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Existence and Continuity of Solutions of Systems of Fractional Differential Equations

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Abstract: In this work we present some results for systems of fractional differential equations described by Caputo-Fabrizio fractional derivative. Existence and uniqueness of solutions are extended to the vectorial case and continuity of solutions with respect to data is stated. Some characteristics of this fractional operator are highlighted. Numerical simulations of the evolution of a system that models the epidemiological phenomenon of a disease with vaccination and treatment in a variable population is presented.

Keywords: Fractional differential equations, Caputo-Fabrizio derivate, approximate solutions.

1 Introduction

Fractional derivatives are defined as integral operators with different kernels. They have memory which makes them suitable to describe certain processes where the global evolution of the system is to be considered. There exist different definitions named after Riemann-Liouville, Caputo, Atangana-Baleanu and Caputo-Fabrizio among others. Formula, properties and theoretical results of these fractional derivative can be found in [1,2,3,4,5,6]. In the last decades diverse applications of this theory were developed in different areas: diffusion problems, hydraulics, potential theory, control theory, electrochemistry, viscoelasticity and nanotechnology (see for example [7,8,9,10,11,12,13,14,15,16,17].

Since the analytical calculus of fractional operators is, in general, not easy, numerical schemes have been proposed to find approximate solutions to fractional differential equations (see [18,19,20,21,22,23,24,25]. Each fractional derivative presents characteristics that make them suitable to describe different phenomena. In the case of Caputo-Fabrizio fractional derivative, it has the advantage of being an integral operator with regular kernel. It has been used to describe some physical systems [26,27,28,29,30,31], hysteresis phenomena [13], difussion problems, predator prey models [25] and disease models [32,33,34]. Despite the numerous applications that have been developed using this operator, there are some details that must be considered when choosing this fractional derivative to describe the evolution of a system. We will comment on them. It is worth mentioning that some critical works regarding some fractional derivatives have also been published (see [35,36,37,38]).

In this work we extend some results regarding existence uniqueness and continuity of solutions to system of ordinary differential equations, to the case of Caputo-Fabrizio fractional derivative. It is organised as follows: in the next section we present some definitions, properties, and point out some characteristics of the Caputo-Fabrizio operator. Existence, uniqueness and continuity of solutions to fractional differential equations are extended to the vectorial case in Section 3. Numerical simulations of the evolution of a system that models the epidemiological phenomenon of a disease with vaccination and treatment in a variable population is presented in Section 4. Finally we state some conclusions.

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2 Mathematical preliminaires

Definition 1. *The Caputo-Fabrizio Fractional derivative(CFFD)*

$$ {}^{CF}D_0^{\alpha} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t f'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} d\tau, \ t \geq 0 \tag{1} $$

where $\alpha \in (0,1)$, $f \in H^1(0,b)$, $b > 0$, *and* $M(\alpha)$ *is a normalization constant (see [31]).*

In this work we consider $M(\alpha) = \frac{2}{2-\alpha}$, thus ${}^{CF}D_0^{\alpha} f(t) = \frac{1}{1-\alpha} \int_0^t f'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} d\tau$.
Note that ${}^{CF}D_0^{\alpha} f(0) = 0$.

Definition 2. *The Caputo-Fabrizio Fractional integral of order* α *For* $f \in H^1(0,b)$ *and* $0 < \alpha < 1$*:*

$$ {}^{CF}I^{\alpha} f(t) := \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(\tau) d\tau, \ t \geq 0 $$

(see [32]).

For the chosen value of $M(\alpha)$ it results ${}^{CF}I^{\alpha} f(t) = (1-\alpha) f(t) + \alpha \int_0^t f(\tau) d\tau$.
Similary to the ordinary differential case, an equivalence between Fractional Initial Value Problem and an integral equation can be stated. Based on this equivalence, existence and uniqueness of solutions to the initial value problem can be proved (see [39,40,41]).

