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Exponential Stability for a Transmission Problem with Locally Indirect Stabilization

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Abstract: In this manuscript we consider the transmission problem, in one space dimension, for linear dissipative waves with locally indirect stabilization. We study the wave propagation in a medium with a component with attrition and another being simply elastic. We show that for this type of material, the dissipation produced by the frictional part is strong enough to produce exponential decay of the solution.

Keywords: Transmission problem, exponential stability, localized damping, indirect stabilization.

1 Introduction

Wave equation with localized damping has been studied by many people, for instance see [1]. For locally indirect stabilization or indirect control, see [2]. For semilinear wave equation with localized damping in unbounded domain see [3]. Transmission problem to wave equation with localized frictional damping forms the centre of this work. In this paper, we consider the following model where the material has one component purely elastic u and another has frictional localized damping $a(x)v_t$ that produces a locally indirect stabilization

$$\rho_1 u_{tt} - \kappa_1 u_{xx} = 0 \text{ in } (-L,0)\times(0,\infty), \tag{1}$$

$$\rho_2 v_{tt} - \kappa_2 v_{xx} + a(x)v_t = 0 \text{ in } (0,L)\times(0,\infty), \tag{2}$$

with boundary conditions

$$u(-L,t) = v(L,t) = 0 \text{ in } (0,\infty), \tag{3}$$

transmission conditions

$$u(0,t) = v(0,t), \;\; \kappa_1 u_x(0) = \kappa_2 v_x(0) \text{ in } (0,\infty), \tag{4}$$

and the initial data

$$\begin{aligned} u(x,0) &= u^0(x) \;\; u_t(x,0) = u^1(x) \text{ in } (-L,0),\\ v(x,0) &= v^0(x) \;\; v_t(x,0) = v^1(x) \text{ in } \;(0,L). \end{aligned} \tag{5}$$

Here, $\rho_1, \kappa_1, \rho_2, \kappa_2$ are positive constants, which represent the density and tension in each part of the material respectively and the function $a = a(x) \in L^\infty(0,L)$, satisfies

$$0 < a_0 \le a(x) \le a_1, \text{ a.e. on } [0,L], \tag{6}$$

with a_0 and a_1 are positive constants.

The energies $E_u = E_u(t)$ and $E_v = E_v(t)$ associated to equations (1) and (2) are given by

$$E_u(t) = \frac{1}{2}\int_{-L}^{0} (\rho_1|u_t|^2 + \kappa_1|u_x|^2)dx \tag{7}$$

$$E_v(t) = \frac{1}{2}\int_{0}^{L} (\rho_2|v_t|^2 + \kappa_1|v_x|^2)dx, \tag{8}$$

We denote by $E(t) = E_u(t) + E_v(t)$ the total energy associated to the system (1)-(5).

From the mathematical point of view, a transmission problem for wave propagation consists on a hyperbolic equation for which the corresponding elliptic operator has discontinuous coefficients, see [4]. Recently, in the work [5], the authors have pointed out that the system (1)-(5) arises in many applications in the engineering and evolution models of the displacement of an elastic body consisting of two different types of materials, one of them simply elastic and the other is subject to the action of an external force. The system (1)-(2) with $a(x) = a$ positive

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constant has been investigated in [6] by Bastos and Raposo and has shown the exponential stability. Rivera and Oquendo [7] looked at transmission problem of viscoelastic waves and established that the dissipation produced by the viscoelastic part is strong enough to produce the exponential stability, no matter small its size is. See also [8]. In [9] the transmission problem for the longitudinal displacement of an Euler-Bernoulli beam, where one small part of the beam made of a viscoelastic material with Kelvin-Voigt constitutive relation was considered. The authors proved existence, uniqueness and exponential stability by semigroup approach and also numerical scheme was presented. About system with frictional damping and delay we mention A. Benseghir [10] where the transmission problem in a bounded domain was analyzed. Under suitable assumptions on the weight of the damping and the weight of the delay, the existence and the uniqueness of the solution using the semigroup theory and the exponential stability by energy method was analyzed. In this work, we establish the exponential stability of the semigroup associated to the system (1)-(5). The technique used here besides offering the advantage of the semigroup theory, also allows to obtain information about the infinitesimal generator associated to the system. This type of approach allows establishing, for example, the idea that the spectrum of the infinitesimal generator associated to (1)-(5) is constituted only by isolated eigenvalues. We use the Sobolev spaces and its properties as in [11] and semigroup theory, see [12]. We apply the semigroup technique for dissipative systems, see Liu and Zheng [13], which is different from some others in the literature, like as the energy method, see Rivera [14], the direct method, see Kormonik [15,16] and Nakao's method, see [17]. This manuscript is organized as follows. Section 2 deals with setting of the semigroup where we prove the well-posedness of the system. In section 3, we show the exponential stability using the Gearhart-Huang-Pruss theorem, [18,19,20].

