve the precision of estimators of population parameters. The use of auxiliary information at the estimation stage appears to have started with the work of Cochran (1940). He envisaged the ratio estimator to estimate the population mean or total of the study variate \( y \) by using supplementary information on an auxiliary variate \( x \), positively correlated with \( y \). The ratio estimator is most effective when the relationship between study variate \( y \) and auxiliary variate \( x \) is linear passing through the origin and the mean square error of \( y \) is proportional to \( x \). When the auxiliary variate \( x \) is negatively correlated with the study variate \( y \), Robson (1957) and Murthy (1964) proposed the product estimator of the population mean or total. In fact, for the better utilization of information on an auxiliary variate \( x \), Murthy (1964) has suggested the use of

- Ratio estimator \( \bar{y}_R \) if, \( \rho C_y / C_x > 1/2 \),
- Product estimator \( \bar{y}_P \) if, \( \rho C_y / C_x < -1/2 \),
- Unbiased estimator \( \bar{y} \) if, \( -1/2 \leq \rho C_y / C_x \leq 1/2 \),

Where \( (C_y, C_x) \) are coefficients of variation of \( (y, x) \) respectively and \( \rho \) is the correlation coefficient between \( y \) and \( x \) respectively.

Consider a finite population \( U = \{U_1, U_2, ..., U_N\} \) of \( N \) units. A sample of size \( n(n < N) \) is drawn using simple random sampling without replacement (SRSWOR) method to estimate the population mean \( \overline{Y} = N^{-1} \sum_{i=1}^{N} y_i \) of the study variate \( y \). Let the sample means \( (\bar{x}, \bar{y}) \) be the unbiased estimators of the population means respectively \( (\overline{X}, \overline{Y}) \) based on \( n \) observations.

The classical ratio and product estimators of population mean \( \overline{Y} \) are respectively given by

\[
\bar{y}_R = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right), \quad \bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right).
\]  

(1.1)  

(1.2)

The biases and mean squared errors (MSEs) of \( \bar{y}_R \) and \( \bar{y}_P \) to the first degree of approximation, are respectively given by

