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On the Fine Structure of Spectra of Upper Triangular Double-Band Matrices as Operators on ℓ_p Spaces

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Abstract: In the present paper, we study the fine structure of spectra of infinite upper triangular double-band matrices as operators on ℓ_p, where $1 \leq p < \infty$. Three methods for classifying the spectrum are considered. Moreover, the obtained results are used to study the eigenvalue problem associated with certain infinite matrices. Our results improve and generalize many known results in the current literature.

Keywords: Spectrum, Infinite matrices, Sequence spaces, Eigenvalue problem

1. Introduction and preliminaries

Several authors have studied the fine structure of spectra of linear operators defined by some particular infinite matrices as operators over sequence spaces [1–20]

Karakaya and Altun [21] have studied the fine spectra of upper triangular double-band matrices over the sequences spaces c_0 and c; see also [22]. Recently Fathi and Lashkaripour [23] have studied the fine structure of the spectra of upper triangular double-band matrices as operators over the sequence space ℓ_1. Very recently Karaisa [24, 25] have studied the fine structure of spectra of the upper triangular double-band matrices as operators over the sequence space ℓ_p, where $1 \leq p < \infty$. All the results in [21–25] have been given under strong conditions that must be fulfilled for the matrices under consideration.

In this paper, by omitting all the conditions on the matrices, we obtain new results in the general case. Our results not only improve and generalize the results of [23–25], but also give results for some more operators. Results are illustrated by considering the eigenvalue problem associated with certain infinite matrices. A similar treatment can be given to generalize and improve the results of [21, 22].

Let X be a complex Banach space and T a bounded linear operator with domain $D(T)$ and range $R(T)$ in X. By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$, for all $f \in X^*$ and $x \in X$. With T we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator with domain $D(T)$. All of the points λ in the complex plane $\mathbb{C}$ are divided into two mutually exclusive and complementary sets: the *resolvent set* $\rho(T,X)$ and the *spectrum* $\sigma(T,X)$. The set $\rho(T,X)$ consists of all $\lambda \in \mathbb{C}$ for which the following conditions are satisfied:

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X.

The spectrum $\sigma(T,X)$ is the complement of $\rho(T,X)$ in the complex plane $\mathbb{C}$.

It is useful to make a finer classification of points by subdividing $\sigma(T,X)$ in some way. One such method of subdivision is well-known; the spectrum $\sigma(T,X)$ can be analyzed into three disjoint sets as follows:

The point (*discrete*) *spectrum* $\sigma_p(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

The continuous spectrum $\sigma_c(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.

The residual spectrum $\sigma_r(T,X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists (and may be bounded or not) but does

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not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.

Therefore, these three subspectra form a disjoint subdivision

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \quad (1)$$

This subdivision is the customary subdivision (see, for example, Stone [26], where the definitions are given in the context of Hilbert spaces). An advantage of this classification is the division of the spectrum into disjoint sets. It is based on the consideration of $T-\lambda I$. Let us say that some of the sets in the above definition may be empty. For instance, we may have $\sigma_c(T,X) = \sigma_r(T,X) = \varnothing$ even if X is infinite dimensional space (see Example 1).

Another classification of the spectrum is also considered. Following Taylor and Halberg [27, 28], a linear operator T with domain and range in a normed space X, is classified *I*, *II* or *III*, according as its range, $R(T)$, is all of X; is not all of X, but is dense in X; or is not dense in X. In addition T is classified 1, 2 or 3 according as T^{-1} exists and is continuous; exists, but is not continuous; or does not exist. The state of an operator is the combination of its Roman and Arabic numerical classifications and is denoted by the Roman numeral with the Arabic numeral as a subscript (*cf.* [27], p.94, [28], p.235-236).

For a bounded linear operator T on a complex Banach space X, we partition the complex plane into subsets corresponding to the states of the operator $T-\lambda I$. For example, the subset consisting of those λ for which the state of the operator $T-\lambda I$ is II_3 will be denoted by $II_3(T,X)$. Thus the resolvent set, $\rho(T,X)$, of the operator T consists of the union of $I_1(T,X)$ and $II_1(T,X)$, the point spectrum consists of the union of $I_3(T,X)$, $II_3(T,X)$ and $III_3(T,X)$, the residual spectrum consists of the union of $III_1(T,X)$ and $III_2(T,X)$ and the continuous spectrum consists of $II_2(T,X)$ (*cf.* [27], p.109, [28], p. 264).

