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$\mathscr{Z}$-Tensor on Mixed Generalized Quasi-Einstein GRW Space-times

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Abstract: The object of the present paper is to study mixed generalized quasi-Einstein manifolds by using the properties of the $\mathscr{Z}$-tensor. Also we prove that mixed generalized quasi-Einstein GRW space-times admitting $\mathscr{Z}$-tensor reduce to Einstein space-times or perfect fluid space-times provided ϕ=constant. Finally, a non-trivial example of MG(QE)$_4$-space-times are given.

Keywords: Mixed generalized quasi-Einstein manifolds, Generalized Robertson-Walker space-times, $\mathscr{Z}$-tensor, Torse-forming vector field, $\varphi(\widetilde{Ric})$-vector field.

1 Introduction

The warped product Θ=$I \times_{\psi} \widetilde{\Theta}$ of an open connected interval $(I, -dt^2)$ of $\mathscr{R}$ and a Riemannian manifold $\widetilde{\Theta}$ with warping function $\psi : I \to \mathscr{R}^+$ is called a generalized Robertson-Walker space-time (or GRW space-times) [25, 16]. This family of Lorentzian space-times broadly extends the CRW space-times, Friedmann cosmological models, Einstein-de Sitter space-times and many others [4, 16]. The CRW space-time is regarded as cosmological models since it is spatially homogeneous and spatially isotropic whereas GRW space-times serve as in-homogeneous extension of RW space-times that admit an isotropic radiation [4, 16]. A Lorentzian manifold is called a PFST if the Ricci tensor $\widetilde{\mathscr{R}}_{ij}$ takes the for

$$\widetilde{\mathscr{R}}_{ij} = \alpha g_{ij} + \beta \mathscr{T}_i \mathscr{T}_j, \tag{1}$$

where α, β are scalars and $\mathscr{T}$ is a 1-form metrically equivalent to a unit time-like vector field [17]. The PFST in the language of differential geometry are called QES where $\mathscr{T}$ is metrically equivalent to a unit space-like vector field. Recently, in [17], it is proven that a PFST with divergence-free conformal curvature tensor is a GRW space-time with Einstein fibers given that the scalar curvature is constant.
Recently, in [24] proved that the Ricci tensor of a GRW space-time in all classes of Gray's decomposition [14], but $\widetilde{\mathscr{C}} \oplus \widetilde{\mathscr{D}}$ is either Einstein or takes the form of a perfect fluid whereas $\widetilde{\mathscr{C}} \oplus \widetilde{\mathscr{D}}$ is not restricted. The class $\widetilde{\mathscr{C}} \oplus \widetilde{\mathscr{D}}$ is characterized by $\nabla\mathscr{R}$=0, that is, the scalar curvature is constant. Now, the following question arises. Does the Ricci tensor of all GRW space-times in $\widetilde{\mathscr{C}} \oplus \widetilde{\mathscr{D}}$ reduce to be Einstein or take the form of a perfect fluid ?.
A (pseudo-) Riemannian manifold $(\widetilde{\Theta}, g)$ is called a generalized quasi-Einstein manifold (briefly, G(QE)$_{\widetilde{n}}$) if its Ricci tensor satisfies

$$\widetilde{\mathscr{R}}_{ij} = \alpha g_{ij} + \beta \mathscr{T}_i \mathscr{T}_j + \gamma \mathscr{N}_i \mathscr{N}_j, \tag{2}$$

where α, β and γ are non-zero constants, $\mathscr{T}$ and $\mathscr{N}$ are 1-forms corresponding to two orthonormal vector field [6, 13, 19]. If γ=0, then $(\widetilde{\Theta}, g)$ reduces to a quasi-Einstein manifold.
A non-flat Riemannian manifold $(\widetilde{\Theta}, g)$is said to be a mixed generalized quasi-Einstein manifolds (briefly, MG(QE)$_{\widetilde{n}}$), if its $\mathscr{R}_{ij} \neq 0$ subjected to [3]:

$$\widetilde{\mathscr{R}}_{ij} = \alpha g_{ij} + \beta \mathscr{T}_i \mathscr{T}_j + \gamma \mathscr{N}_i \mathscr{N}_j + \delta(\mathscr{T}_i \mathscr{N}_j + \mathscr{T}_j \mathscr{N}_i), \tag{3}$$

where α, β, γ and δ are non-zero scalars and $\mathscr{T}$ and $\mathscr{N}$ are 1-forms, such that $\mathscr{T}_i\mathscr{T}^i$=$\mathscr{N}_j\mathscr{N}^j$=1 and $\mathscr{T}_i\mathscr{N}^j$=0. After that MG(QE)$_{\widetilde{n}}$ and GRW space-times have been studied by various geometer in different ways [10, 11, 1, 8,

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5,27,28,29,30,31] and many others. The trace of equation (3) gives,

$$\tau = \widetilde{n}\alpha + \beta + \gamma$$

. A vector field ξ on $(\widetilde{\Theta}, g)$ is called torse-forming vector field (briefly,TFVF) if it fulfill the condition $\nabla_{\rho_1}\xi = \rho\rho_1 + \sigma(\rho_1)\xi$, where $\rho_1 \in T\Theta$, $\sigma(\rho_1)$ is a linear form and ρ is a function, [26]. In term of local transcription, it can be written as

$$\xi^l_{,i} = \rho\delta^l_i + \xi^l\sigma_i, \tag{4}$$

where ξ^l and σ_i are the components of ξ and σ respectively, and δ^l_i is the Kronecker symbol. A TFVF ξ is called [26]:

i) recurrent if ρ=0, i.e.,

$$\xi^l_{,i} = \xi^l\sigma_i, \tag{5}$$

ii) concircular if the σ_i is gradient covector (i.e., σ_i=$\sigma_{,i}$), i.e.,

$$\xi^l_{,i} = \rho\delta^l_i, \tag{6}$$

iii) convergent if it is concircular and ρ=const.exp(σ).

A $\varphi(\widetilde{Ric})$-vector field is a vector field on $(\widetilde{\Theta}^{\widetilde{n}}, g)$ with metric g and Levi-Civita connection $\widetilde{\nabla}$, which satisfies the condition [15]:

$$\widetilde{\nabla}\varphi = \theta\widetilde{Ric}, \tag{7}$$

where θ is a constant and $\widetilde{Ric}$ is the Ricci tensor. Obviously, when (Θ, g) is an Einstein space, the vector field φ is concircular. Moreover, if θ=0, the vector field φ is covariantly constant. Thus in our study we suppose that $\theta \neq 0$ and (Θ, g) is neither an Einstein space nor a vacuum solution of the Einstein equations. In a locally coordinate neighborhood $\mathscr{U}(\rho_1)$, the equation (7) takes the form

$$\varphi^l_{,i} = \theta\widetilde{\mathscr{R}}^l_i, \tag{8}$$

where φ^l and $\widetilde{\mathscr{R}}^l_i$ are components of φ and $\widetilde{Ric}$, respectively. In term of lowering indices, i.e.,

$$\varphi_{i,j} = \theta\widetilde{\mathscr{R}}_{ij}, \tag{9}$$

where $\varphi_i = \varphi^a g_{ia}$, $\quad \widetilde{\mathscr{R}}_{ij} = g_{ia}\widetilde{\mathscr{R}}^a_j$.

