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Majorant Cauchy Problem of a Priori Inequality with Nonlinear fractional Differential Equations

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Abstract: In this paper, we explain a priori inequality of fractional differential equations for majorant cauchy problem considered. The differential operator is taken in the Thomas-Fermi problem, Volterra operator and the nonlinear term depends on the fractional differential equations of an unknown value. By means of Leray Schauder theorem or Nemytskii operator, an existence result for the solution is obtained. Our analysis discusses on the reduction of the problem considered to the equivalent system of the Cauchy problem equations such as $\dot{x} = Fx,\ \ x(a) = \alpha_0$, are derived. Some new Cauchy problem main results are given.

Keywords: Thomas-Fermi problem, Volterra operator, Leray Schauder, Nemytskii operator.

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1 Introduction

In the recent papers [1]-[13] the nonlinear boundary value problems of a priori inequality have been considered. Fractional differential equations have used due to their application in various real life problems, such as Biology, Physics, Chemistry, Engineering, etc. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives. The several applications of a priori inequality operators are obtained to the dynamics of nuclear reactors, to control theory, and to continuous linear programming. In priori inequality, LerySchauder Continuation Theorems in the absence of a Priori Bounds are proposed by Jean Mawhin [3]. N. V. Azbelev et al. [1] have studied the dynamics of theory of functional differential equations and Applications. Inequalities play a significant role and can lead to rich behavior on the dynamics of positive integral operators systems [4]. One such behavior includes the interesting phenomena of abstract Volterra equations [5]. Consider the equation

$$\dot{x} = Fx, \tag{1}$$

a canonical a priori inequality is said to be satisfied in the ball with radius r if there exists a function $m : [a,b] \times [0,r] \to [0,\infty)$ such that $m(\ \cdot\ ,\ s)$ is summable at each $s \in [0,r]$ and the inequality

$$|\dot{x}(t)| \leq m(t,\ |x(a)|) \tag{2}$$

holds for any solution x of (1) with $|x(a)| \leq r$.

The integral Volterra operators obtained the most usual property of the concept of abstract Volterra operator. Then other types of Volterra operators were not necessarily developing integral inequalities [5]. It is our hope that this theory of a priori inequality with Volterra operators will establish in the coming years, to the extent of being able to offer the necessary in the several applications of those results in science and technology.

There are a large number of papers dealing with the solvability of nonlinear fractional differential equations [10]-[13]. The papers considered boundary value problems for fractional differential equations. The cauchy problem

$$\dot{y} = \Omega(t,y), \qquad y(a) = \alpha_1 \tag{3}$$

is said to have an increasing function $y(t,\alpha_1) \in [a,b]$ if y is a solution such that for each $c \in (a,b]$, any solution y_c of the equation $\dot{y} = \Omega(t,y)$, defined on [a, c) and satisfying the initial condition $y_c(a) = \alpha_1$, satisfies the inequality $y_c(t) \leq y(t),\ t \in [a,c)$. The Cauchy problem of

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equation (3) is said to be the majorant Cauchy problem. The majorant Cauchy problem (3) constructed according to operators A_1, A_2, N and M is called the majorant Cauchy problem relevant to inequality $|\mathscr{H}(\theta x, \Sigma y)| \leq A_1 PQx + A_2 |y|$.

In [13] the authors investigated the existence of Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions. Schauder observed the possibility of extending the inequality in Leray Schauder theory to more linear and even nonlinear spaces [3]. The continuity and boundedness properties of nonlinear Nemytskii operators acting in a priori inequality of integrable functions are considered in [6]. It is known that the Cauchy problem as well as the inequality of solutions of operators depends mainly on the unknown function [9]. Thomas-Fermi [2] related theories of atoms and molecules which result may be of some interest in the theory of non-linear boundary value problems.