Following Appel et al. [29], three more subdivisions of the spectrum can be defined; the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X a *Weyl sequence* for T if $\|x_\kappa\| = 1$ and $\|Tx_\kappa\| \to 0$, as $\kappa \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T,X) := \{\lambda \in \mathbb{C} : \exists \text{ a Weyl sequence for } \lambda I - T\} \quad (2)$$

the approximate point spectrum of T. Moreover, the subspectrum

$$\sigma_\delta(T,X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \quad (3)$$

is called *defect spectrum* of T. The two subspectra given by (2) and (3) form a (not necessarily disjoint) subdivision of the spectrum, that is

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X).$$

There is another subspectrum

$$\sigma_{co}(T,X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\}, \quad (4)$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$$

of the spectrum. Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, comparing these subspectra with those in (1) we note that

$$\sigma_r(T,X) = \sigma_{co}(T,X) \setminus \sigma_p(T,X)$$

$$\sigma_c(T,X) = \sigma(T,X) \setminus [\sigma_p(T,X) \cup \sigma_{co}(T,X)].$$

Next proposition is required in our study.

Proposition 1. *[29] The following statements hold:*
(a) $\sigma(T^*,X^*) = \sigma(T,X)$,
(b) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$,
(c) $\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$,
(d) $\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$,
(e) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$,
(f) $\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$,
(g) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*)$
$= \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. By $\ell_1$and ℓ_p we denote the spaces of all absolutely summable sequences and p-absolutely summable sequences, which are the Banach spaces with the norms $\|x\| = \sum_k |x_k|$ and $\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$, where $1 < p < \infty$, respectively. Also we write ℓ_∞, c and c_0 for the spaces of all abounded, convergent and null sequences, which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$, where $\mathbb{N} = \{0,1,2,\cdots\}$.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ an infinite matrix of real or complex numbers a_{nk}, where $n,\ \kappa \in \mathbb{N}$. We say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \ , \ (n \in \mathbb{N}).$$

Now, let (a_k) and (b_k) be two convergent sequences of nonzero real numbers with

$$\lim_{k\to\infty} a_k = a$$

and

$$\lim_{k\to\infty} b_k = b \neq 0.$$

We define the operator $\triangle^{ab}$ on the sequence space ℓ_p, where $1 \leq p < \infty$ as follows:

$$\triangle^{ab}x = \triangle^{ab}(x_k) = (a_k\, x_k + b_k\, x_{k+1}\,)_{k=0}^{\infty}.$$

Clearly, the operator $\triangle^{ab}$ can be represented by the upper triangular double-band matrix $A = \triangle^{ab}$, where

$$A = \begin{bmatrix} a_0 & b_0 & 0 & \cdots \\ 0 & a_1 & b_1 & \cdots \\ 0 & 0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In this paper, the spectrum and the fine structure of spectrum of $\triangle^{ab}$ on the sequence space ℓ_p, where $1 \leq p < \infty$ are considered. In Section 2, the spectrum and the Stone's classification of the spectrum are given. These are the main results of the paper. Some additional results concerning other classifications of the spectrum are given in Section 3. In Section 4, it may be helpful to provide some illustrative examples to support the results. We make use of the main results in Section 2 to study the eigenvalue problem

$$\triangle^{ab}x = \lambda x, \tag{5}$$

where $\lambda \in \mathbb{C}$ and $x \in \ell_p$. We can find, in general, all the values of λ for which *Eq*. (5) has nontrivial solutions. Section 5 briefly concludes suggestions for future work.

2. Main results

In this section we focus on the spectrum and the fine structure of spectrum of the operator $\triangle^{ab}$ on ℓ_p with respect to the first classification of Stone.

The following theorem is one of the main results, which gives the bounded linearity of the operator $\triangle^{ab}$ on ℓ_p.