*Corresponding author e-mail: suryakantpal6676@gmail.com

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\[
B(\bar{y}_R) = \frac{(1-f)}{n} \bar{Y}C^{-2}_{x'}[1-K],
\]
(1.3)
\[
B(\bar{y}_r) = \frac{(1-f)}{n} \bar{Y}C^{-2}_{x}K.
\]
(1.4)
\[
MSE(\bar{y}_R) = \frac{(1-f)}{n} \bar{Y}^2 \left[ C^{-2}_{x'} + C^{-2}_{x}(1 - 2K) \right].
\]
(1.5)
\[
MSE(\bar{y}_r) = \frac{(1-f)}{n} \bar{Y}^2 \left[ C^{-2}_{x'} + C^{-2}_{x}(1 + 2K) \right].
\]
(1.6)
where \( f = \frac{n}{N} \), \( K = \frac{C_{x_i}^2}{C_{x_i}^2 + \rho \frac{S_{x_i}^2}{S_Y^2} C_{x_i}^2} \).
\[
S^2_{x'} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{(N-1)} \quad \text{and} \quad S^2_x = \frac{\sum_{i=1}^{N} (y_i - \bar{Y})^2}{(N-1)}.
\]
Consider a transformation
\[
x^*_i = \frac{N\bar{x} - nx_i}{N-n} = (1+g)\bar{X} - g\bar{x}_i, \quad i = 1, 2, ..., N,
\]
where \( g = n/(N-n) \). Then \( \bar{x}^* = (1+g)\bar{X} - g\bar{x} \) is an unbiased estimator for \( \bar{X} = N^{-1} \sum_{i=1}^{N} x_i \) and the correlation between \( \bar{y} \) and \( \bar{x}^* \) is negative. Using the transformation \( x^*_i \), Srivkenkaratama (1980) and Bandyopadhyaya (1980) obtained dual to ratio and dual to product estimators respectively as
\[
\bar{y}^*_R = \bar{Y} \left( \frac{x^*}{X} \right),
\]
(1.7)
\[
\bar{y}^*_r = \bar{Y} \left( \frac{X}{\bar{x}} \right).
\]
(1.8)
The biases and mean squared errors (MSEs) of \( \bar{y}^*_R \) and \( \bar{y}^*_r \) to the first degree of approximation are respectively given as
\[
B(\bar{y}^*_R) = -\frac{(1-f)}{n} \bar{Y}C^{-2}_{x'} gK,
\]
(1.9)
\[
B(\bar{y}^*_r) = \frac{(1-f)}{n} \bar{Y}C^{-2}_{x} g(K + 1).
\]
(1.10)
\[
MSE(\bar{y}^*_R) = \frac{(1-f)}{n} \bar{Y}^2 \left[ C^{-2}_{x'} + C^{-2}_{x} g(1 - 2K) \right].
\]
(1.11)
\[
MSE(\bar{y}^*_r) = \frac{(1-f)}{n} \bar{Y}^2 \left[ C^{-2}_{x'} + C^{-2}_{x} g(1 + 2K) \right].
\]
(1.12)
Singh and Agnihotri (2008) defined a family of product-cum-ratio estimators of population mean \( \bar{Y} \) in simple random sampling (SRS) as
\[
\bar{y}_{PR} = a_B \left( \frac{a\bar{X} + b}{a\bar{X} + b} \right) + (1-a)\bar{Y} \left( \frac{a\bar{X} + b}{a\bar{X} + b} \right).
\]
(1.13)
where ‘a’ and ‘b’ are known characterizing positive scalars and \( \alpha \) is a real constant to be determined such that the MSE of \( \bar{y}_{PR} \) is minimum. The bias and MSE of \( \bar{y}_{PR} \) to the first degree of approximation are respectively given as
\[
B(\bar{y}_{PR}) = -\frac{(1-f)}{n} \bar{Y}C^{-2}_{x'} \alpha(K + \alpha(\delta - 2K) - \delta),
\]
(1.14)
\[
MSE(\bar{y}_{PR}) = \frac{(1-f)}{n} \bar{Y}^2 \left[ C^{-2}_{x'} + \delta(2\alpha - 1)C^{-2}_{x} \right].
\]
(1.15)
where \( \delta = \frac{a\bar{X}}{a\bar{X} + b} \).
The aim of this paper is to suggest a generalised class of dual to product-cum-dual to ratio estimators for population mean \( \bar{Y} \) in SRSWOR and their properties are studied under large sample approximation. It is shown that the proposed dual to product-cum dual to ratio estimator includes several known estimators based on transformation in auxiliary variable \( x \). An empirical study is carried out to discuss the superiority of the proposed class of estimators.

2 A Generalized Class of Dual to Product-Cum-Dual to Ratio Type Estimators of Finite Population Mean

We suggest a class of dual to product-cum-dual to ratio type estimators in SRSWOR for population mean \( \bar{Y} \) as
\[
\bar{y}_{PR} = \eta [\frac{a\bar{X} + b}{a\bar{X} + b} + (1-\eta)\bar{Y} \left( \frac{a\bar{X} + b}{a\bar{X} + b} \right)],
\]
(2.1)
where \( (\alpha, b) \) are same as defined earlier, \( \eta \) being a suitably chosen scalar and \( \bar{X} = \frac{N\bar{x} - nx}{N-n} = (1+g)\bar{X} - g\bar{x} \) with \( g = \frac{n}{N-n} \).

To obtain the bias and MSE of \( y_{PR}^* \) to the first degree of approximation, we write
\[
e_0 = \frac{\bar{y} - \bar{y}}{\bar{Y}} \quad \text{and} \quad e_1 = \frac{(\bar{x} - \bar{X})}{\bar{X}},
\]
such that \( E(e_0) = 0, E(e_1) = 0 \) and
\[
E(e_0^2) = \frac{(1-f)}{n} C^{-2}_{y}, \quad E(e_1^2) = \frac{(1-f)}{n} C^{-2}_{y_1}, \quad E(e_0 e_1) = \frac{(1-f)}{n} K C^{-2}_{y}.
\]
(2.2)
Table 2.1 shows the members of the proposed class of estimators \( y_{PR}^* \) for different choices of \( (a, b, \delta, \eta) \).