In the whole paper we use the concise as PFST:Perfect fluid space-time, CRW:Classical Robertson-Walker space-time, GRR:Generalized Ricci recurrent, QES: Quasi-Einstein space, RVF:Recurrent vector field, CVF: Concircular vector field and TEM:Total energy momentum.

In this work we characterize MG(QE)$_{\widetilde{n}}$ in view of the $\mathscr{Z}$-tensor. As follows: After introduction in Section 2, we deduce the basics results of $\mathscr{Z}$-tensor on MG(QE)$_{\widetilde{n}}$. Section 3 is devoted to the study of MG(QE)$_{\widetilde{n}}$ using the properties of $\mathscr{Z}$-tensor. In Section 4 we analysis MG(QE)-GRW space-times with $\mathscr{Z}$-tensor and find the crucial results. Finally, we construct an example of MG(QE)$_{\widetilde{4}}$.

2 $\mathscr{Z}$-Tensor on MG(QE)$_{\widetilde{n}}$

A generalized symmetric tensor of type $(0,2)$ on $(\widetilde{\Theta}, g)$ which is called $\mathscr{Z}$-tensor is given by [20]:

$$\mathscr{Z}_{ij} = \widetilde{\mathscr{R}}_{ij} + \phi g_{ij}, \tag{10}$$

where ϕ is an arbitrary scalar function. In Refs. [20,21, 22,23] various properties of the $\mathscr{Z}_{ij}$-tensor were pointed out. The classical $\mathscr{Z}$-tensor is obtained with the choice ϕ=$\frac{\tau}{\widetilde{n}}$. In particular cases, the $\mathscr{Z}$-tensor gives the several well known structures on $(\widetilde{\Theta}, g)$. For example, i) if $\mathscr{Z}_{ij}$=0 (i.e, $\mathscr{Z}$-flat), then this manifold reduces to an Einstein manifold [2]; ii) if $\nabla_k\mathscr{Z}_{ij}$=$\lambda_k\mathscr{Z}_{ij}$(i.e., $\mathscr{Z}$-recurrent), then this manifold reduces to a GRR manifold [12]; iii) if $\nabla_k\mathscr{Z}_{ij}$=$\nabla_i\mathscr{Z}_{kl}$ (i.e., Codazzi tensor), then we find $\nabla_k\widetilde{\mathscr{R}}_{ij}$-$\nabla_i\widetilde{\mathscr{R}}_{kj}$=$\frac{1}{2(\widetilde{n}-1)}(g_{ij}\nabla_k - g_{kj}\nabla_i)\tau$[7]. iv) The relation between the $\mathscr{Z}$-tensor and the energy-stress tensor of Einstein's equations with cosmological constant Γ is $\mathscr{Z}_{kj}$=$\widetilde{\kappa}\mathscr{T}^*{}_{kj}$[9], where ϕ=-$\frac{\tau}{2}$ + Γ and $\widetilde{\kappa}$ is the gravitational constant. In this case, the $\mathscr{Z}$-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ. The vacuum solution ($\mathscr{Z}$=0) determines an Einstein space Γ=$(\frac{(\widetilde{n}-2)}{2\widetilde{n}})\tau$; the conservation of TEM ($\nabla^l\mathscr{T}^*{}_{kl}$=0) gives ($\nabla_j\mathscr{T}^*{}_{kl}$=0) then this space-time gives the conserved energy-momentum density.

In a MG(QE)$_{\widetilde{n}}$, from the equation (10) we have

$$\mathscr{Z}_{ij} = (\alpha+\phi)g_{ij} + \beta\mathscr{T}_i\mathscr{T}_j + \gamma\mathscr{N}_i\mathscr{N}_j + \delta(\mathscr{T}_i\mathscr{N}_j + \mathscr{T}_j\mathscr{N}_i), \tag{11}$$

and scalar $\mathscr{Z}^\star$, one can get

$$\mathscr{Z}^\star = g^{ij}\mathscr{Z}_{ij} = (\alpha+\phi)\widetilde{n} + \beta + \gamma. \tag{12}$$

Also, from (11) we yields

$$\mathscr{T}^i\mathscr{T}^j\mathscr{Z}_{ij} = \alpha+\beta+\phi, \quad \mathscr{T}^i\mathscr{N}^j\mathscr{Z}_{ij} = \delta, \tag{13}$$

$$\mathscr{N}^i\mathscr{N}^j\mathscr{Z}_{ij} = \alpha+\gamma+\phi, \quad (\mathscr{T}^i\mathscr{T}^j - \mathscr{N}^i\mathscr{N}^j)\mathscr{Z}_{ij} = \beta-\gamma. \tag{14}$$

So we state that:

Proposition 1. *The trace of $\mathscr{Z}$-tensor on a MG(QE)$_{\widetilde{n}}$ with generators $\mathscr{T}$ and $\mathscr{N}$, is given by*

$$\mathscr{Z}^\star = (\widetilde{n}-1)\mathscr{T}^i\mathscr{T}^j\mathscr{Z}_{ij} + \mathscr{N}^i\mathscr{N}^j\mathscr{Z}_{ij} - \beta(\widetilde{n}-2)$$

or,

$$\mathscr{Z}^\star = (\widetilde{n}-1)\mathscr{N}^i\mathscr{N}^j\mathscr{Z}_{ij} + \mathscr{T}^i\mathscr{T}^j\mathscr{Z}_{ij} - \beta(\widetilde{n}-2)$$