2 The Semigroup Setting

In this section, we prove the existence and the uniqueness of solution of system (1)-(5) by using the semigroup theory. So let us define

$$\mathbb{H}^m = H_0^m(-L,0)\times H_0^m(0,L), \quad m=1,2.$$
$$\mathbb{L}^2 = L^2(-L,0)\times L^2(0,L).$$
$$\mathbb{H}_L^1 = \{(u,v)\in\mathbb{H}^1;\ u(-L)=v(L)=0,\ u(0)=v(0)\}.$$

Now the energy space is defined by

$$\mathscr{H} = \mathbb{H}_L^1 \times \mathbb{L}^2. \tag{9}$$

Let $\phi = u_t, \psi = v_t$. Denoting $(u,v,\phi,\psi)\in\mathscr{H}$ we define the inner product in $\mathscr{H}$ as follows:

$$\langle (u,v,\phi,\psi),(\bar{u},\bar{v},\bar{\phi},\bar{\psi})\rangle_{\mathscr{H}} := \int_{-L}^{0}[\kappa_1 u_x\bar{u}_x+\rho_1\phi\bar{\phi}]dx + \int_0^L[\kappa_2 v_x\bar{v}_x+\rho_2\psi\bar{\psi}]dx.$$

We can write (1)-(5) as a Cauchy problem

$$\begin{cases} U_t = \mathscr{A}U,\ t>0, \\ U(0)=U_0=(u_0,v_0,u_1,v_1)^T, \end{cases} \tag{10}$$

where the operator A is defined by $A: D(\mathscr{A})\subset\mathscr{H}\to\mathscr{H}$

$$\mathscr{A} = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \frac{\kappa_1}{\rho_1}\partial_x^2 & 0 & 0 & 0 \\ 0 & \frac{\kappa_2}{\rho_2}\partial_x^2 & 0 & -\frac{a(x)}{\rho_2}I \end{bmatrix} \tag{11}$$

with

$$D(\mathscr{A}) = \left\{ \begin{array}{c} (u,v)\in\mathbb{H}^2,\ (\varphi,\psi)\in\mathbb{H}_L^1, \\ \kappa_1 u_x(0)=\kappa_2 v_x(0). \end{array} \right\} \tag{12}$$

It is easy to prove that $\mathscr{A}$ is dissipative, that is,

$$\langle \mathscr{A}U,U\rangle_{\mathscr{H}} = -\rho_2\int_0^L a(x)|\psi|^2dx \text{ for all } U\in D(\mathscr{A}). \tag{13}$$

The goal is to show that $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup, thus proving that (10) is well-posed and consequently, the system (1)-(5) would have a unique solution with regularity depending on where U_0 is located. In this direction we consider the following corollary of the Lummer-Phillips theorem.

Corollary 1.*Let $\mathscr{A}$ be a linear operator with domain $D(\mathscr{A})$ dense in a Hilbert space $\mathscr{H}$. If $\mathscr{A}$ is dissipative and $0\in\rho(\mathscr{A})$ (where $\rho(\mathscr{A})$ is the resolvent set of $\mathscr{A}$). Then $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup of contractions in $\mathscr{H}$.*

The next lemma then ensures that the operator $\mathscr{A}$ is in the conditions of corollary 1.