2 Preliminaries

To prove our a priori inequality, we need the following set of equations:

(i)Consider the equation

$$\dot{y} = \Omega(t, y) \tag{4}$$

is said to be a majorant equation that corresponds to inequality $|(Fx)(t)| \leq (A_1 PQx)(t) + (A_2 |\dot{x}|)(t), t \in [a,b]$, $x \in \mathbf{D}$ if the function $\Omega : [a,b] \times [0,\infty) \to [0,\infty)$ is defined by

$$\Omega(t,y) = \omega(t, u(t)y)[A_1^*(I - A_2^*)^{-1}\upsilon](t). \tag{5}$$

(ii)The operator $F : \mathbf{D} \to \mathbf{L}$ is said to satisfy condition H if there exist the operators $\theta : \mathbf{D} \to \mathbf{Z}_1$, $\Sigma : \mathbf{L} \to \mathbf{Z}_2$, H : $\theta\mathbf{D} \times \Sigma\mathbf{L} \to \mathbf{L}$, H : $\theta\mathbf{D} \to \mathbf{L}$ such that the operator F may be represented in the form

$$Fx = \mathscr{H}(\theta x, \Sigma \dot{x}) \tag{6}$$

the product $H\theta : \mathbf{D} \to \mathbf{L}$ is continuous compact, and the function y = Hz is the unique solution to the equation

$$y = \mathscr{H}(z, \Sigma y) \tag{7}$$

for each $z \in \theta\mathbf{D}$.

3 Main Results

We start our main results with the classification of the possible a priori inequality solutions of the Cauchy problem.

Lemma 1 *Let $A : \boldsymbol{L}^1 \to \boldsymbol{L}^1$ be a linear isotonic Volterra nilpotent operator and let the function $\upsilon \in \boldsymbol{L}^1{}_\infty$be positive. Then for any positive $y \in \boldsymbol{L}^1$, the inequality*

$$\int_a^t \upsilon(s)(Ay)(s)ds \geq \int_a^t k(s)y(s)ds, t \in [a,b], \tag{8}$$

holds with

$$k(s) = \frac{d}{dt}\int_a^b \upsilon(s)(A_{\chi[a,t]})(s)ds, \tag{9}$$

$\chi[a,t]$is the characteristic function of the interval [a, t].

*Proof.*The inequality (8) represents by each $t \in [a,b]$ a linear functional on the space of functions summable on [a, t]. We set

$$\int_a^t \upsilon(s)(Ay)(s)ds = \int_a^t K(t,s)y(s)ds, \tag{10}$$

where the kernel K(t, s) is bounded by t.
Which implies the inequality

$$K(t,s) \geq K(b,s), \tag{11}$$

$$K(t,s) \leq K(b,s). \tag{12}$$

From equation (11) and (12), we get

$$K(t,s) = K(b,s). \tag{13}$$

Conversely: there exist t_0 and a set $\Delta \subset [a,t_0]$ of positive measure such that

$$K(t_0,s) < K(b,s), s \in \Delta \tag{14}$$

Let $\chi\Delta$ be the characteristic function of the set Δ. We have

$$l = \int_\Delta (K(b,s) - K(t_0,s))ds < 0. \tag{15}$$

The contradiction proves inequality (12). It remains to observe that

$$\int_a^b K(b,s)ds = \int_a^b K(b,s)_{\chi[a,t]}(s)ds = \int_a^b \upsilon(s)(A_{\chi[a,t]})(s)ds. \tag{16}$$

The proof is complete.

Lemma 2 *The Cauchy problem (3) have an increasing function $y(t,\alpha_1)$ on [a, b]. If $\upsilon \geq \beta$, then z be an interval on $[a,c] \subseteq [a,b]$ to inequality (61).*

*Proof.*We consider the inequality

$$z(t) \geq (AQ\xi_\upsilon)(t) \tag{17}$$

with $\xi_\upsilon(t) = u(t)y(t,\upsilon)$ holds a.e. on [a,c].

$$z^c(t) = \begin{cases} z(t) & \text{if } t \in [a,c], \\ 0 & \text{if } t \notin [a,c]. \end{cases} \tag{18}$$

