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# Nonparametric Estimation of a Conditional Quantile Density Function for Time Series Data

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**Abstract:** The aim of this paper is to estimate nonparametrically the conditional quantile density function. A non-parametric estimator of a conditional quantile function density is presented, its asymptotic properties are derived via the estimation of the conditional distribution, as of the conditional quantile in the case of dependent data. To obtain the asymptotic properties we consider some concentration hypotheses acting on the distribution of the conditional functional variable.

**Keywords:** Conditional quantile, Conditional quantile density function, Functional variable, Kernel density estimators, $\alpha$-mixing

The problem of quantile estimation has a very long history, estimating quantiles of any distribution is an important part of Statistics. This allows to derive many applications in various fields as chemistry, geophysics, medicine, meteorology,.... On the other hand, functional random variables are becoming more and more important. The recent literature in this domain shows the great potential of these new functional statistical methods. The most popular case of functional random variable corresponds to the situation when we observe random curve on different statistical units. Such data are called Functional Data. Many multivariate statistical technics, mainly parametric in the functional model terminology, have been extended to functional data and good overviews on this topic can be found in Ramsay [21,22] and or Bosq [3].

More recently, nonparametric methods taking into account functional variables have been developed with very interesting practical motivations dealing with environmetrics (see Damon and Guillas [5], Fernandez et al. [7], Aneiros et al. [1]), chemometrics (see Ferraty and Vieu [9]), meteorological sciences (see Besse et al. [2], Hall and Heckman [15]), speech recognization problem (see Ferraty and Vieu [10]), radar range profile (see Hall et al. [16], Dabo-Niang et al. [4]), medical data (see Gasser et al. [14]), ...

Estimating the conditional quantile constitutes an important statistical topic. It is used to build predictive intervals, as a prediction method by the conditional median and to determine reference curves, predictive intervals etc. It has been widely studied, when the explanatory variable lies within a finite-dimension space (see, e.g., Gannoun et al. [13] and the references therein).
Jones [17] estimated the quantile density function by kernel means, via two alternative approaches. One is the derivative of the kernel quantile estimator, the other is essentially the reciprocal of the kernel density estimator, he gave ways in which the former method has certain advantages over the latter. In his paper, Jones discussed various closely related smoothing issues.

Soni et al. [20] defined a new nonparametric estimator of quantile density function and studied its asymptotic properties are studied. The comparison of the proposed estimator has been made with estimators given in [17].

The goal of this paper is to estimate nonparametrically the conditional quantile density function. A non-parametric estimator of a conditional quantile function density is presented, its asymptotic properties are derived via the estimation of the conditional distribution, as of the conditional quantile in dependent data. In a nonparametric context, it is known that the rate of convergence decreases with the dimension of the space in which the conditional variable is valued. But here, the conditional variable takes its values in an infinite dimensional space. So to override this problem is to consider

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some concentration hypotheses acting on the distribution of the conditional functional variable which allows to obtain the asymptotic properties.

## 1 The model

We consider a random pair $(X,Y)$ where $Y$ is valued in $\mathbb{R}$ and $X$ is valued in some infinite dimensional semi-metric vector space $(\mathscr{F},d(.,.))$. Let $(X_i,Y_i)$, $i=1,\ldots,n$ be the statistical sample of pairs which are identically distributed like $(X,Y)$, but not necessarily independent. From now on, $X$ is called functional random variable f.r.v. Let $x$ be fixed in $\mathscr{F}$ and let $F_{Y|X}(\cdot,x)$ be the conditional cumulative distribution function (cond-cdf) of $Y$ given $X=x$, namely:

$$\forall y\in\mathbb{R}, F_{Y|X}(x,y)=\mathbb{P}(Y\leq y|X=x).$$

Let $Q_{Y|X}(\gamma)$ be the $\gamma-$order quantile of the distribution of $Y$ given $X=x$. From the cond-cdf $F_{Y|X}(.,x)$, it is easy to give the general definition of the $\gamma$-order quantile:

$$Q(\gamma|X=x)\equiv Q_{Y|X}(\gamma)=\inf\left\{t:F_{Y|X}(t,x)\geq\gamma\right\},\ 0\leq\gamma\leq 1 \tag{1}$$

Then, the definition of conditional quantile implies that

$$F_{Y|X}\left(Q_{Y|X}(\gamma)\right)=\gamma.$$

On differentiating partially w.r.t. $\gamma$ we get

$$f_{Y|X}\left(Q_{Y|X}(\gamma)\right)=\frac{1}{\frac{\partial}{\partial\gamma}\left(Q_{Y|X}(\gamma)\right)}.$$