Theorem 1. *The operator $\triangle^{ab}: \ell_p \longrightarrow \ell_p$ is a bounded linear operator satisfying the inequality*

$$\left\|\triangle^{ab}\right\|_{\ell_p} \leq sup_k\, |a_k| + sup_k\, |b_k|,$$

for $1 < p < \infty$ and

$$\left\|\triangle^{ab}\right\|_{\ell_1} = sup_k\,(|a_k| + |b_{k-1}|)\,.$$

Proof. Firstly, we consider the case for which $1 < p < \infty$. The linearity of $\triangle^{ab}$ is trivial and so is omitted. Let us take any $x = (x_k) \in \ell_p$. Then

$$\left\|\triangle^{ab}x\right\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |a_k x_k + b_k x_{k+1}|^p\right)^{\frac{1}{p}}.$$

Therefore, by Minkowski's inequality, we have

$$\begin{aligned}\left\|\triangle^{ab}x\right\|_{\ell_p} &\leq \left(\sum_{k=0}^{\infty} |a_k x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{\kappa=0}^{\infty} |b_\kappa x_{\kappa+1}|^p\right)^{\frac{1}{p}} \\ &\leq sup_k\,|a_\kappa| \left(\sum_{\kappa=0}^{\infty} |x_\kappa|^p\right)^{\frac{1}{p}} + sup_k\,|b_\kappa| \left(\sum_{\kappa=0}^{\infty} |x_\kappa|^p\right)^{\frac{1}{p}} \\ &= (sup_k\,|a_\kappa| + sup_k\,|b_\kappa|)\,\|x\|_{\ell_p}.\end{aligned}$$

Thus,

$$\left\|\triangle^{ab}\right\|_{\ell_p} \leq sup_k\,|a_k| + sup_k\,|b_k|\,.$$

This completes the proof where $1 < p < \infty$. Analogously, we can deal with the case $p = 1$. ∎

Remark 1. One can prove that

$$\left\|\triangle^{ab}\right\|_{\ell_p} \geq \sup_k\,(|a_k|^p + |b_{k-1}|^p)^{\frac{1}{p}} \quad ,$$

for $1 < p < \infty$.

If $T: \ell_p \to \ell_p$, where $1 \leq p < \infty$, is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_p^* \longrightarrow \ell_p^*$ is defined by the transpose of the matrix A. It is well-known that the dual space ℓ_p^* of ℓ_p is isomorphic to ℓ_q with $p^{-1} + q^{-1} = 1$, for $1 < p < \infty$. Also, the dual space ℓ_1^* of ℓ_1 is isomorphic to ℓ_∞.

In [6], the spectrum of the lower triangular double-band matrix $\triangle_{ab} = (\triangle^{ab})^t$ has been studied, and it is proved that

$$\sigma(\triangle_{ab}, \ell_p) = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\} \cup E,$$

where

$$E = \{a_k : \kappa \in \mathbb{N}, |a_k - a| > |b|\}. \tag{6}$$

Also, the spectrum of the operator $\triangle_{ab}$ on ℓ_∞ equals $\sigma(\triangle_{ab}, \ell_p)$, which can be derived by analogy to that on the space ℓ_p. Now, since $\triangle^{ab}$ is a bounded linear operator on the Banach space ℓ_p into itself, the spectra of the operator $\triangle^{ab}$ and its adjoint $\triangle^{ab\,*} = \triangle_{ab} : \ell_q \to \ell_q$, are equal (by Proposition 1(a)). Then, we have the following main result:

Theorem 2. *For $1 \leq p < \infty$,*

$$\sigma\left(\triangle^{ab}, \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\} \cup E,$$

where E is given as in (6).

The next theorem gives the point spectrum of the operator $\triangle^{ab}$ on ℓ_p.