Theorem 1. *Let's* $\Omega = (0,T) \times (y_0 - \delta, y_0 + \delta)$, $f \in H^1(0,T)$ *and* $g : \Omega \longrightarrow \mathbb{R}$ *Lipschitz,* $g(0, f_0) = 0$, *then the initial value problem*

$$ \begin{cases} {}^{CF}D_0^{\alpha} f(t) = g(t, f(t)), \ t \in (0,T), \ \alpha \in (0,1) \\ f(0) = f_0, \quad f_0 \in \mathrm{R} \end{cases} \tag{2} $$

is equivalent to

$$ f(t) = f_0 + \alpha \int_0^t g(\tau, f(\tau)) d\tau + (1-\alpha) g(t, f(t)). \tag{3} $$

That is: $f \in H^1(0,T)$ *is a solution to (2) if and only if* f *satisfies (3)*

Remark. Since ${}^{CF}D_0^{\alpha} f(0) = 0$, the initial value f_0 in (2) must satisfy $g(0, f_0) = 0$. This restriction about the admissible initial values is a consequence of the definition of CFFD and differs from the case of interger order derivatives or other fractional derivatives.

In order to extend the results to the vectorial case we introduce some definitions.
Related to (3) we define $\psi : H^1(0,T) \to H^1(0,T)$

$$ \psi(f)(t) = f_0 + \alpha \int_0^t g(\tau, f(\tau)) d\tau + (1-\alpha) g(t, f(t)) \tag{4} $$

and consider that g is a smooth function, $g(\cdot, f(\cdot)) \in H^1(0,T)$ that satisfies the following hypothesys
Hypothesis: there exits M such that

$$ \|g(\cdot, f_1(\cdot)) - g(\cdot, f_2(\cdot))\|_{H^1(0,T)} \leq M \|f_1 - f_2\|_{H^1(0,T)}. \tag{5} $$

Under this hypothesis, for a given $0 < \alpha < 1$ there exists values of T and M for which ψ is a contraction and existence and uniqueness of solutions to (3) and, consequently, to (2) follow from the fixed point theorem. We include the proof of the following result with the sole purpose of extending it afterwards to the vector case.

Proposition 1. *If* g *is a smooth function satisfying (5), for a given* $0 < \alpha < 1$, ψ *satisfies*

$$ \|\psi(\cdot, f_1(\cdot)) - \psi(\cdot, f_2(\cdot))\|_{H^1(0,T)} \leq L \|f_1 - f_2\|_{H^1(0,T)} $$

with $L = \alpha M \sqrt{T^3 + 1} + (1-\alpha) M$.

Proof

$$\|\psi(\cdot,f_1(\cdot))-\psi(t,f_2(\cdot))\|_{H^1(0,T)}\leq\alpha\|\int_0^t g(s,f_1(s))-g(s,f_2(s))\,ds\|_{H^1(0,T)}+$$
$$+(1-\alpha)\|g(\cdot,f_2(\cdot))-g(\cdot,f_2(\cdot))\|_{H^1(0,T)}.$$

Under the *Hypothesis*, the second term satisfies

$$\|g(\cdot,f_1(\cdot))-g(\cdot,f_2(\cdot))\|_{H^1(0,T)}\leq M\|f_1-f_2\|_{H^1(0,T)}. \tag{6}$$

For the first term we note that

$$\|\int_0^t (g(s,f_1(s))-g(s,f_2(s)))ds\|^2_{H^1(0,T)}$$
$$\leq\int_0^T(\int_0^T|g((s,f_1(s))-g(s,f_2(s))|\,ds)^2dt+M\|f_1-f_2\|_{H^1(0,T)}$$
$$\leq T^3M^2\|f_1-f_2\|^2_{H^1(0,T)}+M\|f_1-f_2\|_{H^1(0,T)}.$$

Choosing $L=\alpha M\sqrt{T^3+1}+(1-\alpha)M$ we have

$$\|\psi(\cdot,f_1(\cdot))-\psi(\cdot,f_2(\cdot)))\|_{H^1(0,T)}\leq L\|f_1-f_2\|_{H^1(0,T)}$$

and the resut follows.