Parzen [19] and Jones [17] defined the quantile density function as the derivative of $Q(\gamma)$, that is, $q(\gamma)=Q'(\gamma)$. Note that the sum of two quantile density functions is again a quantile density function. Thus, the conditional quantile density function can be written as follows (see [26] )

$$\begin{aligned} q(\gamma|X=x)\equiv q_{Y|X}(\gamma) &= \frac{\partial}{\partial\gamma}\left(Q_{Y|X}(\gamma)\right) \\ &= \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)} \end{aligned} \tag{2}$$

Let us now, define the kernel estimator $\widehat{F}_{Y|X}(.,x)$ of $F_{Y|X}(.,x)$

$$\widehat{F}_{Y|X}(x,y)=\frac{\sum_{i=1}^{n}K\left(h_K^{-1}d(x,X_i)\right)H\left(h_H^{-1}(y-Y_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}d(x,X_i)\right)} \tag{3}$$

where $K$ is a kernel function, $H$ a cumulative distribution function and $h_K=h_{K,n}$ (resp. $h_H=h_{H,n}$) a sequence of positive real numbers. Note that using similar ideas, Roussas [23] introduced some related estimate but in the special case when $X$ is real, while Samanta [24] produced previous asymptotic study. As a by-product of (1) and (3), it is easy to derive an estimator $\widehat{Q}_{Y|X}$ of $Q_{Y|X}$ :

$$\widehat{Q}_{Y|X}(\gamma)=\inf\left\{t:\widehat{F}_{Y|X}(t,x)\geq\gamma\right\}=\widehat{F}_{Y|X}^{-1}\left(Q_{Y|X}(\gamma)\right) \tag{4}$$

Let

$$\widehat{F}_{Y|X}^{(j)}(x,y)=\frac{h_H^{-j}\sum_{i=1}^{n}K\left(h_K^{-1}d(x,X_i)\right)H^{(j)}\left(h_H^{-1}(y-Y_i)\right)}{\sum_{i=1}^{n}K\left(h_K^{-1}d(x,X_i)\right)} \tag{5}$$

be the $jth$ successive derivative of $\widehat{F}_{Y|X}(x,y)$, $f_{Y|X}(x,y)$ is conditional density function, such that $f_{Y|X}(x,y)=F_{Y|X}^{(1)}(x,y)$.

Nair and Sankaran [18] defined the hazard quantile function as follows:

$$H(\gamma) = h(Q(\gamma)) = \frac{f(Q(\gamma))}{S(Q(\gamma))} = ((1-\gamma)q(\gamma))^{-1}.$$

Thus hazard rate of two populations would be equal if and only if their corresponding quantile density functions are equal. This has been used to construct tests for testing equality of failure rates of two independent samples. Now, from this definition, let us introduce the $\gamma$-order conditional quantile of the conditional hazard function

$$H_{Y|X}(\gamma) = h_{Y|X}\left(Q_{Y|X}(\gamma)\right) = \frac{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}{S_{Y|X}\left(Q_{Y|X}(\gamma)\right)} = \left((1-\gamma)q_{Y|X}(\gamma)\right)^{-1}.$$

The smooth estimator of the conditional quantile density functional defined as follows:

$$\widehat{q}_{Y|X}(\gamma) = \frac{1}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)}. \tag{6}$$

where $\widehat{f}_{Y|X}(x,y)$ is a conditional kernel density estimator of $f_{Y|X}(x,y)$ and $\widehat{Q}_{Y|X}(\gamma)$ is the conditional empirical estimator of the conditional quantile function $Q_{Y|X}(\gamma)$. Let

$$\widehat{f}_{Y|X}(x,y) = \frac{h_H^{-1}\sum_{i=1}^{n} K\left(h_K^{-1}d(x,X_i)\right) H^{(1)}\left(h_H^{-1}(y-Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}d(x,X_i)\right)},$$

and

$$\widehat{f}^{(j)}_{Y|X}(x,y) = \frac{h_H^{-j-1}\sum_{i=1}^{n} K\left(h_K^{-1}d(x,X_i)\right) H^{(j+1)}\left(h_H^{-1}(y-Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}d(x,X_i)\right)}$$

$h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers which goes to zero as n tends to infinity, and with the convention $0/0 = 0$.

Let's now derive the asymptotic properties of our conditional quantile density function.