In Table 2.1, \( C_{x} \) and \( \beta_2(x) \) respectively are known coefficient of variation and coefficient of kurtosis respectively of an auxiliary variable \( x \).
Expressing (2.1) in terms of $e^i$s, we have

$$
\bar{y}_{PR}^* = \bar{y}(1 + e_0)(1 - \delta e_1)^{-1} + (1 - \eta)(1 - \delta e_1) .
$$

(2.3) We assume that $|\delta e_1| < 1$ so that $(1 - \delta e_1)^{-1}$ is expandable. From (2.3) we have
Neglecting terms of e’s having power greater than two we have
\[
\hat{y}_{PR}^* = \hat{y} + c_0 + \delta c_1(2\eta - 1) + \delta^2 c_2 + \ldots \]
Taking expectation of both sides of (2.4), we obtain the bias of \( \hat{y}_{PR}^* \) to the first degree of approximation as
\[
B(\hat{y}_{PR}^*) = \hat{y} \left( 1 - \frac{1}{n} \right) \delta C_2^* \left[ K(2\eta - 1) + \eta \delta \right] \]
Taking expectation of both sides of (2.6) we get the MISE of \( \hat{y}_{PR}^* \) to the first degree of approximation as
\[
MSE(\hat{y}_{PR}^*) = \hat{y} \left( 1 - \frac{1}{n} \right) \delta C_2^* \left[ K(2\eta - 1) + \eta \delta \right] \]
which is minimized when
\[
\eta = \frac{1}{2} \left( \frac{1}{2} - \frac{K}{\delta^2} \right) \]
Substituting the value of \( \eta_{\text{opt}} \) in (2.1) yields the asymptotically optimum estimator (AOE) as
\[
\hat{y}_{PR,\text{opt}}^* = \hat{y} \left( 1 - \frac{1}{n} \right) \delta C_2^* \left[ K(2\eta_{\text{opt}} - 1) + \eta_{\text{opt}} \delta \right].
\]
Thus, the resulting bias and MISE of \( \hat{y}_{PR,\text{opt}}^* \) respectively are respectively given by
\[
B(\hat{y}_{PR,\text{opt}}^*) = \hat{y} \left( 1 - \frac{1}{n} \right) \delta C_2^* \left[ K(2\eta_{\text{opt}} - 1) + \eta_{\text{opt}} \delta \right].
\]
\[
\min MSE(\hat{y}_{PR}^*) = MSE(\hat{y}_{PR,\text{opt}}^*) = \hat{y} \left( 1 - \frac{1}{n} \right) \delta C_2^* \left[ \frac{1}{(1 - \rho^2)} \right].
\]