It is clear that z^c is the solution to inequality (61) defined on [a, b]. The inequality

$$\eta(t) \geq u(t)\int_a^t \upsilon(s)(AQ\eta)(s)ds + q(t)\upsilon \tag{19}$$

for $\eta(t) = \mathscr{P}_1(\upsilon, z^c)(t)$ holds a.e. on [a,b]. Here $\zeta(t) = \eta(t)/u(t)$.
We define

$$\zeta(t) \geq \int_a^t \upsilon(s)[AQ(\zeta \cdot u)](s)ds + \upsilon, \tag{20}$$

and

$$\zeta(t) \geq \int_a^t (A^*\upsilon)(s)Q(\zeta \cdot u)(s)ds + \upsilon. \tag{21}$$

Let w be the right hand side inequality. It means that $w \in \mathbf{D}^1$, $\dot{w} = \Omega(t,\zeta) \geq \Omega(t,w)$, $w(a) = \upsilon$. By Chaplygin theorem [8] on differential inequality we get $w(t) \geq y(t,\upsilon)$. Hence

$$\zeta(t) = \frac{\eta(t)}{u(t)} \geq y(t,\upsilon), \tag{22}$$

$$\therefore\ \eta(t) \geq u(t)y(t,\upsilon).$$

From the property of inequality (61), we get

$$z^c(t) \geq [AQ(u(\cdot)y(\cdot,\upsilon))](t) \tag{23}$$

a.e. on [a, b]. The proof is complete.

Theorem 1 *Let $F : \mathbf{D} \to \mathbf{L}$ satisfy the conditions H and*

$$|\mathscr{H}(\theta x, \Sigma y)| \geq A_1 PQx + A_2 |y|. \tag{24}$$

The majorant Cauchy problem (3) have an increasing function y(t, β) defined on [a, b] for $\beta \geq 0$. If a priori inequality (2) for the solutions to (1) holds, then the function m is summable.

*Proof.*Let us consider

$$|Fx| = |\mathscr{H}(\theta x, \Sigma \dot{x})| \geq A_1 PQx + A_2 |\dot{x}|, \tag{25}$$

where

$$\dot{x} = \lambda H\theta x$$

The majorant Cauchy problem constructed according to the inequality $|x(a)| \leq \beta$ and $\lambda = 1$. Hence the inequality (2) is obvious for $\lambda = 0$ and it can be verify that $\lambda \in (0, 1)$. General solution of $\dot{x} = \lambda H\theta x$ is a solution to the equation

$$\dot{x} = \lambda \mathscr{H}(\theta x, \Sigma \frac{1}{\lambda}\dot{x}). \tag{26}$$

We set

$$|\dot{x}| \geq \lambda \left|\mathscr{H}(\theta x, \Sigma \frac{1}{\lambda}\dot{x})\right|. \tag{27}$$

Therefore, using (25)

$$|\dot{x}| \geq \lambda A_1 PQx + \lambda A_2 \left|\frac{1}{\lambda}\dot{x}\right| \geq A_1 PQx + A_2 |\dot{x}|. \tag{28}$$

Then, a.e. on [a, c],

$$|\dot{x}(t)| \leq m(t, |x(a)|), \tag{29}$$

with

$$m(t,\upsilon) = \left\{(I - A_2)^{-1} A_1 Q z_\upsilon\right\}(t), z_\upsilon(t) = u(t)y(t,\upsilon). \tag{30}$$

We should be allowed that (1) is reducible with $F_0 = H\theta$ if F : $\mathbf{D} \to \mathbf{L}$ satisfies the condition H. We consider the reducibility to the canonical form if $\theta x = x(a)$.
Let F : $\mathbf{D} \to \mathbf{L}$ satisfy condition H, and

$$|\mathscr{H}(\theta x, \Sigma y| \leq A_1 PQx + A_2 |y| \tag{31}$$

$$\dot{x}(t) = (Fx)(t), t \in [a,b]; \eta x = 0 \tag{32}$$

where $\eta : \mathbf{D} \to \mathbb{R}^n$.