(H1) $\forall h > 0$, $\mathbb{P}(d(x,X) \le h) = \mathbb{P}(X \in B(x,h)) = \phi_x(h) > 0$, (with $B(x,h)$ the ball of center $x$ and radius $h$)

(H2) $\sup_{i\neq j}\mathbb{P}((X_i,X_j) \in B(x,h)\times B(x,h)) = \mathbb{P}(W_i \le h, W_j \le h) \le \psi_x(h)$, where $\psi_x(h) \to 0$ as $h \to 0$. Furthermore, we assume that $\psi_x(h) = O(\phi_x^2(h))$.

(H3) $H$ is such that, for all $(y_1,y_2) \in \mathbb{R}^2$, $|H(y_1) - H(y_2)| \le C|y_1 - y_2|$ and its first derivative $H^{(1)}$ verifies $\int |t|^{b_2} H^{(1)}(t)dt < \infty$,

(H4) $K$ is a function with support $(0,1)$ such that $0 < C_1 < K(t) < C_2 < \infty$,

(H5) $\lim_{n\to\infty} h_K = 0$ with $\lim_{n\to\infty} \frac{\log n}{n\phi_x(h_K)} = 0$

(H6) $\exists j > 0$, $\forall l$, $1 \le l < j$, $f^{(l)}_{Y|X}\left(Q_{Y|X}(\gamma)\right) = 0$ and $\left|f^{(j)}_{Y|X}\left(Q_{Y|X}(\gamma)\right)\right| > 0$

## 2 Main result

### *2.1 Estimation of conditional quantile density function*

**Theorem 1.** *Let $q_{Y|X}(\gamma)$ be the conditional density function corresponding to a density function $f_{Y|X}\left(Q_{Y|X}(\gamma)\right)$ and $\widehat{q}_{Y|X}(\gamma)$ denote the estimator of $q_{Y|X}(\gamma)$. Then under the assumptions (H1)-(H6) and as n tends to infinity, we have*

$$\sup_{\gamma}\left|\widehat{q}_{Y|X}(\gamma) - q_{Y|X}(\gamma)\right| \longrightarrow 0 \qquad a.co.$$

**Proof.** At first let us consider

$$\begin{aligned}\widehat{q}_{Y|X}(\gamma) &= \frac{1}{\widehat{f}_{Y|X}(\widehat{Q}_{Y|X}(\gamma))}\\ &= \frac{1}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right) - f_{Y|X}\left(Q_{Y|X}(\gamma)\right) + f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\\ &= \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\left(\frac{1}{1+\frac{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}}\right)\end{aligned}$$

Then, we get

$$\begin{aligned}\widehat{q}_{Y|X}(\gamma) &= \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\left(1-\frac{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\right)\\ &+\frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\left(\frac{\left(\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)\right)^2}{f^2_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\right)\\ &-\frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\left(\frac{\left(\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)\right)^3}{f^3_{Y|X}\left(Q_{Y|X}(\gamma)\right)}+\dots\right),\end{aligned}$$

hence

$$\begin{aligned}\widehat{q}_{Y|X}(\gamma)-q_{Y|X}(\gamma) &= \frac{-\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)+f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}{f^2_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\\ &+\frac{\left(\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)\right)^2}{f^3_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\\ &-\frac{\left(\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)\right)^3}{f^4_{Y|X}\left(Q_{Y|X}(\gamma)\right)}+\dots\end{aligned}$$

With

$$\begin{aligned}\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right) = \ &\widehat{f}_{Y|X}\left(Q_{Y|X}(\gamma)\right)+\left(\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right)\widehat{f'}_{Y|X}\left(Q_{Y|X}(\gamma)\right)+\\ &\frac{\left(\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right)^2\widehat{f}''_{Y|X}(Q_{Y|X}(\gamma))}{2!}+\\ &\frac{\left(\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right)^3\widehat{f^{(3)}}_{Y|X}(Q_{Y|X}(\gamma))}{3!}+\dots\end{aligned}$$

Therefore

$$\begin{aligned}\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right) = \ &\widehat{f}_{Y|X}\left(Q_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)+\\ &\left(\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right)\widehat{f}'_{Y|X}\left(Q_{Y|X}(\gamma)\right)+\\ &\frac{\left(\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right)^2\widehat{f}''_{Y|X}\left(Q_{Y|X}(\gamma)\right)}{2!}+\\ &\frac{\left(\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right)^3\widehat{f}^{(3)}_{Y|X}\left(Q_{Y|X}(\gamma)\right)}{3!}+\dots \quad (7)\end{aligned}$$