Also, let $\mathscr{T}$ is an eigenvector of the $\mathscr{Z}$-tensor with eigenvalue λ_1, i.e., $\mathscr{T}^i\mathscr{Z}_{ij}$=$\lambda_1\mathscr{T}^j$. Then from (11), we get

$$\begin{aligned}\mathscr{T}^i\mathscr{Z}_{ij} &= (\alpha+\phi)\mathscr{T}^i g_{ij} + \beta\mathscr{T}^i\mathscr{T}_i\mathscr{T}_j + \gamma\mathscr{T}^i\mathscr{N}_i\mathscr{N}_j \\ &\quad + \delta(\mathscr{T}^i\mathscr{T}_i\mathscr{N}_j + \mathscr{T}^i\mathscr{T}_j\mathscr{N}_i),\end{aligned} \tag{15}$$

which implies that $(\lambda_1-\alpha-\phi-\beta)\mathscr{T}_j=\delta\mathscr{N}_j$, that is, $\delta=0$ and $\lambda_1=\alpha+\beta+\phi$. Similarly, for the eigenvector $\mathscr{N}$ corresponding to the eigenvalue λ_2, we have

$$\begin{aligned}\mathscr{N}^i\mathscr{Z}_{ij} &= (\alpha+\phi)\mathscr{N}^i g_{ij}+\beta\mathscr{N}^i\mathscr{T}_i\mathscr{T}_j+\gamma\mathscr{N}^i\mathscr{N}_i\mathscr{N}_j\\ &+\delta(\mathscr{N}^i\mathscr{T}_i\mathscr{N}_j+\mathscr{N}^i\mathscr{T}_j\mathscr{N}_i), \end{aligned}\tag{16}$$

which is equivalent to $(\lambda_2-\alpha-\gamma-\phi)\mathscr{N}_j=\delta\mathscr{N}_j$. In this sequel we get $\lambda_2=\alpha+\gamma+\phi$ and $\delta=0$. For conversely, we get from (2) that

$$\begin{aligned}\mathscr{T}^i\mathscr{Z}_{ij} &= (\alpha+\phi)\mathscr{T}^i g_{ij}+\beta\mathscr{T}^i\mathscr{T}_i\mathscr{T}_j+\gamma\mathscr{T}^i\mathscr{N}_i\mathscr{N}_j\\ &=(\alpha+\beta+\phi)\mathscr{T}_j\end{aligned}\tag{17}$$

Also,

$$\begin{aligned}\mathscr{N}^i\mathscr{Z}_{ij} &= (\alpha+\phi)\mathscr{N}^i g_{ij}+\beta\mathscr{N}^i\mathscr{T}_i\mathscr{T}_j+\gamma\mathscr{N}^i\mathscr{N}_i\mathscr{N}_j\\ &=(\alpha+\gamma+\phi)\mathscr{N}_j.\end{aligned}$$

Thus we set up the result:

Theorem 1. *If a MG(QE)$_{\widetilde{n}}$ admitting $\mathscr{Z}$-tensor then the manifolds reduces to G(QE)$_{\widetilde{n}}$ iff one of the generators is an eigenvector of the $\mathscr{Z}$-tensor.*

Again, taking covariant derivative of the equation (11), we have

$$\begin{aligned}\nabla_k\mathscr{Z}_{ij} &= (\nabla_k\phi)g_{ij}+\beta[(\nabla_k\mathscr{T}_i)\mathscr{T}_j+\mathscr{T}_i(\nabla_k\mathscr{T}_j)]\\ &+\gamma[(\nabla_k\mathscr{N}_i)\mathscr{N}_j+\mathscr{N}_i(\nabla_k\mathscr{N}_j)]\\ &+\delta[(\nabla_k\mathscr{T}_i)\mathscr{N}_j+\mathscr{T}_i(\nabla_k\mathscr{N}_j)+(\nabla_k\mathscr{T}_j)\mathscr{N}_i\\ &+\mathscr{T}_j(\nabla_k\mathscr{N}_i)],\end{aligned}\tag{18}$$

also we write

$$\begin{aligned}\nabla_i\mathscr{Z}_{kj} &= (\nabla_i\phi)g_{kj}+\beta[(\nabla_i\mathscr{T}_k)\mathscr{T}_j+\mathscr{T}_k(\nabla_i\mathscr{T}_j)]\\ &+\gamma[(\nabla_i\mathscr{N}_k)\mathscr{N}_j+\mathscr{N}_k(\nabla_i\mathscr{N}_j)]\\ &+\delta[(\nabla_i\mathscr{T}_k)\mathscr{N}_j+\mathscr{T}_k(\nabla_i\mathscr{N}_j)+(\nabla_i\mathscr{T}_j)\mathscr{N}_k\\ &+\mathscr{T}_j(\nabla_i\mathscr{N}_k)]\end{aligned}\tag{19}$$

So, the Codazzi deviation tensor $\mathscr{H}$ is given by

$$\begin{aligned}\mathscr{H}_{kij} &= \nabla_k\mathscr{Z}_{ij}-\nabla_i\mathscr{Z}_{kj}=(\nabla_k\phi)g_{ij}-(\nabla_i\phi)g_{kj}\\ &+\beta[(\nabla_k\mathscr{T}_i)\mathscr{T}_j+\mathscr{T}_i(\nabla_k\mathscr{T}_j)-(\nabla_i\mathscr{T}_k)\mathscr{T}_j-\mathscr{T}_k(\nabla_i\mathscr{T}_j)]\\ &+\gamma[(\nabla_k\mathscr{N}_i)\mathscr{N}_j+\mathscr{N}_i(\nabla_k\mathscr{N}_j)-(\nabla_i\mathscr{N}_k)\mathscr{N}_j-\mathscr{N}_k(\nabla_i\mathscr{N}_j)]\\ &+\delta[(\nabla_k\mathscr{T}_i)\mathscr{N}_j+\mathscr{T}_i(\nabla_k\mathscr{N}_j)+(\nabla_k\mathscr{T}_j)\mathscr{N}_i+\mathscr{T}_j(\nabla_k\mathscr{N}_i)]\\ &-\delta[(\nabla_i\mathscr{T}_k)\mathscr{N}_j+\mathscr{T}_k(\nabla_i\mathscr{N}_j)+(\nabla_i\mathscr{T}_j)\mathscr{N}_k+\mathscr{T}_j(\nabla_i\mathscr{N}_k)],\end{aligned}\tag{20}$$

which implies that

$$\begin{aligned}\mathscr{T}^j\mathscr{H}_{kij} &= (\nabla_k\phi)\mathscr{T}_i-(\nabla_i\phi)\mathscr{T}_k+\beta[(\nabla_k\mathscr{T}_i)-(\nabla_i\mathscr{T}_k)]\\ &+\delta[(\nabla_k\mathscr{N}_i)-(\nabla_i\mathscr{N}_k)].\end{aligned}$$

and

$$\begin{aligned}\mathscr{N}^j\mathscr{H}_{kij} &= (\nabla_k\phi)\mathscr{N}_i-(\nabla_i\phi)\mathscr{N}_k+\gamma[(\nabla_k\mathscr{N}_i)-(\nabla_i\mathscr{N}_k)]\\ &+\delta[(\nabla_k\mathscr{T}_i)-(\nabla_i\mathscr{T}_k)].\end{aligned}$$

So, we conclude that:

Theorem 2. *Let a MG(QE)$_{\widetilde{n}}$ is Einstein-like of class $\mathscr{D}$(i.e. the $\mathscr{Z}$-tensor is a Codazzi tensor), then the generators $\mathscr{T}$ and $\mathscr{N}$ are closed, provided ϕ=constant.*

3 Properties of $\mathscr{Z}$-tensor on MG(QE)$_{\widetilde{n}}$

Throughout this section, we obtain some interesting outcomes by using the $\mathscr{Z}$-tensor on a MG(QE)$_{\widetilde{n}}$. First, we examine the following findings.