Theorem 3. *For $1 \leq p < \infty$,*

$$\sigma_p\left(\triangle^{ab}, \ell_p\right) = \{\lambda \in \mathbb{C} : \sum_{k=0}^{\infty} \left|\prod_{i=0}^{k} \frac{\lambda - a_i}{b_i}\right|^p < \infty\}.$$

Proof. Consider the equation $\triangle^{ab}x = \lambda x$ for $x = (x_0, x_1, x_2, \ldots) \neq \theta$ in ℓ_p. Then, we obtain that

$$x_{k+1} = \frac{\lambda - a_k}{b_k} x_k = x_0 \prod_{i=0}^{k} \frac{\lambda - a_i}{b_i}, \quad \kappa \in \mathbb{N}.$$

For $\lambda = a_k$, $\kappa \in \mathbb{N}$, there exists a nonzero vector $x = (x_0, x_0\frac{\lambda-a_0}{b_0}, x_0\prod_{i=0}^{1}\frac{\lambda-a_i}{b_i}, \ldots, x_0\prod_{i=0}^{k}\frac{\lambda-a_i}{b_i}, 0, \ldots)$ with $x_0 \neq 0$, which presents an eigenvector corresponding to the eigenvalue $\lambda = a_k$, and in this case we have $\sum_{k=0}^{\infty}\left|\prod_{i=0}^{k}\frac{\lambda-a_i}{b_i}\right|^p < \infty$. Also, for all $\lambda \neq a_\kappa$, $\kappa \in \mathbb{N}$, we obviously have $\lambda \in \sigma_p(\triangle^{ab}, \ell_p)$ if and only if there exists $x \neq \theta$ in ℓ_p with $\triangle^{ab}x = \lambda x$, that is, if and only if

$$\sum_{k=0}^{\infty}\left|\prod_{i=0}^{k}\frac{\lambda - a_i}{b_i}\right|^p < \infty.$$

Thus, the required result follows. ■

Remark 2. One can easily observe that $\lambda \in \sigma_p\left(\triangle^{ab}, \ell_p\right)$ *if and only if the sequence* $\left(\prod_{i=0}^{k}\frac{\lambda-a_i}{b_i}\right)$ *belongs to* ℓ_p. *Also, the point spectrum can be given in the form*

$$\sigma_p\left(\triangle^{ab}, \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup E \cup H,$$

where E *is given as in* (6) *and*

$$H = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|, \sum_{k=0}^{\infty}\left|\prod_{i=0}^{k}\frac{\lambda - a_i}{b_i}\right|^p < \infty\}.$$

The following theorem gives the point spectrum of the adjoint operator $\triangle^{ab\,*}$.

Theorem 4. $\sigma_p(\triangle^{ab\,*}, \ell_p^*) = E \cup K_p$ *, where*

$$K_p = \{a_j : j \in \mathbb{N}, \left|a_j - a\right| = |b|, \sum_{k=m}^{\infty}\left|\prod_{i=m}^{k}\frac{b_{i-1}}{a_j - a_i}\right|^q < \infty, \text{ for some } m \in \mathbb{N}\},$$

for $1 < p < \infty$ *and*

$$K_1 = \{a_j : j \in \mathbb{N}, \left|a_j - a\right| = |b|, \sup_{k \geq m}\left|\prod_{i=m}^{k}\frac{b_{i-1}}{a_j - a_i}\right| < \infty, \text{ for some } m \in \mathbb{N}\}$$

and E *is given as in* (6).

Proof. We consider the case for which $1 < p < \infty$. Suppose $\triangle^{ab\,*}f = \lambda f$ for $f \in \ell_p^* \cong \ell_q$ with $p^{-1} + q^{-1} = 1$. Then, we obtain that

$$(a_0 - \lambda) f_0 = 0 \text{ and } b_k f_k + (a_{k+1} - \lambda) f_{k+1} = 0,$$

for all $\kappa \in \mathbb{N}$. Indeed, we have two cases:
Case (i): if the sequence (a_k) is a constant sequence, then we obtain that $f = \theta$, and so, $\sigma_p(\triangle^{ab\,*}, \ell_p^*) = \varnothing$. In this case $E = K_p = \varnothing$.
Case (ii): if the sequence (a_k) is not a constant sequence, then for all $\lambda \notin \{a_\kappa : \kappa \in \mathbb{N}\}$, we have $f_k = 0$ for all $\kappa \in \mathbb{N}$. Hence $\lambda \notin \sigma_p(\triangle^{ab\,*}, \ell_p^*)$. As well as, if $\lambda = a$, we can prove that $\lambda \notin \sigma_p\left(\triangle^{ab\,*}, \ell_p^*\right)$. Therefore

$$\sigma_p\left(\triangle^{ab\,*}, \ell_p^*\right) \subseteq \{a_\kappa : \kappa \in \mathbb{N}\} \setminus \{a\}.$$