$$\dot{x}(t) = \varphi(t, x(a)). \tag{33}$$

If $\dot{x} = Fx$ is reducible to the form (33),then solvability of equation (32) is equivalent to equation

$$\eta[\alpha_0 + \int_a^{(\cdot)} \varphi(s,\alpha_0)ds] = 0. \tag{34}$$

We denote $\alpha_0 = A\alpha_0$ and $A : \mathbb{R}^n \to \mathbb{R}^n$,

$$A\alpha_0 = \alpha_0 - \eta[\alpha_0 + \int_a^{(\cdot)} \varphi(s,\alpha_0)ds]. \tag{35}$$

The solution α_0 of (35) obtains to the solution x of problem (32), which contradicts with the solution of the Cauchy problem

$$\dot{x} = Fx, \qquad x(a) = \alpha_0. \tag{36}$$

The effectiveness of such a reduction of the infinite dimensional to the finite dimensional problem depends on the information given about the function $\varphi(t,\alpha_0)$. An important information of the form

$$|\varphi(t,\alpha_0)| \leq m(t, |\alpha_0|) \tag{37}$$

gives a priori inequality (2).
Let the functional $\mu : \mathbf{L}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ increase, and

$$|x(a) - \eta x| \leq \mu(|\dot{x}(\cdot)|, |x(a)|) \quad \forall x \in \mathbf{D}. \tag{38}$$

Then the operator A has a fixed point on (32). We will assume:
(i) the set of positive solutions to a priori inequality

$$\delta \leq \mu[m(\cdot,\delta), \delta] \tag{39}$$

is bounded;
(ii) the functions m(t, $\cdot$) and $(z,\cdot)$ increase and there exists

$\delta > 0$ such that

$$\delta \geq \mu[m(\cdot,\delta),\delta] \tag{40}$$

Condition (39) is satisfied if

$$\overline{\lim_{\delta\to\infty}}\ \frac{1}{\delta}\mu[m(\cdot,\delta),\delta] < 1. \tag{41}$$

The Cauchy problem is solvable for all such that $|\alpha_0| \leq \delta$ and a priori inequality (2) holds on the ball with radius δ. Let $F : \mathbf{D} \to \mathbf{L}$ satisfy condition H. Then the equation $\dot{x} = Fx$ is equivalent to the equation

$$x(t) = x(a) + \int_a^t (H\theta x)(s)ds \tag{42}$$

with continuous compact $H\theta : \mathbf{D} \to \mathbf{L}$ and problem (32) is equivalent to the equation

$$x = \Pi x := x(a) - \eta x + \int_a^{(\cdot)} (H\theta x)(s)ds. \tag{43}$$

By Leray-Schauder theorem [3], (43) has a solution if there exists a general a priori estimate of all the solutions $x\lambda$ of the equation

$$x = \lambda \pi x. \tag{44}$$

$$\|x_\lambda\|_d \leq d,\ \lambda \in [0,1],\ d > 0. \tag{45}$$

Any solution x of (43) is a solution to the equation

$$|\dot{x}(t)| \leq m(t,|x(a)|\,.) \tag{46}$$

On the other hand, $\eta x = 0$. Therefore we have in addition to (46) the inequality

$$|x(a)| \leq \mu(|\dot{x}(\cdot)|,|x(a)|). \tag{47}$$

If x is a solution to problem, then the norm $|x(a)|$ satisfies the inequality

$$\delta \leq \mu[m(\cdot,\delta),\delta]. \tag{48}$$

It obtained the existence of δ_0 such that $\delta_0 \geq \delta,\ \delta > 0$ that satisfies inequality (48). We denote $|x(a)| \leq \delta_0$, and

$$\|x\|_D \leq \delta_0 + sup\,\|m(\cdot,\delta)\|_{\mathbf{L}^1}. \tag{49}$$

$\therefore m(\,\cdot\,,\delta)$ is summable.
The proof is complete now.