Now, it rests to show the following convergence

$$\sup_{\gamma}\left|\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right|\underset{n\to\infty}{\longrightarrow}0 \qquad a.co.$$
$$\sup_{\gamma}|\widehat{f}_{Y|X}\left(Q_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)|\underset{n\to\infty}{\longrightarrow}0 \qquad a.co.$$

It was shown in [8] the following results

$$\sup_{\gamma}|\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)|\underset{n\to\infty}{\longrightarrow}0\ a.co.$$
$$\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)=O\left(h_K^{\frac{b_1}{j}}+h_H^{\frac{b_2}{j}}\right)+O_{a.co.}\left(\frac{\log n}{n\phi_x(h_K)}\right)^{\frac{1}{2j}},$$

and

$$|\widehat{F}^{(j)}_{Y|X}(x,y)-F^{(j)}_{Y|X}(x,y)|=O\left(h_K^{b_1}+h_H^{b_2}\right)+O_{a.co.}\left(\sqrt{\frac{\log n}{nh_H^{2j-1}\phi_x(h_K)}}\right) \tag{8}$$

Note that, $\widehat{f}_{Y|X}=\widehat{F}^{(1)}_{Y|X}$, so applying (8) for $j=1$, we get

$$|\widehat{f}_{Y|X}\left(Q_{Y|X}(\gamma)\right)-f_{Y|X}\left(Q_{Y|X}(\gamma)\right)|=O\left(h_K^{b_1}+h_H^{b_2}\right)+O_{a.co.}\left(\sqrt{\frac{\log n}{nh_H\phi_x(h_K)}}\right), \tag{9}$$

Based on data $X_1,X_2,\dots,X_n$, we propose a smooth estimator of the conditional quantile density function, Ferraty et al. [8] proposed a kernel-type estimator of conditional quantile which is is a conditional version of Parzen's estimator in the univariate case (see Parzen [19]).

For an appropriate kernel function $H'$ and a bandwidth sequence $h_H$. We suggest an estimator of $q_{Y|X}(\gamma)$;

$$\widehat{q}^{1}_{Y|X}(\gamma)=\frac{1}{h_H}\int_0^1\frac{H'\left(h_H^{-1}(v-\gamma)\right)}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)}dv \tag{10}$$

The next theorem proves consistency of the proposed estimator of the conditional quantile density function.

**Theorem 2.** *Let $q_{Y|X}(\gamma)$ be the conditional density function corresponding to a density function $f_{Y|X}\left(Q_{Y|X}(\gamma)\right)$ and $\widehat{q}^{1}_{Y|X}(\gamma)$ given by (10) the proposed estimator of $q_{Y|X}(\gamma)$, the conditional quantile density function. Then under hypotheses (H1)-(H6) as n tends to infinity, we have*

$$\sup_{\gamma}\left|\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma)\right|\longrightarrow 0 \qquad a.co.$$

**Proof.**

(10) gives the estimator of the conditional quantile density function $q(\gamma)$ as

$$\widehat{q}^{1}_{Y|X}(\gamma)=\frac{1}{h_H}\int_0^1\frac{H'\left(h_H^{-1}(v-\gamma)\right)}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)}dv.$$

Hence

$$\begin{aligned}
\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma) &= \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)}dv - q_{Y|X}(\gamma)\\
&= \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)}dv - \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv\\
&+ \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv - \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\\
&= -\frac{1}{h_H}\int_0^1 H'\left(h_H^{-1}(v-\gamma)\right)\left[\frac{1}{f_{Y|X}\left(Q_{Y|X}(v)\right)} - \frac{1}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)}\right]dv\\
&+ \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv - \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\\
&= -\frac{1}{h_H}\int_0^1 H'\left(h_H^{-1}(v-\gamma)\right)\left[\frac{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right) - f_{Y|X}\left(Q_{Y|X}(v)\right)}{\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right) f_{Y|X}\left(Q_{Y|X}(v)\right)}\right]dv\\
&+ \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv - \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}.
\end{aligned}$$