Now, if $\lambda \in \sigma_p\left(\triangle^{ab\,*}, \ell_p^*\right)$, then $\lambda = a_j \neq a$ for some $j \in \mathbb{N}$ and there exists $f \in \ell_p^*$, $f \neq \theta$ such that $\triangle^{ab\,*}f = a_j f$. So, we have

$$\lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{b}{a_j - a}\right|^q \leq 1$$

This implies that $\lambda = a_j \in E$ or $\left|a_j - a\right| = |b|$. In the case when $\left|a_j - a\right| = |b|$ we have, for some $m \in \mathbb{N}$

$$f_k = \frac{b_{k-1}b_{k-2}\ldots b_{m-1}}{(a_j - a_k)(a_j - a_{k-1})\ldots(a_j - a_m)} f_{m-1}$$
$$= f_{m-1}\prod_{i=m}^{k}\frac{b_{i-1}}{a_j - a_i}, \quad \kappa \geq m.$$

Therefore

$$\sum_{k=0}^{\infty}|f_k|^q < \infty$$

if

$$\sum_{k=m}^{\infty}\left|\prod_{i=m}^{k}\frac{b_{i-1}}{a_j - a_i}\right|^q < \infty, \text{ for some } m \in \mathbb{N}.$$

Then $\lambda = a_j \in K_p$. So, $\lambda = a_j \in E \cup K_p$. Thus $\sigma_p\left(\triangle^{ab\,*}, \ell_p^*\right) \subseteq E \cup K_p$.

Conversely, let $\lambda \in E \cup K_p$. If $\lambda \in E$, then there exists $i \in \mathbb{N}$ such that $\lambda = a_i \neq a$ and so we can take $f \neq \theta$ such that $\triangle^{ab\,*}f = a_i f$ and

$$\lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{b}{a - a_i}\right|^q < 1,$$

that is $f \in \ell_p^* \cong \ell_q$. Also, if $\lambda \in K_p$, then there exists $j \in \mathbb{N}$ such that $\lambda = a_j \neq a$, $\left|a_j - a\right| = |b|$ and $\sum_{k=m}^{\infty}\left|\prod_{i=m}^{k}\frac{b_{i-1}}{a_j - a_i}\right|^q < \infty$ for some $m \in \mathbb{N}$. Then we can take $f \in l_p^* \cong \ell_q$, $f \neq \theta$ such that $\triangle^{ab\,*}f = a_j f$. Thus

$$E \cup K_p \subseteq \sigma_p(\triangle^{ab\,*}, \ell_p^*).$$

This completes the proof in the case for which $1 < p < \infty$. Analogously, we can deal with the case $p = 1$. ■

Theorem 5. *For* $1 \leq p < \infty$,

$$\sigma_r\left(\triangle^{ab}, \ell_p\right) = \varnothing.$$

Proof. It is clear that, by Proposition 1, $\sigma_r\left(\triangle^{ab},\ \ell_p\right) = \sigma_p\left(\triangle^{ab\,*}, \ell_p^*\right) \backslash \sigma_p\left(\triangle^{ab}, \ell_p\right)$. Also, by Theorems 3 and 4, we have

$$\sigma_p(\triangle^{ab\,*},\ \ell_p^*) \subseteq \{a_k : k \in \mathbb{N}\} \subseteq \sigma_p\left(\triangle^{ab}, \ell_p\right).$$

Thus $\sigma_r\left(\triangle^{ab},\ \ell_p\right) = \varnothing$. This completes the proof. ■

Theorem 6. *For* $1 \le p < \infty$,

$$\sigma_c\left(\triangle^{ab}, \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \backslash H.$$