□

If $L<1$, ψ is a contraction in $H^1(0,T)$ and existence and uniqueness of solutions to (2) follows.

Corollary 1. *If $\alpha M\sqrt{T^3+1}+(1-\alpha)M<1$, ψ is a contraction in $H^1(0,T)$ and (2) can be solved uniquely.*

3 Systems of fractional differential equations

Consider for $\alpha\in(0,1)$, a system of fractional differential equations where $v:[0,T]\to\mathbb{R}^n$, $v(t)=(v_1(t),\cdots,v_n(t)), v_i\in H^1(0,T)$ and ${}^{CF}D_0^\alpha v=({}^{CF}D_0^\alpha v_1,\ldots,{}^{CF}D_0^\alpha v_n)$

$$\begin{cases}{}^{CF}D_0^\alpha v(t)=g(t,v(t)),\ t\in(0,T)\\ v(0)=v_0\qquad v_0\in\mathbb{R}^n.\end{cases} \tag{7}$$

We extend the result of Corollary 1 to the vectorial case.

3.1 Existence and uniqueness of solutions

Consider $v(t)=(v_1(t),v_2(t),\cdots,v_n(t))$, $v_i\in H^1(0,T)$ and

$$g(t,v(t))=(g_1(t,v(t)),g_2(t,v(t)),\cdots,g_n(t,v(t))) \tag{8}$$

where $g_i:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are smooth functions, g_i Lipschitz, $g(\cdot,v(\cdot))\in H^1(0,T)\times H^1(0,T)\times\cdots\times H^1(0,T)$.
If $v_0=(v_{01},\cdots,v_{0n})=v(0)$, system (7) can be written as

$${}^{CF}D^\alpha v(t)=g(t,v(t))$$

and its integral (equivalent) form results

$$v(t)=v_0+\alpha\int_0^t g(s,v(s))\,ds+(1-\alpha)g(t,v(t)) \tag{9}$$

where $\int_0^t v(t)\,dt = (\int_0^t v_1(t)\,dt, \int_0^t v_2(t)\,dt, \cdots, \int_0^t v_n(t)\,dt)$.

We denote by X the Banach space $X = H^1(0,T) \times H^1(0,T) \times \cdots H^1(0,T)$ endowed with the norm

$$\|v\|_X^2 = \sum_{i=1}^{n} \|v_i\|_{H^1}^2.$$

The integral operator $\psi : X \to X$ associated to (9) is

$$\psi(v)(t) = (\psi_1(v)(t), \psi_2(v)(t), \cdots \psi_n(v)(t)) \tag{10}$$

where

$$\psi_i(v)(t) = v_{0i} + \alpha \int_0^t g_i(s, v(s))\,dt + (1-\alpha) g_i(t, v(t)) \qquad i = 1, 2, \cdots n. \tag{11}$$

As in the one dimensional case, we will state certain hypothesis on g under which ψ is a contraction, i.e., for $v, w \in X$

$$\|\psi(v) - \psi(w)\|_X \le L\|v - w\|_X, \qquad L < 1$$

and existence and uniqueness of solutions will be guaranteed via the fixed point theorem.

Hypothesis: there exits $M \in \mathbb{R}$ that

$$\|g(\cdot, v(\cdot)) - g(\cdot, w(\cdot))\|_X \le M\|v - w\|_X. \tag{12}$$

Proposition 2. *For a given $0 < \alpha < 1$, there exist $L > 0$ such that ψ satisfies*

$$\|\psi(v) - \psi(w)\|_X \le L\|v - w\|_X. \tag{13}$$

Proof

$$\|\psi(v) - \psi(w)\|_X \le \alpha \|\int_0^t g_1(s, v(s)) - g_1(s, w(s))\,ds\|_X + (1-\alpha)\|g(\cdot, v(\cdot)) - g(\cdot, w(\cdot))\|_X.$$