Theorem 3. *The vector field ϕ_l and the RVF λ_l on a MG(QE)$_{\widetilde{n}}$ admitting $\mathscr{Z}$-tensor must be parallel and given by*

$$\phi_k=\left(\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}\right)\lambda_k.$$

Proof. If the $\mathscr{Z}$-tensor is recurrent with λ_k as the RVF. Then from (10) we have

$$\lambda_k\mathscr{Z}_{ij}=\widetilde{\mathscr{R}}_{ij,k}+\phi_k g_{ij},\tag{21}$$

Multiplying (21) by g^{ij}, we get

$$\lambda_k\mathscr{Z}^\star=\tau_{,k}+\widetilde{n}\phi_k.\tag{22}$$

Since τ must be constant so from (22) and (12) we yields

$$\lambda_k[(\alpha+\phi)\widetilde{n}+\beta+\gamma]=\widetilde{n}\phi_k,\tag{23}$$

which implies that

$$\phi_k=\left(\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}\right)\lambda_k.\tag{24}$$

Hence, the proof is finished.

Theorem 4.*If a MG(QE)$_{\widetilde{n}}$ admitting $\mathscr{Z}$-tensor, then $\frac{\tau}{\widetilde{n}}$ is an eigenvalue of the Ricci tensor $\widetilde{\mathscr{R}}_{jk}$ as the eigenvector ρ_1 given by $\lambda(\rho_1)$=$g(\rho_1,\upsilon)$.*

Proof. Let $\mathscr{Z}$-tensor is recurrent, we have

$$\mathscr{Z}_{ij,k}=\lambda_k\mathscr{Z}_{ij}.\tag{25}$$

Multiplying (25) by g^{ik}, we get

$$\mathscr{Z}^k_{j,k}=\lambda^k\mathscr{Z}_{jk}.\tag{26}$$

Using the Ricci identity, $\widetilde{\mathscr{R}}^k_{j,k}$=0. Then we have

$$\mathscr{Z}^k_{j,k}=\phi_j.\tag{27}$$

In view of (10),(26) and (27), we yields

$$\phi_j=\lambda^k(\widetilde{\mathscr{R}}_{jk}+\phi g_{jk}).\tag{28}$$

Using (24) in (28), we obtain

$$\lambda^k\widetilde{\mathscr{R}}_{jk}=\left(\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}\right)\lambda_j.\tag{29}$$

So the proof is done.

Theorem 5. *A necessary and sufficient condition for a vector field ϕ^k generated by the scalar function ϕ on a MG(QE)$_{\widetilde{n}}$ to be divergence-free is that the divergence of λ_k is of negative value and has the form*

$$\lambda^k_{,k} = -\|\lambda\|^2.$$

Proof. Taking the covariant derivative of (24), we get

$$\phi_{k,p} = \phi_p\lambda_k + [\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}]\lambda_{k,p}. \tag{30}$$

Again using (24) in (30), we have

$$\phi_{k,p} = [\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}](\lambda_k\lambda_p+\lambda_{k,p}). \tag{31}$$

Multiplying (31) by g^{kp}, we obtain

$$\phi^k_{,k} = [\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}](\|\lambda\|^2+\lambda^k_{,k}). \tag{32}$$

If the vector field ϕ_k is divergence-free and $\phi \neq -\frac{\tau}{\widetilde{n}}$, then (32) implies that

$$\lambda^k_{,k} = -\|\lambda\|^2. \tag{33}$$

Conversely, from (33) and (32) we get $\phi^k_{,k}$=0. Now the proof is finished.

Theorem 6. *If the vector field λ_k on a MG(QE)$_{\widetilde{n}}$ is divergence-free then the divergence of the vector field ϕ_k has the form*

$$\phi^k_{,k} = \frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\|\phi\|^2.$$

Proof. Since the relation (30) holds on a MG(QE)$_{\widetilde{n}}$, using (24) in (30), we get

$$\phi_{k,p} = \frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\phi_p\phi_k + \frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\lambda_{k,p} \tag{34}$$

Multiplying g^{kp} in (34), we obtain

$$\phi^k_{,k} = \frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\|\phi\|^2 + \frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\lambda^k_{,k}. \tag{35}$$

If the vector field λ_k is divergence-free then from (35) the divergence of the vector field ϕ_k has the form

$$\phi^k_{,k} = \frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\|\phi\|^2. \tag{36}$$

So, the proof is over out.

Theorem 7. *If a MG(QE)$_{\widetilde{n}}$ admits a TFVF as per the 1-form ϕ_k in the relation $\phi_{k,p}$=εg_{kp}+$\mu_p\phi_k$, then the vector field λ_k is also TFVF satisfies the following equation*

$$\lambda_{k,p} = \widetilde{\pi}g_{kp} + \widetilde{\pi}_p\lambda_k$$

where $\widetilde{\pi}$=$\frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}$ *and* $\widetilde{\pi}_p$=$(\mu_p-\lambda_p)$.

Proof. Let ϕ_k is a TFVF with a scalar function ε and a vector field μ_k. Then from (4) we have

$$\phi_{k,p} = \varepsilon g_{kp} + \mu_p\phi_k \tag{37}$$

By virtue of (30), we get

$$\phi_p\lambda_k + [\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}]\lambda_{k,p} = \varepsilon g_{kp} + \mu_p\phi_k \tag{38}$$

Again, using (24), we obtain

$$\lambda_{k,p} = \frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}g_{kp} + (\mu_p-\lambda_p)\lambda_k \tag{39}$$

In this case, let $\widetilde{\pi}$=$\frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}$ and $\widetilde{\pi}_p$=$(\mu_p-\lambda_p)$, then (39) takes the form

$$\lambda_{k,p} = \widetilde{\pi}g_{kp} + \widetilde{\pi}_p\lambda_k. \tag{40}$$

which implies that λ_k is a TFVF. Thus, the proof is finished. Next, we suppose that μ_p=λ_p, then (39) reduces to as

$$\lambda_{k,p} = \frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}g_{kp}. \tag{41}$$

For fix; $\widetilde{\pi}$=$\frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}$, we get from (41) that

$$\lambda_{k,p} = \widetilde{\pi}g_{kp}, \tag{42}$$

which implies that λ_k is a CVF. This leads to the result:

Corollary 1. *If a MG(QE)$_{\widetilde{n}}$ admits a TFVF in view of the 1-form ϕ_k in the relation $\phi_{k,p}$=εg_{kp}+$\mu_p\phi_k$, then λ_k forms a CVF given by $\lambda_{k,p}$=$\widetilde{\pi}g_{kp}$, where $\widetilde{\pi}$=$\frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}$.*

Theorem 8. *If ϕ_k of a MG(QE)$_{\widetilde{n}}$ is a CVF, then the vector field λ_k forms a TFVF given by (45).*

Proof. Let ϕ_k is a CVF with a scalar function ε then we have

$$\phi_{k,p} = \varepsilon g_{kp}. \tag{43}$$

Using (43) in (31), we get

$$\varepsilon g_{kp} = [\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}]\lambda_k\lambda_p + [\frac{(\alpha+\phi)\widetilde{n}+\beta+\gamma}{\widetilde{n}}]\lambda_{k,p} \tag{44}$$

which implies that

$$\lambda_{k,p} = [\frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}]g_{kp} - \lambda_k\lambda_p. \tag{45}$$

It notify that λ_k forms a TFVF. So the proof is over out.

Theorem 9. *If the vector field λ_k of a MG(QE)$_{\widetilde{n}}$ has constant length and the ϕ_k is a CVF, then the relation (47) holds.*

Proof. We suppose that λ_k is of constant length, i.e., $\lambda_k\lambda^k$=σ^2, then multiplying (50) by λ^k, we have

$$\lambda^k\lambda_{k,p} = [\frac{\widetilde{n}\varepsilon}{\widetilde{n}(\alpha+\phi)+\beta+\gamma} - \lambda^k\lambda_k]\lambda_p. \quad (46)$$

Since $\lambda^k\lambda_{k,p}$=0 due to constant length λ_k, then (46) implies that

$$\varepsilon = [\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}]\sigma^2. \quad (47)$$

where $\|\lambda\|$=σ. So the proof is finished.

Theorem 10. *If λ_k of a $MG(QE)_{\widetilde{n}}$ is a CVF in the form $\lambda_{k,p}$=εg_{kp}, then the vector field ϕ_k is a TFVF satisfies the relation (49).*

Proof. Let λ_k of a $MG(QE)_{\widetilde{n}}$ is a CVF. Then

$$\lambda_{k,p} = \varepsilon g_{kp}, \quad (48)$$

holds. In view of (48), equation (44) turn up

$$\phi_{k,p} = \frac{\widetilde{n}}{(\alpha+\phi)\widetilde{n}+\beta+\gamma}\phi_p\phi_k + \frac{\varepsilon\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}g_{kp}, \quad (49)$$

which means that ϕ_k is a TFVF. Therefore, the proof is finished.

Theorem 11. *If the vector field λ_k of a $MG(QE)_{\widetilde{n}}$ is a CVF in the form $\lambda_{k,p}$=εg_{kp} and ϕ_k has constant length, then the scalar function ε generating by λ_k has negative value and it satisfies the equation (52).*

Proof. Suppose that λ_k be a CVF and the vector field ϕ_k is of constant length. Then multiplying (49) by ϕ^k, we have

$$\phi^k\phi_{k,p} = \frac{\widetilde{n}}{(\alpha+\phi)\widetilde{n}+\beta+\gamma}\phi^k\phi_p\phi_k + \frac{\varepsilon\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\phi_p. \quad (50)$$

Since $\phi^k\phi_{k,p}$=0 and using $\|\phi\|$=σ, equation (50) reduces

$$0 = \frac{\widetilde{n}}{(\alpha+\phi)\widetilde{n}+\beta+\gamma}\phi_p + \frac{\varepsilon\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\phi_p, \quad (51)$$

which implies that

$$\varepsilon = \left(\frac{\widetilde{n}\sigma}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)^2. \quad (52)$$

Therefore, the proof is turn out.

Theorem 12. *If the vector field ϕ_k of a $MG(QE)_{\widetilde{n}}$ is a RVF in the $\phi_{k,p}$=$\rho_p\phi_k$, then the vector field λ_k is also RVF in the form $\lambda_{k,p}$=$(\rho_p-\lambda_p)\lambda_k$.*

Proof. Let ϕ_k is RVF, i.e.,

$$\phi_{k,p} = \rho_p\phi_k \quad (53)$$

In view of (53) and (30), we get

$$\rho_p\phi_k = \phi_p\lambda_k + [\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}]\lambda_{k,p}. \quad (54)$$

With the help of (24), equation (54) implies that

$$\lambda_{k,p} = (\rho_p - \lambda_p)\lambda_k. \quad (55)$$

Thus, the proof is over out. Also, if ρ_p=λ_p, then from (55), we get $\lambda_{k,p}$=0 which means the vector field λ_k is covariantly constant. Conversely, if the relation $\lambda_{k,p}$=0 is holds on MG(QE)$_{\widetilde{n}}$. Then from (55) we have ρ_p=λ_p. Similarly, let the λ_k has constant length then multiplying the equation (55) by λ^k, we have ρ_p=λ_p. The converse is also true. Thus we have:

Corollary 2. *A RVF ϕ_p with the RVF ρ_p of MG(QE)$_{\widetilde{n}}$ admits the relation ρ_p=λ_p iff the vector field λ_p is covariantly constant, or is of constant length.*

Theorem 13. *If the vector field λ_k on a $MG(QE)_{\widetilde{n}}$ be a $\lambda\widetilde{Ric}$ vector field then necessary and sufficient condition for a vector field ϕ_k to be divergence free is that the scalar function θ to be in the form*

$$\theta = -\left(\frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)^2\frac{\|\phi\|^2}{\tau}.$$

Proof. Let λ_k is a $\lambda\widetilde{Ric}$ vector field then from (7), we obtain

$$\lambda_{k,p} = \theta\widetilde{\mathscr{R}}_{kp}, \quad (56)$$

In view of (56), equation (34) reduces

$$\phi_{k,p} = \frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\phi_p\phi_k + \theta\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\widetilde{\mathscr{R}}_{kp} \quad (57)$$

Multiplying (57) by g^{kp}, we yields

$$\phi^k_{,k} = \frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\|\phi\|^2 + \theta\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\tau. \quad (58)$$

If the vector field ϕ_k is divergence-free, then (58) takes the form

$$\theta = -\left(\frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)^2\frac{\|\phi\|^2}{\tau}. \quad (59)$$

Conversely, statement is obvious from (59) and (58). Thus the proof is finished.
Again, in view of Theorem 13, if ϕ_k is of constant length. Then multiplying (57) by ϕ^k we have

$$\phi^k\phi_{k,p} = \frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\|\phi\|^2\phi_p + \theta\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\phi^k\widetilde{\mathscr{R}}_{kp} \quad (60)$$