Proof. Since $\sigma(\triangle^{ab}, \ell_p)$ is the union of the disjoint sets $\sigma_p\left(\triangle^{ab}, \ell_p\right)$, $\sigma_r(\triangle^{ab}, \ell_p)$ and $\sigma_c\left(\triangle^{ab}, \ell_p\right)$, then Theorems 2, 3 and 5 imply that

$$\sigma_c\left(\triangle^{ab},\ \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \backslash H. ■$$

Although the point spectrum $\sigma_p(\triangle^{ab},\ \ell_p)$ and the continuous spectrum $\sigma_c\left(\triangle^{ab}, \ell_p\right)$ depend on the index p, the spectrum itself does not depend on p. Also, for all p, where $1 \le p < \infty$, the residual spectrum $\sigma_r(\triangle^{ab}, \ell_p)$ is empty.

3. Further Results

In this section, we investigate the fine structure of the spectrum of the operator $\triangle^{ab}$, with respect to the other classification schemes. Firstly, we give the following lemma which is needed in the proof of the next theorem.

Lemma 1. *[27] If T is a bounded linear operator on a normed space X into a normed space Y, then T has a dense range in Y if and only if $(T^*)^{-1}$ exists.*

We note that, for the operator $\triangle^{ab} : \ell_p \to \ \ell_p$ we have

$$III_1\left(\triangle^{ab}, \ell_p\right) = III_2\left(\triangle^{ab}, \ell_p\right) = \varnothing,$$

since

$$\sigma_r\left(\triangle^{ab}, \ell_p\right) = \varnothing.$$

Also, $I_2\left(\triangle^{ab},\ \ell_p\right) = \varnothing$, by the closed graph theorem. Next, we calculate $I_3\left(\triangle^{ab}, \ell_p\right), II_2\left(\triangle^{ab},\ \ell_p\right)$, $II_3\left(\triangle^{ab},\ \ell_p\right)$ and $III_3\left(\triangle^{ab},\ \ell_p\right)$.

Theorem 7. *For $1 \le p < \infty$, the parts, $I_3\left(\triangle^{ab}, \ell_p\right), II_2\left(\triangle^{ab},\ \ell_p\right)$, $II_3\left(\triangle^{ab},\ \ell_p\right)$ and $III_3\left(\triangle^{ab},\ \ell_p\right)$ are given as follows:*
(i) $I_3\left(\triangle^{ab}, \ell_p\right) \cup II_3\left(\triangle^{ab}, \ell_p\right)$

$$\begin{aligned} &= \sigma_p(\triangle^{ab},\ \ell_p) \backslash \sigma_p(\triangle^{ab\,*}, \ell_p^*) \\ &= \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup (H \backslash K_p), \end{aligned}$$

(ii) $III_3\left(\triangle^{ab}, \ell_p\right) = \sigma_p(\triangle^{ab\,*}, \ell_p^*) = E \cup K_p$,

(iii) $II_2\left(\triangle^{ab}, \ell_p\right) = \sigma_c\left(\triangle^{ab}, \ell_p\right)$

$$= \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \backslash H,$$

where E is given as in (6), and K_p and H are given in Theorem 4 and Remark 2, respectively.

Proof. (i) Let $\lambda \in \sigma_p(\triangle^{ab},\ \ell_p) \backslash \sigma_p(\triangle^{ab\,*}, \ell_p^*)$. Then, the operator $\triangle^{ab} - \lambda I$ has no inverse and $\triangle^{ab\,*} - \lambda I$ is one to one. So, by Lemma 1, $\left(\triangle^{ab} - \lambda I\right)^{-1}$ does not exist and $\triangle^{ab} - \lambda I$ has a dense range. Thus, $\lambda \in I_3\left(\triangle^{ab}, \ell_p\right) \cup II_3\left(\triangle^{ab}, \ell_p\right)$. The converse can be proved analogously.
(ii) Let $\lambda \in \sigma_p(\triangle^{ab\,*}, \ell_p^*)$. Then, $\triangle^{ab\,*} - \lambda I$ is not one to one, and so, by Lemma 1, $\triangle^{ab} - \lambda I$ has not a dense range. Since $III_1\left(\triangle^{ab}, \ell_p\right) = III_2\left(\triangle^{ab},\ \ell_p\right) = \varnothing$, then $\lambda \in III_3\left(\triangle^{ab}, \ell_p\right)$. Conversely, for $\lambda \notin \sigma_p(\triangle^{ab\,*}, \ell_p^*)$, $\triangle^{ab\,*} - \lambda I$ is one to one, and so $\triangle^{ab} - \lambda I$ has a dense range. Then, $\lambda \notin III_1\left(\triangle^{ab}, \ell_p\right) \cup III_2\left(\triangle^{ab},\ \ell_p\right) \cup III_3\left(\triangle^{ab}, \ell_p\right)$. Thus, $\lambda \notin III_3\left(\triangle^{ab}, \ell_p\right)$. This completes the proof.
(iii) By definition, the continuous spectrum of the bounded linear operator $\triangle^{ab}$ on the Banach space ℓ_p consists of $II_2\left(\triangle^{ab}, \ell_p\right)$. Then