Lemma 1.*Let $\rho(\mathscr{A})$ be the resolvent set of $\mathscr{A}$. Then, $0\in\rho(\mathscr{A})$.*

*Proof.*In fact, given $F=(f^1,f^2,f^3,f^4)\in\mathscr{H}$ we must get $U=(u,v,\varphi,\psi)\in D(\mathscr{A})$, with $U\neq 0$ such that

$$\mathscr{A}U = F \tag{14}$$

and

$$||U||_{\mathscr{H}} \le C||F||_{\mathscr{H}}, \tag{15}$$

for some positive constant C independent of U and F. Equation (14) leads to

$$\varphi = f^1 \text{ in } H^1(-L,0), \tag{16}$$
$$\psi = f^2 \text{ in } H^1(0,L), \tag{17}$$
$$\kappa_1 u_{xx} = \rho_1 f^3 \text{ in } L^2(-L,0), \tag{18}$$
$$\kappa_2 v_{xx} - a(x)\psi = \rho_2 f^4 \text{ in } L^2(0,L). \tag{19}$$

From definition of $D(\mathscr{A})$ in (12) one must still have

$$u(-L) = v(L) = 0, \ \ u(0) = v(0), \tag{20}$$
$$\varphi(-L) = \psi(L) = 0, \ \ \varphi(0) = \psi(0), \tag{21}$$
$$\kappa_1 u_x(0) = \kappa_2 v_x(0). \tag{22}$$

Consider the bilinear form $J : \mathbb{H}^1_L \times \mathbb{H}^1_L \to \mathbb{C}$ and the linear functional $h : \mathbb{H}^1_l \to \mathbb{C}$ given by

$$J((u,v),(w,\phi)) = \kappa_1 \int_{-L}^{0} u_x w_x dx + \kappa_2 \int_0^L v_x \phi_x dx,$$
$$h(w,\phi) = -\rho_1 \int_{-L}^{0} f^3 w dx - \rho_2 \int_0^L f^4 \phi dx - \int_0^L a(x)\psi\phi dx.$$

It is easy to get that J is continuous and coercive, and h is continuous. By Lax-Milgram lemma

$$J((u,v),(w,\phi)) = h(w,\phi) \ \text{ for all } \ (w,\phi) \in \mathbb{H}^1_L \tag{23}$$

has a unique solution $(u,v) \in \mathbb{H}^1_L$. From Agmon-Douglis-Niremberg theorem (see [21], page 135) it follows from equations (18), (19) that $(u,v) \in \mathbb{H}^2$ and (15) is assured, so $0 \in \rho(A)$.

Theorem 1. *The operator $\mathscr{A}$ is the infinitesimal generator of C_0-semigroup of contractions $S(t) = e^{t\mathscr{A}}$ in $\mathscr{H}$.*

Proof. By (13) we have that $\mathscr{A}$ is a dissipative operator. $D(\mathscr{A})$ is dense in $\mathscr{H}$. Lemma 1 ensures that $0 \in \rho(\mathscr{A})$. The conditions of corollary 1 are satisfied and so $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup of contractions in $\mathscr{H}$.

The existence and uniqueness result is stated as follows.

Theorem 2. *Let $U_0 \in \mathscr{H}$, then the system*

$$\begin{cases} U_t = \mathscr{A}U, \ t > 0, \\ U(0) = U_0, \end{cases}$$

has a unique weak solution $U \in C((0,\infty);\mathscr{H})$. Moreover, if $U_0 \in D(\mathscr{A})$ then $U \in C((0,\infty);D(\mathscr{A})) \cap C^1((0,\infty);\mathscr{H})$.

Proof. The proof is a direct consequence of the theorem 1 and standard semigroup theory.

3 Exponential Stabilization

In this section we prove the exponential stability by using the semigroup theory.

Lemma 2. *$\sigma(\mathscr{A})$ the spectrum of $\mathscr{A}$ consists only of isolated eigenvalues with finite multiplicity.*

Proof. From the previous section, we have $0 \in \rho(\mathscr{A})$. From Rellich-Kondrachov theorem $D(\mathscr{A}) \subset \mathscr{H}$ compactly and thus, $\mathscr{A}$ has compact resolvent set, in addition $D(\mathscr{A})$ is closed because it is infinitesimal generator of a C_0-semigroup, so the result follows from [22], theorem 6.29.

Remark. The lemma 2 asserts that $\mu \in \sigma(\mathscr{A})$, if and only if, there exists $U \in D(\mathscr{A})$, with $U \neq 0$ such that

$$(\mu I - \mathscr{A})U = 0.$$

We then present the necessary and sufficient conditions for exponential stability of a C_0-semigroup of contractions on a Hilbert space. This result was obtained by Gearhart [18] and Huang [19] independently (see also Pruss [20]).