Since $\phi^k\phi_{k,p}$=0, so equation (60) implies that

$$\phi^k\widetilde{\mathscr{R}}_{kp} = -\frac{1}{\theta}\left(\frac{\widetilde{n}\|\phi\|}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)^2\phi_p. \quad (61)$$

In view of (61), we achieve the following:

Corollary 3. *If λ_k is a $\lambda\widetilde{Ric}$ vector field related by $\lambda_{k,p}=\theta\widetilde{\mathscr{R}}_{kp}$ and the vector field ϕ_k of a $MG(QE)_{\widetilde{n}}$ has constant length, then the value $-\frac{1}{\theta}\left(\frac{\widetilde{n}\|\phi\|}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)^2$ is an eigenvalue of the Ricci tensor $\widetilde{\mathscr{R}}_{kp}$ as per the eigenvector ρ_1 defined by $\mathscr{T}(\rho_1)=g(\rho_1,\delta)$.*

Theorem 14. *If λ_k is a $\lambda\widetilde{Ric}$ vector field given by $\lambda_{k,p}=\theta\widetilde{\mathscr{R}}_{kp}$ and the vector field ϕ_k of $MG(QE)_{\widetilde{n}}$ is a CVF, then the manifold reduces to a QE manifold.*

Proof. Let λ_k is a $\lambda\widetilde{Ric}$ vector field, that is $\lambda_{k,p}=\theta\widetilde{\mathscr{R}}_{kp}$ and ϕ_k of a $MG(QE)_{\widetilde{n}}$ is a CVF. Then from (57) and (43), we have

$$\varepsilon g_{kp}=\frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\phi_p\phi_k+\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\theta\widetilde{\mathscr{R}}_{kp}, \tag{62}$$

From (62), one can find

$$\widetilde{\mathscr{R}}_{kp}=\left(\frac{\varepsilon\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)\frac{1}{\theta}g_{kp}-\frac{1}{\theta}\left(\frac{\varepsilon\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)^2\phi_p\phi_k, \tag{63}$$

which implies that

$$\widetilde{\mathscr{R}}_{kp}=\upsilon_1 g_{kp}+\upsilon_2\phi_p\phi_k. \tag{64}$$

where $\upsilon_1=\left(\frac{\varepsilon\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)\frac{1}{\theta}$, $\upsilon_2=-\frac{1}{\theta}\left(\frac{\varepsilon\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\right)^2$. Thus, a MG(QE)$_{\widetilde{n}}$ reduces to a QE manifold. So, the Theorem 14 is finished.

Theorem 15. *If the vector fields λ_k and ϕ_k of a $MG(QE)_{\widetilde{n}}$ are $\lambda\widetilde{Ric}$ and $\phi\widetilde{Ric}$ vector fields in the forms $\lambda_{k,p}=\theta\widetilde{\mathscr{R}}_{kp}$ and $\phi_{k,p}=\nu\widetilde{\mathscr{R}}_{kp}$, respectively. Then the manifold reduces to a QE manifold.*

Proof. Let λ_k and ϕ_k of a MG(QE)$_{\widetilde{n}}$ are $\lambda\widetilde{Ric}$ and $\phi\widetilde{Ric}$ vector fields. Then from (7) that

$$\lambda_{k,p}=\theta\widetilde{\mathscr{R}}_{kp},\ \text{and}\ \phi_{k,p}=\nu\widetilde{\mathscr{R}}_{kp} \tag{65}$$

In view of (65), equation (34) can be written as

$$\nu\widetilde{\mathscr{R}}_{kp}=\frac{\widetilde{n}}{\widetilde{n}(\alpha+\phi)+\beta+\gamma}\phi_p\phi_k+\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\theta\widetilde{\mathscr{R}}_{kp}, \tag{66}$$

which indicate that

$$\widetilde{\mathscr{R}}_{kp}=\vartheta\phi_p\phi_k, \tag{67}$$

where $\vartheta=\frac{\widetilde{n}^2}{\{\widetilde{n}(\alpha+\phi)+\beta+\gamma\}\{\nu\widetilde{n}-\theta(\widetilde{n}((\alpha+\phi)+\beta+\gamma)\}}$ and $\widetilde{n}(\alpha+\phi)+\beta+\gamma\neq 0$,
$\nu\neq\frac{\theta}{\widetilde{n}}[\widetilde{n}(\alpha+\phi)+\beta+\gamma]$. Therefore, the proof is over out.

Theorem 16. *If the vector fields λ_k and ϕ_k of a $MG(QE)_{\widetilde{n}}$ are $\lambda\widetilde{Ric}$ and $\phi\widetilde{Ric}$ vector fields in the forms $\lambda_{k,p}=\theta\widetilde{\mathscr{R}}_{kp}$ and $\phi_{k,p}=\nu\widetilde{\mathscr{R}}_{kp}$, respectively. Then the eigenvalue determined by the vector field θ_l and ν_l are equal to τ.*

Proof. From (65) and (31), we get

$$\widetilde{\mathscr{R}}_{kp}\left[\nu-\theta\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)\right]=\left[\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right]\lambda_k\lambda_p \tag{68}$$

After taking the covariant derivative of (68) and by the use of (24), we obtain

$$\left[\nu_l-\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)(\lambda_l\theta+\theta_l)\right]\widetilde{\mathscr{R}}_{kp}+\left[\nu-\theta\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)\right]\widetilde{\mathscr{R}}_{kp,l}=\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)[\theta(\lambda_k\widetilde{\mathscr{R}}_{pl}+\lambda_p\widetilde{\mathscr{R}}_{kl})+\lambda_k\lambda_p\lambda_l] \tag{69}$$

On multiplying (69) by g^{kp}, we yields

$$\left[\nu_l-\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)\theta_l\right]\tau=\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)[2\theta(\lambda^k\widetilde{\mathscr{R}}_{kl}+\|\lambda\|^2\lambda_l+\lambda_l\theta\tau] \tag{70}$$

Next, multiplying (70) by g^{lk}, we have

$$\left[\nu^k\widetilde{\mathscr{R}}_{kl}-\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)\theta^k\widetilde{\mathscr{R}}_{kl}\right]=\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)[2\theta\lambda^k\widetilde{\mathscr{R}}_{kl}+\|\lambda\|^2\lambda_l+\lambda_l\theta\tau] \tag{71}$$

Subtracting (70) from (71), we get

$$\nu^k\widetilde{\mathscr{R}}_{kl}-\nu_l\tau=\left(\frac{\widetilde{n}(\alpha+\phi)+\beta+\gamma}{\widetilde{n}}\right)(\theta^k\widetilde{\mathscr{R}}_{kl}-\theta_l\tau). \tag{72}$$