3 Efficiency Comparison

(i) Under SRSWOR, the variance of sample mean \( \bar{y} \) is
\[
Var(\bar{y}) = \frac{1}{n} \bar{y}^2 C_\bar{y}^2.
\]
From (2.7) and (3.1), it is found that the proposed dual to product-cum-dual to ratio type estimator \( \hat{y}_{PR}^* \) is more efficient than \( \bar{y} \) if
\[
Var(\bar{y}) - MSE(\hat{y}_{PR}^*) = \frac{1}{n} \bar{y}^2 C_\bar{y}^2 - \frac{1}{n} \left( 1 - \frac{1}{n} \right) \delta^2 C_2^* \left[ \frac{1}{(1 - \rho^2)} \right] > 0.
\]
This condition holds if
\[
either \frac{1}{2} > \eta \quad \text{or} \quad \frac{1}{2} < \eta \quad \text{and} \quad \frac{1}{2} > \frac{K}{\delta^2}.
\]
Therefore, the range of \( \eta \) under which the proposed dual to product-cum-dual to ratio type estimator \( \hat{y}_{PR}^* \) is more efficient than \( \bar{y} \) is
\[
\eta \in \left[ \min \left\{ \frac{1}{2} - \frac{K}{\delta^2}, \max \left\{ \frac{1}{2}, \frac{1}{2} - \frac{K}{\delta^2} \right\} \right. \right].
\]
(ii) The MSEs of dual to Shah and Patel (1984) and Singh and Agnihotri (2008) ratio(product) type estimator \( \hat{y}_{SA}^* \) is obtained by putting \( \eta = 0(1) \) as
\[
MSE(\hat{y}_{SA}^*) = \frac{1}{n} \bar{y}^2 \left[ C_2^* + \delta C_8^2 \left[ \frac{1}{2} \eta \delta \left( \frac{1}{2} - \frac{K}{\delta^2} \right) \right] \right].
\]
From (2.7) and (3.4), it is found that the proposed dual to product-cum-dual to ratio type estimator \( \hat{y}_{PR}^* \) is more efficient than dual to Shah and Patel (1984) and Singh and Agnihotri (2008) class of ratio estimators \( \hat{y}_{SA}^* \) if
\[
MSE(\hat{y}_{SA}^*) - MSE(\hat{y}_{PR}^*) = \frac{1}{n} \bar{y}^2 \left[ C_2^* + \delta C_8^2 \left[ \frac{1}{2} \eta \delta \left( \frac{1}{2} - \frac{K}{\delta^2} \right) \right] \right] - \frac{1}{n} \left( 1 - \frac{1}{n} \right) \delta^2 C_2^* \left[ \frac{1}{(1 - \rho^2)} \right] > 0.
\]
This condition holds if
\[
either 0 < \eta < \left( \frac{1}{2} - \frac{K}{\delta^2} \right), \quad \text{or} \quad \left( \frac{1}{2} - \frac{K}{\delta^2} \right) < \eta < 0.
\]
Therefore, the range of \( \eta \) under which the proposed dual to product-cum-dual to ratio type estimator \( \hat{y}_{PR}^* \) is more efficient than dual to Shah and Patel (1984) and Singh and Agnihotri (2008) class of ratio estimators \( \hat{y}_{SA}^* \) if
\[
\eta \in \left[ \min \left\{ 0, \left( \frac{1}{2} - \frac{K}{\delta^2} \right) \right\}, \max \left\{ 0, \left( \frac{1}{2} - \frac{K}{\delta^2} \right) \right\} \right].
\]
(iii) From (2.7) and (3.5), it is found that the proposed dual to product-cum-dual to ratio type estimator \( \hat{y}_{PR}^* \) is
more efficient than dual to Singh and Agnihotri (2008) class of product estimators $\bar{y}^{*}_{SA2}$ if

$$
MSE(\bar{y}^{*}_{SA2}) - MSE(\bar{y}^{*}_{PR}) = \frac{(1-f)}{n} \left[ C^2_v + \delta g (2\eta - 1) \right]
$$

$$
= \frac{(1-f)}{n} \left[ C^2_v + \delta g (2\eta - 1) + 2K \right] > 0
$$

which is dual to product type estimator.

The biases and $MSE$s of $\bar{y}^{*}_{PR(1)}$ and $\bar{y}^{*}_{PR(2)}$ to the first degree of approximation are respectively given as

$$
B(\bar{y}^{*}_{PR(1)}) = -\bar{y} \left( \frac{(1-f)}{n} \right) \left( \frac{\bar{x}}{\bar{x} + C_x} \right) g C^2_v (\bar{x} - C_x) K,
$$

$$
B(\bar{y}^{*}_{PR(2)}) = -\bar{y} \left( \frac{(1-f)}{n} \right) \left( \frac{\bar{x}}{\bar{x} + C_x} \right) g C^2_v \left[ \left( \frac{\bar{x}}{\bar{x} + C_x} \right) g - K \right],
$$

$$
MSE(\bar{y}^{*}_{PR(1)}) = \bar{y}^2 \left( \frac{(1-f)}{n} \right) \left( \frac{\bar{x}}{\bar{x} + C_x} \right) g C^2_v \left[ \left( \frac{\bar{x}}{\bar{x} + C_x} \right) g - 2K \right],
$$

$$
MSE(\bar{y}^{*}_{PR(2)}) = \bar{y}^2 \left( \frac{(1-f)}{n} \right) \left( \frac{\bar{x}}{\bar{x} + C_x} \right) g C^2_v \left[ \left( \frac{\bar{x}}{\bar{x} + C_x} \right) g + 2K \right].
$$