Theorem 3. *Let $S(t) = e^{t\mathscr{A}}$ be a C_0-semigroup of contractions in a Hilbert space. Then, $S(t)$ is exponentially stable if, and only if,*

$$i\mathbb{R} = \{i\mu \,:\, \mu \in \mathbb{R}\} \subset \rho(\mathscr{A})$$

and

$$\limsup_{|\mu|\to\infty} ||(i\mu I - \mathscr{A})^{-1}|| < \infty.$$

Lemma 3. *$i\mathbb{R} \subset \rho(\mathscr{A})$.*

Proof. From previous results, it is known that $\sigma(\mathscr{A})$ is formed only by eigenvalues of $\mathscr{A}$, so it must be shown that no element in $i\mathbb{R}$ can belong to $\sigma(\mathscr{A})$ implying therefore that such elements belong to $\rho(\mathscr{A}) = \mathbb{C} \setminus \sigma(\mathscr{A})$. By contradiction, suppose there exists $\mu \in \mathbb{R}$, such that $i\mu \in \sigma(\mathscr{A})$, in this way, there exists $U = (u,v,\varphi,\psi) \in D(\mathscr{A})$, with $U \neq 0$ such that

$$(i\mu I - \mathscr{A})U = 0 \tag{24}$$

that is equivalent to

$$i\mu u - \varphi = 0, \tag{25}$$
$$i\mu v - \psi = 0, \tag{26}$$
$$i\mu\rho_1\varphi - \kappa_1 u_{xx} = 0, \tag{27}$$
$$i\mu\rho_2\psi - \kappa_2 v_{xx} + a(x)\psi = 0, \tag{28}$$

together with the conditions (20)-(22). Taking the real part in (24) and using (13) we obtain

$$\int_0^L a(x)|\psi|^2 dx = 0.$$

Therefore, $\psi = 0$. From (25) and (27) results

$$\mu^2\rho_1 u + \kappa_1 u_{xx} = 0,$$

from (20) and (22) we obtain

$$u(-L) = u(0) = u_x(0) = 0,$$

thus $u = 0$. From (25) we have $\varphi = 0$. We conclude that $U = 0$, which contradicts the fact that it is an eigenvector.

Lemma 4.

$$\limsup_{|\mu|\to\infty} ||(i\mu I - \mathscr{A})^{-1}|| < \infty.$$

Proof. The proof is performed using again a contradiction argument. Suppose that

$$\limsup_{|\mu|\to\infty} ||(i\mu I-\mathscr{A})^{-1}|| = \infty.$$

There are sequences $F_n = (f_n^1, f_n^2, f_n^3, f_n^4) \in \mathscr{H}$, $i\mu_n \in \rho(\mathscr{A})$ with $|\mu_n| \to \infty$ and $U_n = (u_n, v_n, \varphi_n, \psi_n) \in D(\mathscr{A})$, with $||U_n||_{\mathscr{H}} = 1$, such that

$$\frac{||(i\mu_n I-\mathscr{A})^{-1}F_n||_{\mathscr{H}}}{||F_n||_{\mathscr{H}}} \geq n,$$

or equivalently

$$||(i\mu_n I-\mathscr{A})^{-1}F_n||_{\mathscr{H}} \geq n||F_n||_{\mathscr{H}} \tag{29}$$

and

$$i\mu_n U_n - \mathscr{A}U_n = F_n. \tag{30}$$

From (29) and (30) we have,

$$||U_n||_{\mathscr{H}} \geq n||F_n||_{\mathscr{H}}, \tag{31}$$

that is,

$$||F_n||_{\mathscr{H}} \leq \frac{1}{n}.$$

Therefore,

$$F_n \to 0 \quad \text{strongly in} \quad \mathscr{H}. \tag{32}$$

From (30) we have

$$i\mu_n||U_n||^2_{\mathscr{H}} - \langle \mathscr{A}U_n, U_n\rangle_{\mathscr{H}} = \langle F_n, U_n\rangle_{\mathscr{H}}$$

and using (13)

$$i\mu_n||U_n||^2_{\mathscr{H}} + \rho_2\int_0^L a(x)|\psi_n|^2dx = \langle F_n, U_n\rangle_{\mathscr{H}}.$$