Using Theorem 1, $\sup_{\gamma}\left|\widehat{q}_{Y|X}(\gamma)-q_{Y|X}(\gamma)\right| \underset{n\to\infty}{\longrightarrow} 0$ a.co. Hence the above expression asymptotically reduces to

$$\begin{aligned}
&-\frac{1}{h_H}\int_0^1 H'\left(h_H^{-1}(v-\gamma)\right)\left(q_{Y|X}(v)\right)^2\left[\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right) - f_{Y|X}\left(Q_{Y|X}(v)\right)\right]dv +\\
&\qquad \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv - \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}\\
&= -\frac{1}{h_H}\int_0^1 H'\left(h_H^{-1}(v-\gamma)\right)q_{Y|X}(v)\left[\widehat{f}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)\widehat{q}_{Y|X}(v) - f_{Y|X}\left(Q_{Y|X}(v)\right)q_{Y|X}(v)\right]dv +\\
&\qquad \frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv - \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}.
\end{aligned}$$

Since $dF_{Y|X}(\left(Q_{Y|X}(v)\right) = f(\left(Q_{Y|X}(v)\right)q_{Y|X}(v)dv$, hence

$$\begin{aligned}
\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma) &= -\frac{1}{h_H}\int_0^1 H'\left(h_H^{-1}(v-\gamma)\right)q_{Y|X}(v)\left[d\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right) - dF_{Y|X}\left(Q_{Y|X}(v)\right)\right]dv\\
&+\frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv - \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}.
\end{aligned}$$

Writing $H^*(\gamma,v) = H'\left(h_H^{-1}(v,\gamma)\right)q_{Y|X}(v)$ and integrating by parts in the first integral, we get

$$\begin{aligned}
\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma) &= \left[-\frac{1}{h_H}\left(H^*(v,\gamma)\right)\left(\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right) - F_{Y|X}\left(Q_{Y|X}(v)\right)\right)\right]_0^1\\
&+\frac{1}{h_H}\int_0^1 dH^*(\gamma,v)\left[\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right) - F_{Y|X}\left(Q_{Y|X}(v)\right)\right]dt\\
&+\frac{1}{h_H}\int_0^1 \frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv - \frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}.
\end{aligned}$$

Since $F_{Y|X}\left(Q_{Y|X}(0)\right)=\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(0)\right)$ and $F_{Y|X}\left(Q_{Y|X}(1)\right)=\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(1)\right)$, the above expression transforms to

$$\begin{aligned}\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma) &= -\frac{1}{h_H}\int_0^1 dH^*(\gamma,v)\left[F_{Y|X}\left(Q_{Y|X}(v)\right)-\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)\right]dt\\ &+\frac{1}{h_H}\int_0^1\frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv-\frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}.\end{aligned}$$

Putting $h_H^{-1}(v-\gamma)=z$ and using (2),

$$\frac{1}{h_H}\int_0^1\frac{H'\left(h_H^{-1}(v-\gamma)\right)}{f_{Y|X}\left(Q_{Y|X}(v)\right)}dv-\frac{1}{f_{Y|X}\left(Q_{Y|X}(\gamma)\right)}=\frac{1}{h_H}\int_{-\gamma/h_H}^{1-\gamma/h_H}H'(z)q_{Y|X}\left(\gamma+zh_H\right)dz-q_{Y|X}(\gamma). \tag{11}$$

Using Taylor series expansion, we can write

$$q_{Y|X}\left(\gamma+zh_H\right)-q_{Y|X}(\gamma)=\sum_{l=1}^{j-1}\frac{(zh_H)^l}{l!}q^{(l)}_{Y|X}(\gamma)+\frac{(zh_H)^j}{j!}q^{(j)}_{Y|X}(\gamma^*)$$

where $\gamma<\gamma^*<\gamma+zh_H$, assuming higher derivatives of $q_{Y|X}(\gamma)$ exist and are bounded.

Hence (11) can be written as

$$\int_{-\gamma/h_H}^{1-\gamma/h_H}H'(z)\left(q_{Y|X}(\gamma)+\sum_{l=1}^{j-1}\frac{(zh_H)^l}{l!}q^{(l)}_{Y|X}(\gamma)+\frac{(zh_H)^j}{j!}q^{(j)}_{Y|X}(\gamma^*)\right)dv-q_{Y|X}(\gamma). \tag{12}$$

For $n\longrightarrow\infty$, $h_K\longrightarrow 0$, (12) converges to $\int_{\mathbb{R}}H'(z)q_{Y|X}(\gamma)dz-q_{Y|X}(\gamma)$ which equals zero as $\int_{\mathbb{R}}H'(z)=1$.