For the first term we note that

$$\|\int_0^t (g((s, v(s)) - g(s, w(s))\,ds\|_X^2 =$$

$$= \sum_{i=1}^{n} \|\int_0^t g_i(s, v(s)) - g_i((s, w(s))\,ds\|_{L^2(0,T)}^2 + \sum_{i=1}^{n} \|g_i(\cdot, v(\cdot)) - g_i(\cdot, w(\cdot))\|_{L^2(0,T)}^2 \le$$

$$\sum_{i=1}^{n} \|\int_0^t g_i((s, v(s)) - g_i(s, w(s))\,ds\|_{L^2(0,T)}^2 + \|g(\cdot, v(\cdot)) - g(\cdot, w(\cdot))\|_X^2.$$

Since

$$\sum_{i=1}^{n} \|\int_0^t g_i(s, v(s)) - g_i(s, w(s))\,ds\|_{L^2([0,T])}^2 \le \sum_{i=1}^{n} T\|g_i(s, v(s)) - g_i(s, w(s))\|_{L^1(0,T)}^2 \le$$

$$\le T^3 \sum_{i=1}^{n} \|g_i(s, v(s)) - g_i(s, w(s))\|_{L^2}^2 \le T^3 M^2 \|v - w\|_X^2$$

we conclude that

$$\|\int_0^t g(s, v(s)) - g(s, w(s))\,ds\|_X^2 \le (T^3 + 1) M^2 \|v - w\|_X^2.$$

Finally

$$\|\psi(v) - \psi(w))\|_X \le \alpha\sqrt{T^3 + 1} M\|v - w\|_X + (1-\alpha) M\|v - w\|_X.$$

Then, choosing $L = \alpha\sqrt{T^3 + 1} M + (1-\alpha) M < 1$, (13) is satisfied.

□

Note that if $L < 1$, ψ is a contraction in X and we have the following result.

Corollary 2. *For given $0 < \alpha < 1$, there exist values for M and T such that the system (7) has a unique solution.*

3.2 Continuity of solutions with respect to data

When modelling real problems, which inevitably contain uncertainties, it is important to analize the sensitivity of the solution with respect to errors in data. In this section we study the continuity of solutions with respect to g and to the initial value v_0.
Consider that systems

$$\begin{cases} {}^{CF}D_0^{\alpha} v(t) = \tilde{g}(t, v(t)), \ t \in (0,T) \\ v(0) = \tilde{v}_0, \quad \tilde{v}_0 \in \mathrm{R}^n \end{cases} \tag{14}$$

and

$$\begin{cases} {}^{CF}D_0^{\alpha} v(t) = g(t, v(t)), \ t \in (0,T) \\ v(0) = v_0, \quad v_0 \in \mathbb{R}^n \end{cases} \tag{15}$$

are close, i.e., for small ε and δ, g and $\tilde{g}$ satisfy:

$$\|g(\cdot, v(\cdot)) - \tilde{g}(\cdot, v(\cdot))\|_X \leq \varepsilon \tag{16}$$

and v_0 and $\tilde{v}_0$

$$|v_0 - \tilde{v}_0|_{\mathbb{R}^n} < \delta. \tag{17}$$

Under these assumptions we can prove that the solutions of both systems remain close in X.
We note that

$$\|v_0 - \tilde{v}_0\|_X \leq \sqrt{T}\delta$$

and for compatibility reasons, $\tilde{g}(0, \tilde{v}_0) = 0$.

Proposition 3.*Let $\tilde{v} \in X$ be the solution to the perturbed system (14) with initial conditions $\tilde{v}_0 \in \mathbb{R}^n$ and $v \in X$ be the solution to (7) with initial condition v_0. If $L < 1$ in (13), the solutions to both systems are close in X, i.e., $\forall \varepsilon > 0$ there exists $\varepsilon > 0$ and $\delta > 0$ such that*

$$\|v - \tilde{v}\|_X < \varepsilon.$$

Proof
The operator that defines the equivalent integral equation for the system (7) is $\psi : X \to X$

$$\psi(v(t)) = x_0 + \alpha \int_0^t g(s, v(s))\, ds + (1-\alpha) g(t, v(t))$$

and the solution satisfies $\psi(v) = v$.
For the system (14), the related operator is $\tilde{\psi} : X \to X$