So the proof is finished.
Also, we get from (12) that

$$\mathscr{Z}^{\star}{}_{,k}=\widetilde{n}\phi_{,k}. \tag{73}$$

Again taking the covariant derivative of (73), we have

$$\mathscr{Z}^{\star}{}_{,kp}=\widetilde{n}\phi_{,kp}. \tag{74}$$

If the vector field ϕ_k of a MG(QE)$_{\widetilde{n}}$ is a $\phi\widetilde{Ric}$, that is, $\phi_{k,p}=\nu\widetilde{\mathscr{R}}_{kp}$, then (74) have the form

$$\mathscr{Z}^{\star}{}_{,kp}=\widetilde{n}\nu\widetilde{\mathscr{R}}_{kp}. \tag{75}$$

After multiplying (75) by g^{kp}, we get

$$g^{kp}\mathscr{Z}^{\star}{}_{,kp}=\nabla\mathscr{Z}^{\star}=\widetilde{n}\nu(\alpha\widetilde{n}+\beta+\gamma). \tag{76}$$

So, we state:

Corollary 4. *If ϕ_k of a MG(QE)$_{\tilde{n}}$ is a $\phi\widetilde{Ric}$ vector field in the form $\phi_{k,p}$=$\nu\widetilde{\mathscr{R}}_{kp}$, then the Laplacian of the trace function of the $\mathscr{Z}$-tensor is give by the relation*

$$\nabla\mathscr{Z}^{\star}=\tilde{n}\nu(\alpha\tilde{n}+\beta+\gamma).$$

Also, let ϕ_k of a MG(QE)$_{\tilde{n}}$ is a $\phi\widetilde{Ric}$, that is, $\phi_{k,p}$=$\nu\widetilde{\mathscr{R}}_{kp}$, then from (31), we yields

$$\nu\widetilde{\mathscr{R}}_{kp}=[\frac{\tilde{n}(\alpha+\phi)+\beta+\gamma}{\tilde{n}}](\lambda_k\lambda_p+\lambda_{k,p}) \tag{77}$$

After multiplying (77) by g^{kp}, it gives

$$\nu\tau=[\frac{\tilde{n}(\alpha+\phi)+\beta+\gamma}{\tilde{n}}](\|\lambda\|^2+\lambda^k_{,k}) \tag{78}$$

which implies that

$$\tau=[\frac{\tilde{n}(\alpha+\phi)+\beta+\gamma}{\tilde{n}\nu}]\varsigma, \tag{79}$$

where ς=$(\|\lambda\|^2+\lambda^k_{,k})$ So, we obtain the next corollary as

Corollary 5. *If ϕ_k of a G(QE)$_{\tilde{n}}$ is a $\phi\widetilde{Ric}$ vector field in the form $\phi_{k,p}$=$\nu\widetilde{\mathscr{R}}_{kp}$, then the scalar curvature satisfies the relation*

$$\tau=[\frac{\tilde{n}(\alpha+\phi)+\beta+\gamma}{\tilde{n}\nu}]\varsigma.$$

4 MG(QE) GRW space-times

A Lorentzian manifold Θ is a *GRW* space-time iff Θ has a unit time-like vector field v_i such that

$$\nabla_k v_j=\varpi(g_{kj}+v_iv_j), \tag{80}$$

which is also an eigenvector of the Ricci tensor, i.e., $\widetilde{\mathscr{R}}_{ij}v^i$=$\zeta v_j$ for some scalar functions ϖ and ζ [24,16,18]. On contracting (10) by v^i yields

$$v^i\mathscr{Z}_{ij}=v^i\widetilde{\mathscr{R}}_{ij}+\phi v^ig_{ij}=(\zeta+\phi)v_j. \tag{81}$$

On contracting (11) by v^i and using (81) we have

$$(\zeta-\alpha)v_j=[\beta(v^i\mathscr{T}_i)+\delta(v^i\mathscr{N}_i)]\mathscr{T}_J+[\gamma(v^i\mathscr{N}_i)+\delta(v^i\mathscr{T}_i)]\mathscr{N}_J. \tag{82}$$

Again taking contractions (82) by two different generators it yields

$$\delta(v^i\mathscr{N}_i)=(\zeta-\alpha-\beta)(v^i\mathscr{T}_i),\ \delta(v^i\mathscr{T}_i)=(\zeta-\alpha)(v^i\mathscr{N}_i). \tag{83}$$

So, in view of (83), equation (82) takes the form

$$(\zeta-\alpha)[(v_j-(v^i\mathscr{T}_i)\mathscr{T}_j-(v^i\mathscr{N}_i)\mathscr{N}_j]=\gamma(v^i\mathscr{N}_i)\mathscr{N}_j. \tag{84}$$

It is obvious that v_j is not a linear combination of the generators only since v^i is time-like. If $\mathscr{T}^i$ and $\mathscr{N}^i$ orthonormal space-like fields, then ζ=α. Thus from (83), we obtain γ=δ=β=0, if $(v^i\mathscr{T}_i)\neq 0$, $(v^i\mathscr{N}_i)\neq 0$, it means Θ is an Einstein. If $\gamma\neq 0$, then from (83) and (84), we conclude that $(v^i\mathscr{N}_i)$=$(v^i\mathscr{T}_i)$=0. Thus we state:

Theorem 17. *Let Θ be a MG(QE) GRW space-time admitting $\mathscr{Z}$-tensor. Then $v^i\widetilde{\mathscr{R}}_{ij}$=$\alpha v_j$, that is, α is the eigenvalue of the eigenvector v^i and Θ reduces to be Einstein spacetime if v^i is orthogonal to both the generators provided $\gamma\neq 0$.*

Finally, if ϕ=constant, $\gamma\neq 0$ and v^i is orthogonal to both the generators then contraction (18) by v^i we have

$$\nabla_k(v^i\mathscr{Z}_{ij})-\mathscr{Z}_{ij}(\nabla_kv^i)=0. \tag{85}$$

Using (80) and (81) in (85), we get

$$(\zeta+\phi)(\nabla_kv_j)-\varpi\delta^i_k\mathscr{Z}_{ij}-\varpi v_k(v^i\mathscr{Z}_{ij})=0,$$

$$(\zeta+\phi)[\varpi(g_{kj}+v_kv_j]-\varpi\mathscr{Z}_{kj}-\varpi v_k(\zeta+\phi)v_j=0,$$

which implies that

$$\varpi[(\zeta+\phi)g_{kj}-\mathscr{Z}_{kj}]=0.$$

Thus we conclude that Θ is Einstein if $\varpi\neq 0$. So, we state that

Theorem 18. *Let Θ be a MG(QE) Lorentzian manifold admitting $\mathscr{Z}$-tensor with a unit time-like non-trivial TFVF. Then Θ reduces to an Einstein GRW space-time, or a perfect fluid GRW space-time, provided ϕ=constant.*