$$\begin{aligned} II_2\left(\triangle^{ab}, \ell_p\right) &= \sigma_c\left(\triangle^{ab},\ \ell_p\right) \\ &= \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \backslash H. ■ \end{aligned}$$

Although, the results in Theorem 7(*i*)-(*ii*) give a finer subclassification of the point spectrum, the two parts $I_3(\triangle^{ab},\ \ell_p)$ and $II_3(\triangle^{ab},\ \ell_p)$ cannot be determined separately in general. In some special cases, one can determine these parts separately.

Theorem 8. *The following statements hold :*
(i) $\sigma_{ap}\left(\triangle^{ab},\ \ell_p\right) = \{\lambda \in \mathbb{C} : |\lambda - a| \le |b|\} \cup E$,
(ii) $\sigma_{co}\left(\triangle^{ab},\ \ell_p\right) = E \cup K_p$,
(iii) $\sigma_\delta\left(\triangle^{ab},\ \ell_p\right) =$

$$[\{\lambda \in \mathbb{C} : |\lambda - a| \le |b|\} \cup E] \backslash I_3\left(\triangle^{ab},\ \ell_p\right),$$

where E is given as in (6), and K_p and H are given in Theorem 4 and Remark 2, respectively.

Proof. (i) Since the equality

$$\sigma_{ap}\left(\triangle^{ab},\ \ell_p\right) = \sigma\left(\triangle^{ab},\ \ell_p\right) \backslash III_1\left(\triangle^{ab}, \ell_p\right)$$

hold, and

$$III_1\left(\triangle^{ab}, \ell_p\right) = \varnothing,$$

then

$$\sigma_{ap}\left(\triangle^{ab},\ \ell_p\right) = \sigma\left(\triangle^{ab},\ \ell_p\right).$$

This completes the proof.

(ii) Follows immediately from Proposition 1 and Theorem 4.
(iii) Since

$$\sigma_\delta\left(\triangle^{ab},\,\ell_p\right)=\sigma\left(\triangle^{ab},\,\ell_p\right)\setminus I_3\left(\triangle^{ab},\ell_p\right),$$

then, the required result follows from Theorem 2. ■

4. The eigenvalue problem associated with the infinite matrix $\triangle^{ab}$

Consider a linear operator $T : X \to X$ that maps a Banach space X into itself. The standard eigenvalue problem involves finding the nontrivial solutions of the equation

$$Tx = \lambda x$$

with $\lambda \in \mathbb{C}$ an eigenvalue, $x \in X$ and $x \neq \theta$ an eigenvector. The standard eigenvalue problem can also be written as

$$(T-\lambda I)x=\theta$$

with I the identity operator on X.