$$\tilde{\psi}(\tilde{v}(t)) = \tilde{x}_0 + \alpha \int_0^t \tilde{g}(s, v(s))\, ds + (1-\alpha) \tilde{g}(t, \tilde{v}(t))$$

and its solution satisfies $\tilde{\psi}(\tilde{v}) = \tilde{v}$.
Then,

$$\|\tilde{v} - v\|_X = \|\tilde{\psi}(\tilde{v}) - \psi(v)\|_X \leq$$
$$\|\psi(v) - \psi(\tilde{v})\|_X + \|\psi(\tilde{v}) - \tilde{\psi}(\tilde{v})\|_X \leq L\|v - \tilde{v}\|_X +$$
$$+\|x_0 - \tilde{x}_0\|_X + \alpha \| \int_0^t g(s, \tilde{v}(s)) - \tilde{g}(s, \tilde{v}(s))\, ds\|_X + (1-\alpha)\|g(t, \tilde{v}(t)) - \tilde{g}(t, \tilde{v}(t))\|_X.$$

The term $\|g(t, \tilde{v}(t)) - \tilde{g}(t, \tilde{v}(t))\|_X$ satisfies

$$\|g(t, \tilde{v}(t)) - \tilde{g}(t, \tilde{v}(t))\|_X \leq \varepsilon$$

and for $\| \int_0^t g(s, \tilde{v}(s)) - \tilde{g}(s, \tilde{v}(s))\, ds\|_X$, we have

$$\| \int_0^t g(s, \tilde{v}(s)) - \tilde{g}(s, \tilde{v}(s))\, ds\|_X^2 = \sum_{i=1}^n \| \int_0^t g_i(s, \tilde{v}(s)) - \tilde{g}_i(s, \tilde{v}(s))\, ds\|_{L^2(0,T)}^2 +$$

$$+\sum_{i=1}^{n}\|g_i(t,\tilde{v}(t))-\tilde{g}_i(t,\tilde{v}(s))\|^2_{L^2(0,T)}\leq$$

$$\leq\sum_{i=1}^{n}\int_0^T(\int_0^t g_i(s,\tilde{v}(s))-\tilde{g}_i(s,\tilde{v}(s))^2\,ds)^2+\|g(t,\tilde{v})-\tilde{g}(t,\tilde{v}(t))\|^2_X.$$

Similarly to the one dimensional case,

$$\sum_{i=1}^{n}\int_0^T(\int_0^t g_i(s,\tilde{v}(s))-\tilde{g}_i(s,\tilde{v}(s))^2\,ds)^2\leq$$

$$\leq T\sum_{i=1}^{n}\|g_i(t,\tilde{v}(t))-\tilde{g}_i(t,\tilde{v}(t))\|^2_{L^1}\leq T^3\|g(\cdot,\tilde{v}(\cdot))-\tilde{g}(\cdot,\tilde{v}(\cdot))\|^2_X,$$

and consequently,

$$\|\int_0^t g(s,\tilde{v}(s))-\tilde{g}(s,\tilde{v}(s))\,ds\|^2_X\leq T^3\|g-\tilde{g}\|^2_X+\varepsilon^2\leq(T^3+1)\varepsilon^2.$$

Finally

$$\|\tilde{v}-v\|_X\leq L\|\tilde{v}-v\|_X+\sqrt{T}\delta+\alpha\sqrt{T^3+1}\varepsilon++(1-\alpha)\varepsilon.$$

Since $L<1$, if $\varepsilon=\frac{\sqrt{T}\delta+\alpha\varepsilon\sqrt{T^3+1}++(1-\alpha)\varepsilon}{1-L}$, we have

$$\|\tilde{v}-v\|_X\leq\varepsilon.$$

□

4 Numerical example

Consider a deterministic model for an infectius disease where the population at each instante t is divided into:

- $I(t)$: the number of infected individuals that can infect the susceptible population,
- $S(t)$:the number of individuals likely to be infected,
- $R(t)$: number of isolated individuals who cannot infect or contract the disease.