This gives

$$\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma)=\frac{1}{h_H^2}\int_0^1 dH^*(\gamma,v)\left[\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)-F_{Y|X}\left(Q_{Y|X}(v)\right)\right]dv. \tag{13}$$

Since $\sup_v\left|\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)-F_{Y|X}\left(Q_{Y|X}(v)\right)\right|\underset{n\to\infty}{\longrightarrow}0$, hence $\sup_v\left|\widehat{q}^{1}_{Y|X}(v)-q_{Y|X}(v)\right|\underset{n\to\infty}{\longrightarrow}0$.

The following theorem proves asymptotic normality of the proposed estimator.

## *2.2 Asymptotic normality*

In this section we give the asymptotic normality of $\widehat{q}^{1}_{Y|X}(\gamma)$.

**Theorem 3.** *Suppose that $F$ is continuous. Assume that $K(\cdot)$ satisfies the conditions $(H1)-(H6)$ given in section 2. For $0<\gamma<1$, we have*

$$\sqrt{n\phi_x(h_K)}\left(\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma)\right)$$

*is asymptotically normal with mean zero and variance $\sigma^2(\gamma)$ where*

$$\sigma^2(\gamma)=\frac{n\phi_x(h_K)}{h_H^2}\mathbb{E}\left(\int_0^1 dH^*(\gamma,v)\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)\right)^2.$$

**Proof.**

Using (13), we have

$$\sqrt{n\phi_x(h_K)}\left(\widehat{q}^{1}_{Y|X}(\gamma)-q_{Y|X}(\gamma)\right)=\frac{\sqrt{n\phi_x(h_K)}}{h_H^2}\int_0^1 dH^*(\gamma,v)\left[\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)-F_{Y|X}\left(Q_{Y|X}(v)\right)\right]dv.$$

Using the results of Ezzahrioui and Ould Saïd [6], for $0<\gamma<1$,

$(n\phi_x(h_K))^{1/2}\left(\widehat{Q}_{Y|X}(\gamma)-Q_{Y|X}(\gamma)\right)$ is asymptotically normal with mean zero and variance $\xi^2\left(x,Q_{Y|X}(\gamma)\right)=\frac{a_2^x}{(a_1^x)^2}\frac{F_{Y|X}\left(Q_{Y|X}(\gamma)\right)\left(1-F_{Y|X}\left(Q_{Y|X}(\gamma)\right)\right)}{f^2_{Y|X}\left(Q_{Y|X}(\gamma)\right)}$.

$(n\phi_x(h_K))^{1/2}\left(\widehat{F}_{Y|X}(x,y)-F_{Y|X}(x,y)\right)$ is asymptotically normal with mean zero and variance $\sigma_F^2=\frac{a_1^x}{(a_1^x)^2}F_{Y|X}(x,y)\left(1-F_{Y|X}(x,y)\right)$.

We have also, $(n\phi_x(h_K)))^{1/2}\left(\widehat{F}_{Y|X}(\widehat{Q}_{Y/X}(t))-F_{Y|X}(Q_{Y/X}(t))\right)$ is asymptotically normal with mean zero and variance $\sigma^2(x)$.

With

$$\sigma^2(x)=\frac{\gamma(1-\gamma)a_2^x(x)}{(f_{y/x}(Q_{Y|X}(\gamma)))^2a_1^x(x)}$$

$a_j^x(x)=K^j(1)-\int_0^1(K^j)'(s)\beta_x(s)ds\ \ j=1,2.$

and

$$\forall s\in[0,1],\ \lim_{h\to 0}\phi_x(sh)/\phi_x(h)=\beta_x(s).$$

Since $\frac{d}{d\gamma}F_{Y|X}\left(Q_{Y|X}(\gamma)\right)=1$, $(n\phi_x(h_K))^{1/2}\left[\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(\gamma)\right)-F_{Y|X}\left(Q_{Y|X}(\gamma)\right)\right]$ is asymptotically normal with mean zero and variance $\xi^2\left(x,Q_{Y|X}(\gamma)\right)$.

Using Delta method and Slutsky's theorem (Serfling [25]), we get that $\sqrt{n\phi_x(h_K)}\left(\widehat{q}^1_{Y|X}(\gamma)-q_{Y|X}(\gamma)\right)$ is asymptotically normal with mean zero and variance $\sigma^2(\gamma)=\frac{n\phi_x(h_K)}{h_H^2}\mathbb{E}\left(\int_0^1 dH^*(\gamma,v)\widehat{F}_{Y|X}\left(\widehat{Q}_{Y|X}(v)\right)\right)^2$.

The expression of $\sigma^2(\gamma)$ in the above theorem cannot be simplified analytically and one can estimate it using bootstrapping.

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