Now, consider the eigenvalue problem

$$\triangle^{ab}x=\lambda x,$$

where $\lambda \in \mathbb{C}$ and $x = (x_k) \in \ell_p$. Then

$$\triangle^{ab}(x_k)=(a_kx_k+b_kx_{k+1})=\lambda(x_k). \tag{7}$$

We obtain the following system of equations

$$\begin{array}{l} a_0x_0+b_0x_1=\lambda x_0,\\ a_1x_1+b_1x_2=\lambda x_1,\\ a_2x_2+b_2x_3=\lambda x_2\\ \qquad\vdots\\ a_kx_k+b_kx_{k+1}=\lambda x_k\\ \qquad\vdots \end{array}$$

If λ is in the resolvent set of $\triangle^{ab}$, then *Eq.* (7) has only the trivial solution, considered in ℓ_p. If λ is in the point spectrum $\sigma_p(\triangle^{ab},\ell_p)$, then *Eq.* (7) has nontrivial solution $x \in \ell_p$. The residual spectrum $\sigma_r(\triangle^{ab},\ell_p)$, is empty (Theorem 5). If $\lambda \in \sigma_c(\triangle^{ab},\ell_p)$, the continuous spectrum of $\triangle^{ab}$, then *Eq.* (7) has no solution $x \in \ell_p$, where $x \neq \theta$. However in this case, *Eq.* (7) may have nonzero solutions which are not in ℓ_p.

It may be helpful to provide the following example.

Example 1. Let $\triangle^{ab}=(c_{ij})$, $i,j=0,1,2,\ldots$, be an infinite matrix defined by $c_{ii}=a_i$, $c_{i,i+1}=b_i$ and $c_{ij}=0$ for all $i\neq j$ and $j\neq i+1$, where $a_0=4, b_0=1$, $a_k=-1$ and $b_k=(k+1)/k$, for all $k\geq 1$. Then $\lim_{k\to\infty}a_k=a=-1$ and $\lim_{k\to\infty}b_k=b=1$. By using the results in Section 2 we can calculate, for $1<p<\infty$, that

$$\sigma\left(\triangle^{ab},\ell_p\right)=\sigma_p\left(\triangle^{ab},\ell_p\right)=\{\lambda\in\mathbb{C}:|\lambda+1|\leq 1\}\cup\{4\}.$$

Also,

$$\sigma_r\left(\triangle^{ab},\ell_p\right)=\sigma_c\left(\triangle^{ab},\ell_p\right)=\varnothing.$$

In this example, we observe that all spectral values are eigenvalues. That is, the eigenvalue problem (7), where $\lambda\in\mathbb{C}$ and $x\in\ell_p$, has nontrivial solutions for all $\lambda\in\{\lambda\in\mathbb{C}:|\lambda+1|\leq 1\}\cup\{4\}$ and it has only the trivial solution for all $\{\lambda\in\mathbb{C}:|\lambda+1|>1\}\setminus\{4\}$.

For $p=1$, we can calculate that

$$\begin{aligned} \sigma(\triangle^{ab},\ell_1)&=\{\lambda\in\mathbb{C}:|\lambda+1|\leq 1\}\cup\{4\},\\ \sigma_p(\triangle^{ab},\ell_1)&=\{\lambda\in\mathbb{C}:|\lambda+1|<1\}\cup\{4\},\\ \sigma_r(\triangle^{ab},\ell_1)&=\varnothing,\\ \sigma_c(\triangle^{ab},\ell_1)&=\{\lambda\in\mathbb{C}:|\lambda+1|=1\}. \end{aligned}$$

In this case, Eq. (7) has nontrivial solutions for all $\lambda\in\{\lambda\in\mathbb{C}:|\lambda+1|<1\}\cup\{4\}$, it has only the trivial solution for all $\lambda\in\{\lambda\in\mathbb{C}:|\lambda+1|>1\}\setminus\{4\}$ and it has no solutions in ℓ_1 for all $\lambda\in\{\lambda\in\mathbb{C}:|\lambda+1|=1\}$.

5. Conclusion and future work

The fine structure of spectra of the operator $\triangle^{ab}$ over the sequence space ℓ_p, where $1 \leq p < \infty$, have been investigated. Moreover, application to the eigenvalue problem has been explained. There are other papers devoted to this problem (see, for example, [24, 25]). But, the new results in this paper cover a wider class of linear operators which are represented by infinite upper triangular double-band matrices over the sequence space ℓ_p.

The fine structure of spectra of many other operators, which are represented by matrices, remain to be studied. For example, the fine structure of the spectrum of the upper and lower triangular triple-band matrices have been studied in some special cases [12, 13]. We intend to study such kind of operators in more general forms by using new techniques.

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