The following system of fractional differential equations models the epidemiological phenomenon of a disease with vaccination and treatment in a variable population (see [35,36])

$$\begin{cases}{}^{CF}D_0^{\alpha}S(t)=b-\beta S(t)I(t)-(d+\mu_1)S(t),\\ {}^{CF}D_0^{\alpha}I(t)=\beta S(t)I(t)-(\mu_2+d+\sigma)I(t),\\ {}^{CF}D_0^{\alpha}R(t)=\mu_1 S(t)+\mu_2 I(t)-dR(t),\end{cases}\tag{18}$$

with initial conditions $S(t_0)=S_0>0$, $I(t_0)=I_0>0$ and $R(t_0)=R_0>0$.

The parameters of the model are:

- b population birth (recruitment) rate
- β rate of disease transmission between the susceptible population and the infected population
- d natural mortality rate σ mortality rate due to disease
- μ_1 proportion of susceptible individuals who are vaccinated per unit of time
- μ_2 proportion of infected undergoing treatment per unit of time.

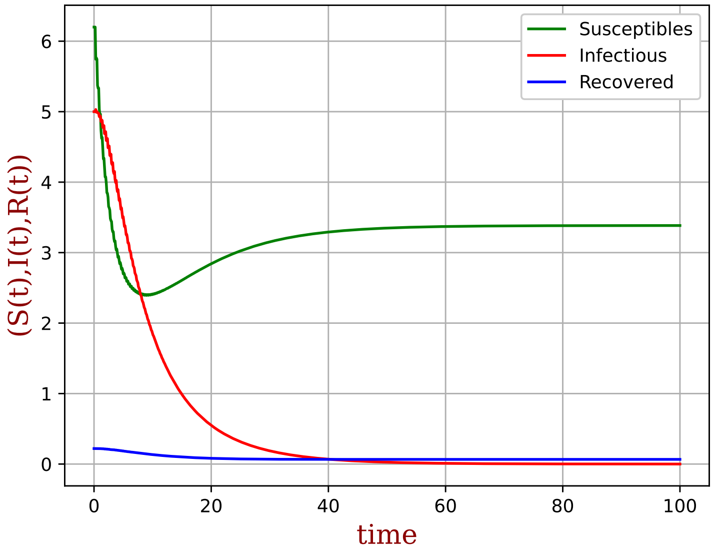
The reason to choose fractional derivatives in (18) instead of the usual integer order ones is that they include memory and contain information about the population's learning mechanism that affects the spread of the disease. In that way, infection and isolation rates and their variation over time is somehow included contributing to reduce the errors that occur when assuming that the values of the parameters are exact and constant. We consider Caputo Fabrizio fractional derivative that is advantageous for numerical implementations, maintaining the appropriate memory property.
Recall that for compatibility reasons, the solutions of the system must satisfy:

$$b-\beta S_0I_0-(d+\mu_1)S_0=0,\quad \beta S_0I_0-(\mu_2+d+\sigma)I_0=0,\quad \mu_1S_0+\mu_2I_0-dR_0=0.$$

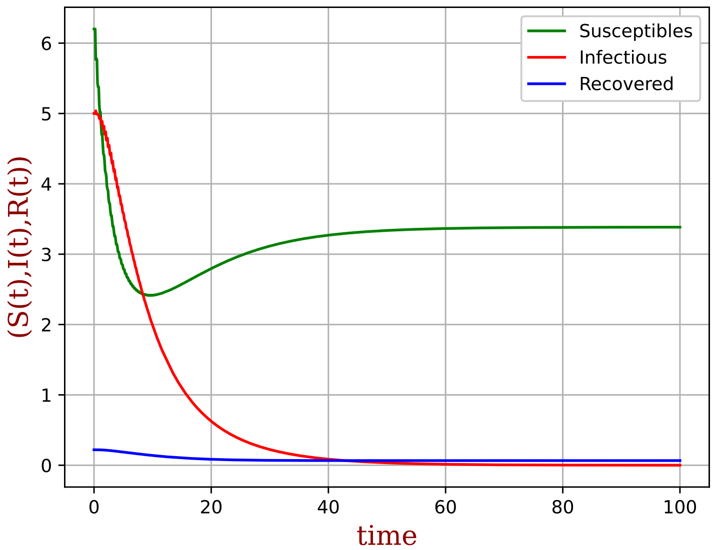
and consequently, the initial values can take only the following values:

–if $I_0 = 0$ then $S_0 = \frac{b}{d+\mu_1}$
–if $I_0 > 0$ then $S_0 = \frac{d+\mu_2+\sigma}{\beta}$

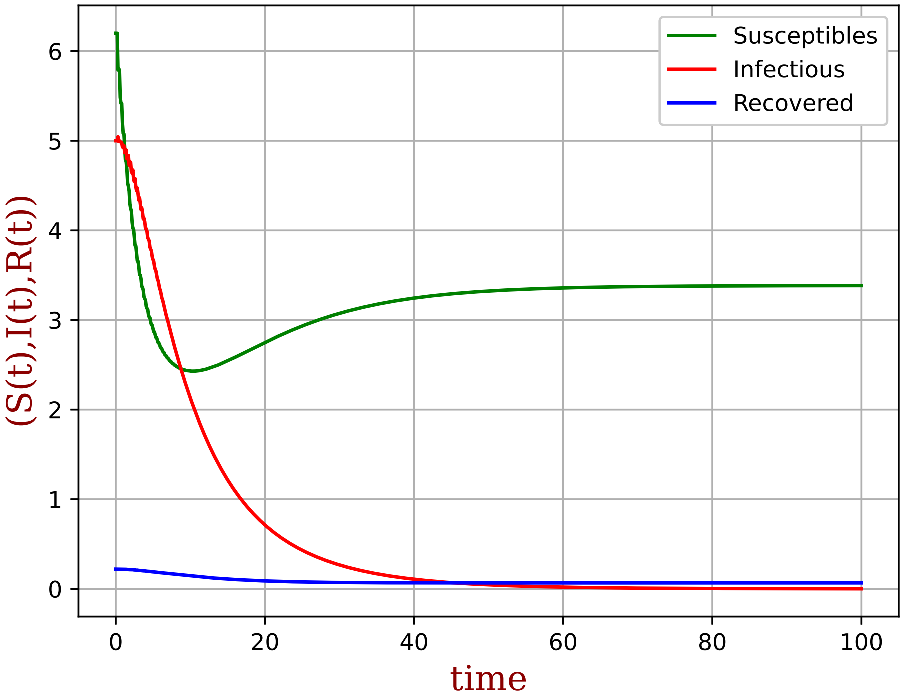
This is a restriction since the model should serve to represent various scenarios of reality and, in general, it reduces the admissible initial conditions to a given system.
Figure 1 shows the results of the numerical approximation of the model using the Three-step Adams-Brashforth scheme, for the Ordinary Differential Equations, i.e. $\alpha = 1$, and for the Fractional Differential Equations for several values of α and for the parameter values: $b = 1,76$, $\beta = 0,1$, $d = 0,51$, $\sigma = 0,1$, $\mu_1 = 0,01$ and $\mu_2 = 0,01$. We choose the addmissible initial conditions as $S_0 = 6,2$, $I_0 = 5$ and $R_0 = 0,22$.