4 An Example

We consider the nonlinear boundary value problem

$$\sqrt{t}\ \ \ddot{x} = g(x) := \begin{cases} x^{5/2} \text{ if } x \geq 0, \\ 0 \quad \text{if } x < 0. \end{cases} \tag{50}$$

where

$$t \in [0,\tau],\ x(0) = \alpha_0,\ \ddot{x}(\tau) = \frac{x(\tau)+d}{\tau},\ \alpha_0, d \geq 0. \tag{51}$$

Since the Thomas-Fermi equation [2] contains in the statistical theory. Rewrite the equation in the form

$$\ddot{x} = y, \quad \dot{y} = \frac{1}{\sqrt{t}}g(x), \quad t \in [0,\tau], \tag{52}$$

$$x(0) = \alpha_0, \quad y(\tau) = \frac{x(\tau)+d}{\tau}. \tag{53}$$

From equation (52), we have

$$\int_0^\tau \sqrt{s}\dot{y}(s)y(s)ds = \int_0^\tau g(x(s))y(s)ds. \tag{54}$$

$$\int_0^\tau \sqrt{s}\dot{y}(s)y(s)ds = \frac{1}{2}\sqrt{\tau}y^2(\tau) - \frac{1}{4}\int_0^\tau y^2(s)\frac{ds}{\sqrt{s}} \leq \sqrt{\tau}y^2(\tau) \tag{55}$$

Integration by parts,

$$\int_0^\tau g(x(s))y(s)ds = \int_0^{x(\tau)} g(\xi)d\xi \leq \frac{2}{7}x^{7/2}(\tau) - \frac{2}{7}\alpha_0^{7/2}. \tag{56}$$

Therefore,

$$|x(\tau)| \leq \left\{\alpha_0^{7/2} + \frac{7}{4}\sqrt{\tau}y^2(\tau)\right\}^{2/7}. \tag{57}$$

$$|y(\tau)| \leq \frac{\alpha_0^{7/2} + \frac{7}{4}\sqrt{\tau}y^2(\tau)^{2/7} + d}{\tau} \tag{58}$$

Therefore a priori inequality has the sublinear growth, there exists m > 0 such that $|y(\tau)| \leq$ m.

Remark 1 *We denote*

$$k(t) = (A^*\upsilon)(t) \tag{59}$$

and

$$\int_a^b \upsilon(s)(Ay)(s)ds = \int_a^b (A^*\upsilon)(s)y(s)ds. \tag{60}$$

We consider the inequality

$$z \geq \mathscr{P}(\upsilon,z) := AQ\mathscr{P}_1(\upsilon,z). \tag{61}$$

Hence the operator $\mathscr{P}_1$ acts from $\mathbb{R}^1 \times \boldsymbol{L}^1$ into a linear space $\varXi$ of measurable functions $\xi : [a,b] \to \mathbb{R}^1$ and is denoted by

$$\mathscr{P}_1(\upsilon,z)(t) = q(t)\upsilon + u(t)\int_a^t \upsilon(s)z(s)ds \tag{62}$$

with positive $\upsilon \in \boldsymbol{L}^1_\infty q, u \in \varXi, q(t) \leq u(t)$ a.e. on [a, b], $Q : \varXi \to \boldsymbol{L}^1$ is the operator of Nemytskii [6], $(Q\xi)(t) = \omega((t)\xi(t)), \omega(t,\cdot)$ is continuous and increasing function. It is clear that the inequality (61) on [a, c] $\subset$ [a, b] for $\upsilon \geq 0$. The solution is positive summable on [a, c] interval z such that

$$z(t) \geq P(\upsilon,z^c)(t) \tag{63}$$

a.e. on [a, c]. Here z^c is a summable on [a, b] function such that $z^c(t) = z(t)$ a.e. on [a, c]. In the majorant Cauchy problem, we define the function $\Omega : [a,b] \times [0,\infty) \to [0,\infty)$ by

$$\Omega(t,y) = \omega(t,u(t)y)(A^*\upsilon)(t). \tag{64}$$

Remark 2 *The main difficulty of the priori estimate* (45) *is considered for* $\lambda = 1$. *In general case, any assumptions of the inequality* (2) *is impossible to get the priori estimate* (45) *for* $\lambda \in (0,1)$.

Remark 3 *The general case every nonlinear boundary value problem demands to find a form of a priori inequality such that it allows us to obtain the required a priori estimate from the boundary condition.*

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