Taking the real part, applying Schwarz's inequality, using that U is limited and (32) we obtain

$$\begin{aligned}\rho_2 a_0\int_0^L |\psi_n|^2dx &\leq \rho_2\int_0^L a(x)|\psi_n|^2dx\\ &= \mathrm{Re}\,\langle F_n, U_n\rangle_{\mathscr{H}} \leq ||F_n||_{\mathscr{H}} \to 0,\end{aligned}$$

thus

$$\psi_n \to 0 \ \text{ in } \ L^2(0,L). \tag{33}$$

Multiplying (30) by i we have

$$-\mu_n U_n - i\mathscr{A}U_n = iF_n,$$

so

$$\mu_n||U_n||^2_{\mathscr{H}} = -i\langle F_n, U_n\rangle_{\mathscr{H}} - i\langle \mathscr{A}U_n, U_n\rangle_{\mathscr{H}},$$

thus

$$|\mu_n|||U_n||^2_{\mathscr{H}} \leq |\langle F_n, U_n\rangle_{\mathscr{H}}| + |\langle \mathscr{A}U_n, U_n\rangle_{\mathscr{H}}|,$$

from (13), (32), (33) and from Schwarz's inequality

$$|\mu_n|||U_n||^2_{\mathscr{H}} \leq ||F_n||_{\mathscr{H}} + \int_0^L a(x)|\psi_n|^2dx \to 0$$

consequently

$$\mu_n|u_{nx}|^2_{L^2} \to 0, \tag{34}$$

$$\mu_n|v_{nx}|^2_{L^2} \to 0, \tag{35}$$

$$\mu_n|\varphi_n|^2_{L^2} \to 0, \tag{36}$$

$$\mu_n|\psi_n|^2_{L^2} \to 0. \tag{37}$$

(30) can be written as

$$i\mu_n u_n - \varphi_n = f_n^1, \tag{38}$$

$$i\mu_n v_n - \psi_n = f_n^2, \tag{39}$$

$$i\rho_1\mu_n\varphi_n - \kappa_1 u_{nxx} = \rho_1 f_n^3, \tag{40}$$

$$i\rho_2\mu_n\psi_n - \kappa_2 v_{nxx} + a(x)\psi_n = \rho_2 f_n^4. \tag{41}$$

Multiplying (38) by $\mu_n u_n$ and integrating on $[-L,0]$ we have

$$i\mu_n^2|u_n|^2_{L^2} = \mu_n\int_{-L}^0 \varphi_n u_n dx + \mu_n\int_{-L}^0 f_n^1 u_n dx.$$

Using Young and Poincarè's inequalities, we get a positive constant c_0 such that

$$\begin{aligned}|\mu_n|^2|u_n|^2_{L^2} \leq{}& \frac{1}{2}|\mu_n||\varphi_n|^2_{L^2} + \frac{c_0}{2}|\mu_n||u_{nx}|^2_{L^2}\\ &+\frac{1}{2}|f_n^1|^2_{L^2} + \frac{1}{2}|\mu_n|^2|u_n|^2_{L^2},\end{aligned}$$

and then

$$|\mu_n|^2|u_n|^2_{L^2} \leq |\mu_n||\varphi_n|^2_{L^2} + c_0|\mu_n||u_{nx}|^2_{L^2} + |f_n^1|^2_{L^2}. \tag{42}$$

From (32), (34) and (36) in (42) we obtain

$$\mu_n u_n \to 0 \ \text{ in } \ L^2(-L,0). \tag{43}$$

Using (32) and (43) in (38) we obtain

$$\varphi_n \to 0 \ \text{ in } \ L^2(0,L). \tag{44}$$

Using (32) and (33) in (39) we have

$$\mu_n v_n \to 0 \ \text{ in } \ L^2(0,L). \tag{45}$$

Now, replacing (38) in (40) and (39) in (41) we get the system

$$-\rho_1\mu_n^2 u_n - \kappa_1 u_{nxx} = \rho_1 f_n^3 + i\rho_1\mu_n f_n^1 \tag{46}$$

$$\begin{aligned}-\rho_2\mu_n^2 v_n - \kappa_2 v_{nxx} + ia(x)\mu_n v_n ={}& \rho_2 f_n^4 - i\rho_2\mu_n f_n^2\\ &+a(x)f_n^2.\end{aligned} \tag{47}$$