5 Example of MG(QE)$_4$ space-times

We define g on Lorentzian manifold (Θ^4,g) as follows

$$ds^2=g_{ij}dx^idy^j=(1+4e^{x^2})[(dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2], \tag{86}$$

where x^1, x^2, x^3, x^4 are standard coordinates of Θ^4, i,j=$1,2,3,4$. Here, the signature of g is $(+,+,+,-)$, which is Lorentzian. Then the non-vanishing components of the Christoffel symbols, the curvature tensor are

$$\Gamma^2_{11}=\Gamma^2_{33}=-\frac{2e^{x^2}}{(1+4e^{x^2})},\ \Gamma^2_{22}=\Gamma^2_{44}=\Gamma^1_{12}=\Gamma^3_{23}=\frac{2e^{x^2}}{(1+4e^{x^2})}.$$

$$\mathscr{R}_{1221}=\mathscr{R}_{2332}=\frac{4e^{x^2}(2+e^{x^2})}{(1+4e^{x^2})},\mathscr{R}_{1331}=\frac{4e^{x^2}}{(1+4e^{x^2})}$$

$$\mathscr{R}_{3443}=\mathscr{R}_{1441}=-\frac{4(e^{x^2})^2}{(1+4e^{x^2})}$$

$$\mathscr{R}_{2442}=-\frac{4e^{x^2}(2+e^{x^2})}{(1+4e^{x^2})}$$

Also the non-vanishing components of the Ricci tensors $\widetilde{\mathscr{R}}_{ij}$ are:

$$\widetilde{\mathscr{R}}_{11}=\frac{4e^{x^2}}{(1+4e^{x^2})},\ \widetilde{\mathscr{R}}_{22}=\widetilde{\mathscr{R}}_{33}=\frac{8e^{x^2}}{(1+4e^{x^2})^2}$$

$$\widetilde{\mathscr{R}}_{44}=-\frac{8e^{x^2}}{(1+4e^{x^2})}. \tag{87}$$

Now, the scalar curvature τ of $(\widetilde{\mathbb{R}}^4, g)$ is $\tau = \frac{4e^{x^2}(3-4e^{x^2})}{(1+4e^{x^2})^3} \neq 0$. We define the associated scalars α, β, γ and δ as

$$\alpha = \frac{e^{x^2}}{(1+4e^{x^2})^2}, \quad \beta = \frac{3e^{x^2}}{(1+4e^{x^2})^2},$$
$$\gamma = \frac{2e^{x^2}}{(1+4e^{x^2})^2}, \quad \delta = \frac{-e^{x^2}}{(1+4e^{x^2})^2}.$$

Also the 1-form

$$\mathscr{T}_i(x) = \mathscr{N}_i = \begin{cases} \sqrt{1+4e^{x^2}}, & \text{if } i = 1. \\ 0, & \text{it } i = 2,3,4. \end{cases}$$

at any point $x \in \widetilde{\mathbb{R}}^4$. Then (3) gives

$$\widetilde{\mathscr{R}}_{11} = \alpha g_{11} + \beta \mathscr{T}_1\mathscr{T}_1 + \gamma \mathscr{N}_1\mathscr{N}_1 + \delta(\mathscr{T}_1\mathscr{N}_1 + \mathscr{T}_1\mathscr{N}_1), \tag{88}$$

$$\widetilde{\mathscr{R}}_{22} = \alpha g_{22} + \beta \mathscr{T}_2\mathscr{T}_2 + \gamma \mathscr{N}_2\mathscr{N}_2 + \delta(\mathscr{T}_2\mathscr{N}_2 + \mathscr{T}_2\mathscr{N}_2), \tag{89}$$

$$\widetilde{\mathscr{R}}_{33} = \alpha g_{33} + \beta \mathscr{T}_3\mathscr{T}_3 + \gamma \mathscr{N}_3\mathscr{N}_3 + \delta(\mathscr{T}_3\mathscr{N}_3 + \mathscr{T}_3\mathscr{N}_3). \tag{90}$$

$$\widetilde{\mathscr{R}}_{44} = \alpha g_{44} + \beta \mathscr{T}_4\mathscr{T}_4 + \gamma \mathscr{N}_4\mathscr{N}_4 + \delta(\mathscr{T}_4\mathscr{N}_4 + \mathscr{T}_4\mathscr{N}_4). \tag{91}$$

Now the R.H.S. of (88)

$$\begin{aligned} &= \alpha g_{11} + \beta \mathscr{T}_1\mathscr{T}_1 + \gamma \mathscr{N}_1\mathscr{N}_1 \\ &+ \delta(\mathscr{T}_1\mathscr{N}_1 + \mathscr{T}_1\mathscr{N}_1) \\ &= \frac{4e^{x^2}}{(1+4e^{x^2})} \\ &= \widetilde{\mathscr{R}}_{11} \\ &= \text{L.H.S. of (88)}. \end{aligned}$$

By same fashion we can also verify (89), (90) and (91). Hence, $(\mathbb{R}^4, g)$ is a MG(QE)$_4$.
Next we define a scalar function ϕ in (10) as $\phi = \frac{1}{1+4e^{x^2}}$. Thus the nonvanishing components of the $\mathscr{Z}$-tensor $\mathscr{Z}_{ij}$ as

$$\mathscr{Z}_{11} = \frac{8e^{x^2}+1}{(1+4e^{x^2})}, \quad \mathscr{Z}_{22} = \mathscr{Z}_{33} = \frac{16(e^{x^2}+1)+1}{(1+4e^{x^2})^2},$$
$$\mathscr{Z}_{44} = \frac{16e^{x^2}+1}{(1+4e^{x^2})^2}.$$

$$\begin{aligned} \text{Now the R.H.S. of (11)} &= (\alpha+\phi)g_{11} + \beta \mathscr{T}_1\mathscr{T}_1 + \gamma \mathscr{N}_1\mathscr{N}_1 \\ &+ \delta(\mathscr{T}_1\mathscr{N}_1 + \mathscr{T}_1\mathscr{N}_1) \\ &= \frac{8e^{x^2}+1}{(1+4e^{x^2})} \\ &= \mathscr{Z}_{11} \\ &= \text{L.H.S.of(11)} \end{aligned}$$

Thus we ensure the following result.

Theorem 19. *Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian manifold with metric g given by*

$$ds^2 = g_{ij}dx^i dy^j = (1+4e^{x^2})[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2],$$

where i=1,2,3,4. Then $(\mathbb{R}^4, g)$ is a M(GQE)$_4$ space-times admitting the $\mathscr{Z}$-tensor.

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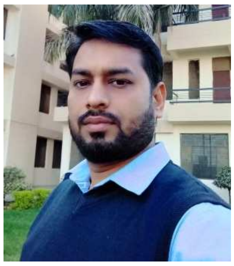

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