It is observed from (4.3) that $MSE(\bar{y}^{*}_{PR(1)}) < Var(\bar{y})$ if

either $\frac{1}{2} < \eta < \frac{1}{2} \left( \frac{(\bar{x} + C_x) K}{g \bar{x}} \right)$

or $\frac{1}{2} \left( \frac{(\bar{x} + C_x) K}{g \bar{x}} \right) < \eta < \frac{1}{2}$.

From (1.11) and (4.3) we see that the proposed class of estimator $MSE(\bar{y}^{*}_{PR(1)})$ will dominate over Srivenkataranama (1980) estimator $\bar{y}^*_{R}$ if

either $\frac{C_x}{2 \bar{x}} < \eta < \frac{1}{2} \left( \frac{1 + \frac{C_x}{\bar{x}} (1 - \frac{2K}{g})}{g \bar{x}} \right)$

or $\frac{1}{2} \left( \frac{1 + \frac{C_x}{\bar{x}} (1 - \frac{2K}{g})}{g \bar{x}} \right) < \eta < -\frac{C_x}{2 \bar{x}}$.

Further, from (4.3) and (4.9) we have that

$MSE(\bar{y}^{*}_{PR(1)}) < MSE(\bar{y}^{*}_{PR(2)})$ if

either $\frac{K (\bar{x} + C_x)}{g \bar{x}} < \eta < 1$

or $\frac{1}{2} \left( \frac{K (\bar{x} + C_x)}{g \bar{x}} \right) < \eta < 0$.

(ii) To illustrate the general results we consider $(a,b) = (1,S_1 = \bar{X}C_1)$.

For $(a,b) = (1,S_1)$ in (2.1), we get a class of estimators for population mean $\bar{y}$ as

$$
\bar{y}^{*}_{PR(1)} = \bar{y} \left( \frac{\bar{x} + C_x}{\bar{x} + C_x} \right),
$$

which is due to Shah and Patel (1984) and Singh and Upadhyaya (1986). We note that the estimator $\bar{y}^{*}_{PR(1)}$ is dual to Sisodia and Dwvedi’s (1981) ratio-type estimator.

For $\eta = 1$, $\bar{y}^{*}_{PR(1)}$ reduces to the estimator for $\bar{y}$ as

$$
\bar{y}^{*}_{PR(2)} = \bar{y} \left( \frac{\bar{x} + C_x}{\bar{x} + C_x} \right),
$$

either $\frac{K (\bar{x} + C_x)}{g \bar{x}} < \eta < 1$

or $\frac{1}{2} \left( \frac{K (\bar{x} + C_x)}{g \bar{x}} \right) < \eta < 0$.

Further, from (4.3) and (4.9) we have that

$MSE(\bar{y}^{*}_{PR(1)}) < MSE(\bar{y}^{*}_{PR(2)})$ if

either $\frac{K (\bar{x} + C_x)}{g \bar{x}} < \eta < 1$

or $\frac{1}{2} \left( \frac{K (\bar{x} + C_x)}{g \bar{x}} \right) < \eta < 0$.
\[ \bar{y}_{PR}^* = \eta \left( \bar{x} + S_x \right) + (1 - \eta) \left( \bar{x}^* + S_x \right), \]
\[ \text{where } \eta \text{ and } x^* \text{ are same as defined earlier.} \]