Since $U_n \in D(\mathscr{A})$, u_n and v_n satisfies (20) and (22), multiplying (46) by u_n and (47) by v_n, integrating on $[-L,0]$ and $[0,L]$ respectively, adding and taking the real part we obtain

$$\begin{aligned}\kappa_1|u_{nx}|^2_{L^2} + \kappa_2|v_{nx}|^2_{L^2} \leq{}& \rho_1\mu_n^2|u_n|^2_{L^2} + \rho_2\mu_n^2|v_n|^2_{L^2}\\ &+\frac{\rho_1^2 c_1}{2k_1}|f_n^3|^2_{L^2}\rho_1\int_{-L}^0 f_n^3 u_n dx\\ &+\rho_2\int_0^L f_n^4 v_n dx\int_0^L a(x)f_n^2 v_n dx.\end{aligned} \tag{48}$$

By Poincarè's inequality we get c_1 such that

$$|u_n|_{L^2}^2 \leq c_1 |u_{nx}|_{L^2}^2$$

and writing

$$\rho_1 \int_{-L}^{0} f_n^3 u_n dx = \int_{-L}^{0} \rho_1 \frac{\sqrt{c_1}}{\sqrt{k_1}} f_n^3 \frac{\sqrt{k_1}}{\sqrt{c_1}} u_n dx$$

by Young's inequality

$$\rho_1 \int_{-L}^{0} f_n^3 u_n dx \leq \frac{\rho_1^2 c_1}{2k_1} |f_n^3|_{L^2}^2 + \frac{k_1}{2} |u_{nx}|_{L^2}^2. \quad (49)$$

Similarly we get c_2, c_3 positive constants such that

$$\rho_2 \int_{0}^{L} f_n^4 v_n dx \leq \frac{\rho_2^2 c_2}{4k_2} |f_n^4|_{L^2}^2 + \frac{k_2}{4} |v_{nx}|_{L^2}^2. \quad (50)$$

$$\int_{0}^{L} a(x) f_n^2 v_n dx \leq \frac{a_1^2 c_3}{4k_2} |f_n^2|_{L^2}^2 + \frac{k_2}{4} |v_{nx}|_{L^2}^2. \quad (51)$$

Using (49), (50), (51) in (48) we have

$$\begin{aligned} \frac{\kappa_1}{2} |u_{nx}|_{L^2}^2 + \frac{\kappa_2}{2} |v_{nx}|_{L^2}^2 \leq \rho_1 \mu_n^2 |u_n|_{L^2}^2 + \rho_2 \mu_n^2 |v_n|_{L^2}^2 \\ + \frac{\rho_1^2 c_1}{2k_1} |f_n^3|_{L^2}^2 + \frac{\rho_2^2 c_2}{4k_2} |f_n^4|_{L^2}^2 \\ + \frac{a_1^2 c_3}{4k_2} |f_n^2|_{L^2}^2. \end{aligned}$$

making $n \to \infty$ in the previous inequality and taking into account (32), (43), (45) we obtain

$$u_{nx} \to 0 \text{ in } L^2(-L, 0), \quad (52)$$

$$v_{nx} \to 0 \text{ in } L^2(0, L). \quad (53)$$

We conclude from (33), (44), (52) and (53) that

$$||U_n||_{\mathscr{H}} \to 0 \text{ in } \mathscr{H},$$

which contradicts the fact that $||U_n||_{\mathscr{H}} = 1$.

Finally we are in position to prove the principal result of this work.

Theorem 4. *The semigroup associated to the system* (1)-(5) *is exponentially stable.*

Proof. The result follows directly from the lemmas 3 and 4 and theorem 3.

4 Conclusion

The asymptotic behaviour for the transmission problem of waves with indirect control form the center of this work. We prove that the wave propagation in a medium with a component with attrition and another being simply elastic is strong enough to produce exponential stability for all system. The spirit of the method used here offers the advantage of combining theorem of Geahart-Huang-Pruss in the semigroup theory with PDE tecniques.

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