The bias and \( \text{MSE} \) of \( \bar{y}_{PR}^* \) to the first degree of approximation are respectively given as
\[ B(\bar{y}_{PR}^*) = -\frac{1}{n} \left( \frac{1}{1 + C_x} \right) g C_x^2 \left( K(2\eta - 1) + g \frac{1}{1 + C_x} \right). \]
\[ \text{MSE}(\bar{y}_{PR}^*) = \bar{y}^2 \left( \frac{1}{n} \right) \left( \frac{1}{1 + C_x} \right) g C_x^2 \left[ C_x^2 + \left( \frac{1}{1 + C_x} \right) g^2 (2\eta - 1) \right]. \]

For \( \eta = 0 \), we get an estimator for \( \bar{y} \) as
\[ \bar{y}_{PR}^* = \bar{y} \left( \bar{x} + S_x \right) \]
\[ \text{which is due to Shah and Gupta (1986). We note that the estimator } \bar{y}_{PR}^* \text{ is dual to Shah and Patel (1984) ratio type estimator. For } \eta = 1, \text{ } \bar{y}_{PR}^* \text{ reduces to the estimator for } \bar{y} \text{ as} \]
\[ \bar{y}_{PR}^* = \bar{y} \left( \bar{x} + S_x \right) \]
\[ \text{which is dual to product-type estimator. The biases and } \text{MSEs of } \bar{y}_{PR}^* \text{ and } \bar{y}_{PR}^* \text{ to the first degree of approximation are respectively given as} \]
\[ B(\bar{y}_{PR}^*) = \bar{y} \left( \frac{1}{n} \right) \left( \frac{1}{1 + C_x} \right) g C_x^2 \left[ C_x^2 + \left( \frac{1}{1 + C_x} \right) g^2 (2\eta - 1) \right] \]
\[ \text{MSE}(\bar{y}_{PR}^*) = \bar{y}^2 \left( \frac{1}{n} \right) \left( \frac{1}{1 + C_x} \right) g C_x^2 \left[ C_x^2 + \left( \frac{1}{1 + C_x} \right) g^2 (2\eta - 1 + 2K) \right]. \]

We note from (4.16) that the proposed class of estimators \( \bar{y}^*_{PR} \) is more efficient than the usual unbiased estimator \( \bar{y} \) if
\[ \text{either } \frac{1}{2} < \eta < \frac{1}{2} \left( 1 + C_x \right) K \]
\[ \text{or } \frac{1}{2} \left( 1 + C_x \right) K < \eta < \frac{1}{2} \]

It follows from (1.11) and (4.16) that the suggested class of estimators \( \bar{y}^*_{PR} \) is better than Srivenkataramana (1980) estimator \( \bar{y}^*_R \) if
\[ \text{either } \frac{C_x}{2} < \eta < \frac{1}{2} \left( 1 + C_x \right) \left( 1 - \frac{2K}{g} \right) \]
\[ \text{or } \frac{1}{2} \left( 1 + C_x \right) \left( 1 - \frac{2K}{g} \right) < \eta < \frac{C_x}{2} \]

From (4.16) and (4.21) that
\[ \text{MSE}(\bar{y}_{PR}^*) < \text{MSE}(\bar{y}_{PR}^*) \text{ if} \]
\[ \text{either } 0 < \eta < \left( 1 - \frac{K(1 + C_x)}{g} \right) \]
\[ \text{or } \left( 1 - \frac{K(1 + C_x)}{g} \right) < \eta < 0 \]

Further, it is observed from (4.16) and (4.22) that
\[ \text{MSE}(\bar{y}_{PR}^*) < \text{MSE}(\bar{y}^*_{PR}) \text{ if} \]
\[ \text{either } - \frac{K(1 + C_x)}{g} < \eta < 1 \]
\[ \text{or } 1 < \eta < - \frac{K(1 + C_x)}{g} \]

5 Empirical Study

To illustrate the performance of the proposed class of estimators \( \bar{y}_{PR}^* \) and \( \bar{y}^*_{PR} \) over usual unbiased estimator \( \bar{y} \), ratio estimator \( \bar{y}_R \), product estimator \( \bar{y}_P \), \( \bar{y}^*_{PR} \), \( \bar{y}^*_{PR} \), \( \bar{y}^*_{PR} \), and \( \bar{y}^*_{PR} \), we consider two natural population data sets.

Population I: (Italian bureau for the environmental protection-APAT Waste 2004)
\[ Y = \text{Total amount (tons) of recyclable-waste collection in Italy in 2003 and} \]
\[ X = \text{amount (tons) of recyclable-waste collection in Italy in 2002} \]
\[ N = 103, n = 40, \bar{Y} = 626.2123, \bar{X} = 557.1909, \rho = 0.9936, C_x = 1.4588, C_y = 1.4683. \]

Population II: (Italian bureau for the environmental protection-APAT Waste 2004)
\[ Y = \text{Total amount (tons) of recyclable-waste collection in Italy in 2003 and} \]
\[ X = \text{Number of inhabitants in 2003} \]
\[ N = 103, n = 40, \bar{Y} = 62.6212, \bar{X} = 556.5541, \rho = 0.7298, C_y = 1.4588, C_x = 1.0963. \]

We have computed the percent relative efficiencies (PREs) of the estimators \( \bar{y}^*_{PR} \) and \( \bar{y}^*_{PR} \) with respect to \( \bar{y}, \bar{y}_R, \)
\( \bar{y}^*_{PR} \) and \( \bar{y}^*_{PR} \) by using the following formulae:
Table 5.1: Range of \( \eta \) in which the proposed class of estimators \( \bar{y}^*_{PR(i)} \) is better than \( \bar{y}, \bar{y}^*_R \) and \( \bar{y}^*_{PR(i)} \).

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Range of ( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Population I</td>
</tr>
<tr>
<td>( \bar{y} )</td>
<td>(-1.727, 0.500)</td>
</tr>
<tr>
<td>( \bar{y}^*_R )</td>
<td>(-1.226, -0.001)</td>
</tr>
<tr>
<td>( \bar{y}^*_{PR(1)} )</td>
<td>(-1.227, 0.000)</td>
</tr>
</tbody>
</table>

Table 5.2: Range of \( \eta \) in which the proposed class of estimators \( \bar{y}^*_{PR(2)} \) is better than \( \bar{y}, \bar{y}^*_R \) and \( \bar{y}^*_{PR(2)} \).

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Range of ( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Population I</td>
</tr>
<tr>
<td>( \bar{y} )</td>
<td>(-4.982, 0.500)</td>
</tr>
<tr>
<td>( \bar{y}^*_R )</td>
<td>(-3.014, -0.734)</td>
</tr>
<tr>
<td>( \bar{y}^*_{PR(2)} )</td>
<td>(-4.482, 0.000)</td>
</tr>
</tbody>
</table>

Tables 5.1 and 5.2 show that there is enough scope of selecting the value of scalar \( \eta \) in obtaining estimators better than \( \bar{y}, \bar{y}^*_R, \bar{y}^*_{PR(i)} \) and \( \bar{y}^*_{PR(2)} \). That is, even if the scalar \( \eta \) deviates from its true optimum value \( \eta_{opt} \), considerable gain in efficiency by using proposed classes of estimators \( \bar{y}^*_{PR(i)} \) and \( \bar{y}^*_{PR(2)} \) over other existing estimators can be obtained. Largest gain in efficiency is observed at the optimum value \( \eta_{opt} \) of \( \eta \). Tables 5.3 and 5.4 exhibit that there is appreciable gain in efficiency by using the proposed class of estimators \( \bar{y}^*_{PR(i)}, i=1, 2; \) over usual unbiased estimator \( \bar{y} \), Srivenkataramana’s (1980) dual to ratio estimator \( \bar{y}^*_R \), Shah and Patel (1984) and Singh and Upadhyaya (1986) estimator \( \bar{y}^*_{PR(i)} \) and Shah and Gupta (1986) estimator \( \bar{y}^*_{PR(2)} \). Findings closed in Tables 5.1 to 5.4 are in the favour of proposed classes of estimators \( \bar{y}^*_{PR(i)} \) and \( \bar{y}^*_{PR(2)} \). Thus we recommend the proposed classes of estimators \( \bar{y}^*_{PR(i)} \) and \( \bar{y}^*_{PR(2)} \) for their use in practice.
Table 5.3: PRe$s of the proposed class of estimators $\tilde{y}_{PR(i)}^*$ with respect to $\tilde{y}, \tilde{y}_R^*$ and $\tilde{y}_{PR(i)}^{**}$ for different values of $\eta$.

<table>
<thead>
<tr>
<th>Population I</th>
<th>Population II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>$E_{11}$</td>
</tr>
<tr>
<td>-1.500</td>
<td>156.61</td>
</tr>
<tr>
<td>-1.276</td>
<td>276.06</td>
</tr>
<tr>
<td>-1.250</td>
<td>298.20</td>
</tr>
<tr>
<td>-1.225</td>
<td>322.05</td>
</tr>
<tr>
<td>-1.000</td>
<td>759.26</td>
</tr>
<tr>
<td>-0.750</td>
<td>3623.70</td>
</tr>
<tr>
<td>-0.500</td>
<td>4344.91</td>
</tr>
<tr>
<td>-0.250</td>
<td>847.71</td>
</tr>
<tr>
<td>0.000</td>
<td>320.06</td>
</tr>
<tr>
<td>0.250</td>
<td>164.89</td>
</tr>
<tr>
<td>0.500</td>
<td>100.00</td>
</tr>
<tr>
<td>-0.613</td>
<td>7837.46</td>
</tr>
</tbody>
</table>

*Stands for PRe$s less than 100%.

Table 5.4: PRe$s of the proposed class of estimators $\tilde{y}_{PR(2)}$ with respect to $\tilde{y}, \tilde{y}_R^*$ and $\tilde{y}_{PR(2)}^{**}$ for different values of $\eta$.

<table>
<thead>
<tr>
<th>Population I</th>
<th>Population II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>$E_{21}$</td>
</tr>
<tr>
<td>-5.000</td>
<td>*</td>
</tr>
<tr>
<td>-4.890</td>
<td>107.00</td>
</tr>
<tr>
<td>-4.500</td>
<td>146.39</td>
</tr>
<tr>
<td>-4.480</td>
<td>148.97</td>
</tr>
<tr>
<td>-3.010</td>
<td>1106.07</td>
</tr>
<tr>
<td>-3.000</td>
<td>1131.18</td>
</tr>
<tr>
<td>-2.000</td>
<td>4901.13</td>
</tr>
<tr>
<td>-1.750</td>
<td>2249.19</td>
</tr>
<tr>
<td>-1.500</td>
<td>1177.30</td>
</tr>
<tr>
<td>-1.000</td>
<td>464.75</td>
</tr>
<tr>
<td>-0.750</td>
<td>327.96</td>
</tr>
<tr>
<td>-0.500</td>
<td>243.26</td>
</tr>
<tr>
<td>-0.250</td>
<td>187.38</td>
</tr>
<tr>
<td>0.000</td>
<td>148.66</td>
</tr>
<tr>
<td>0.250</td>
<td>120.75</td>
</tr>
<tr>
<td>0.500</td>
<td>100.00</td>
</tr>
<tr>
<td>-2.241</td>
<td>7837.58</td>
</tr>
</tbody>
</table>

*Stands for PRe$s less than 100%.

6 Conclusion

This paper proposed a general class of dual to product-cum-dual to ratio type estimator $\tilde{y}_{PR(i)}^*$ estimator based on the transformation in an auxiliary variable $x$ in simple random sampling. The envisaged class of estimators includes several known estimators based on transformation in auxiliary variable $x$. The bias and MSE expressions of the proposed class of estimators have been obtained under large sample approximation. It is interesting to note that this study unifies several estimators with their properties at one place. Empirical study also shows the superiority of the proposed class of estimators over other existing estimators.

Acknowledgements

The authors are thankful to the Editor-in-Chief and to the anonymous learned referees for his valuable suggestions regarding improvement of the paper.

References


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Housila P. Singh (Dean Faculty of Science) Professor School of Studied in Statistics, Vikram University Ujjain, M.P., India. His research interests are in the areas of Statistics, Sampling Theory and Statistical Inference.