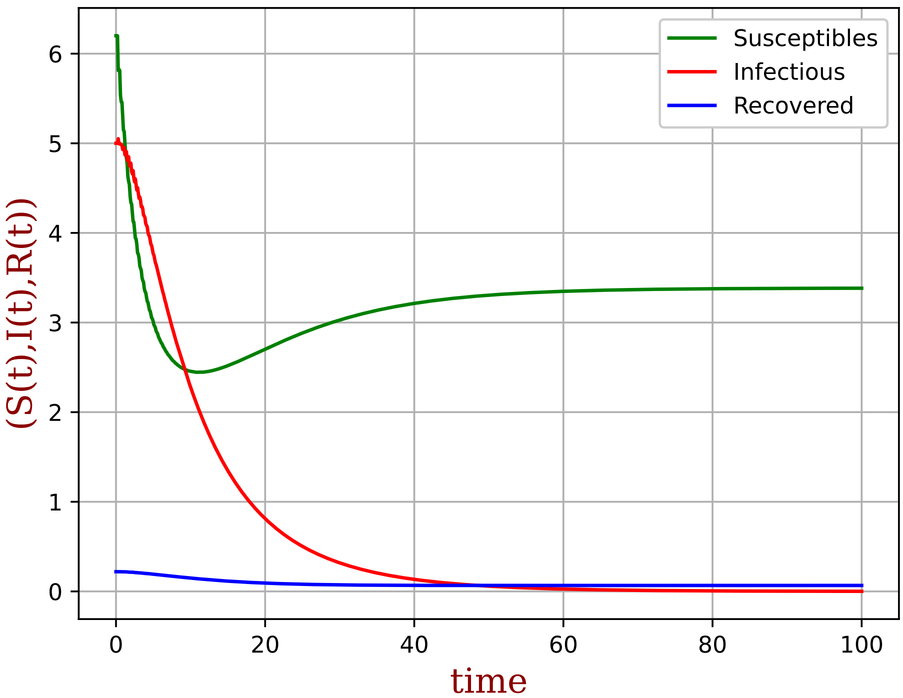
$(S(t), I(t), R(t))$ vs t, $\alpha = 1$



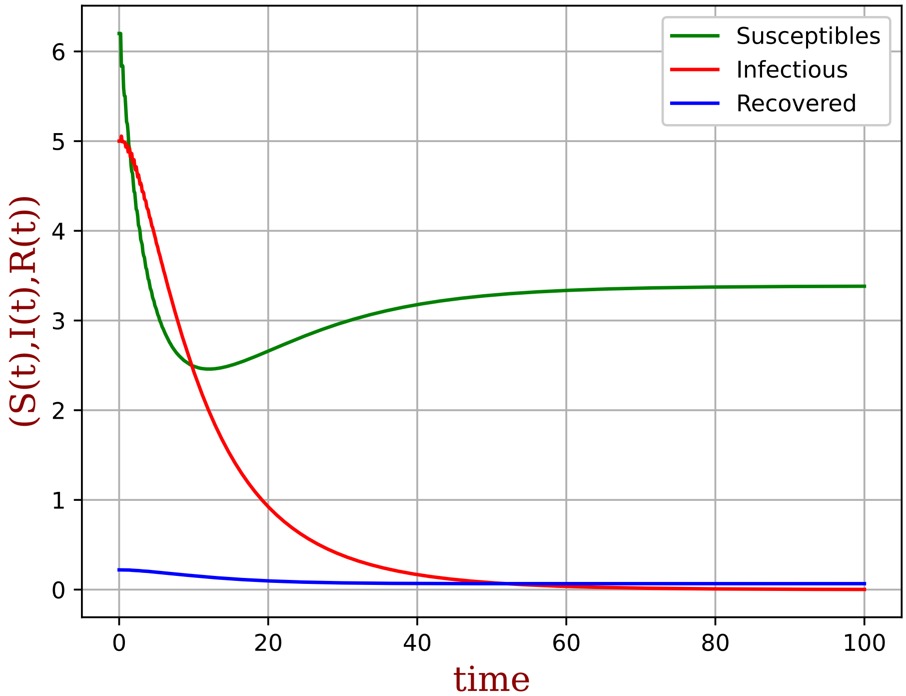
$(S(t), I(t), R(t))$ vs t, $\alpha = 0,95$



$(S(t), I(t), R(t))$ vs t, $\alpha = 0,90$



$(S(t), I(t), R(t))$ vs t, $\alpha = 0,85$



$(S(t), I(t), R(t))$ vs t, $\alpha = 0,80$

Fig. 1: Numerical approximation of the model for different values of α

Note that since the average number of new cases is less than 1, despite the fact that there is nearly no vaccination or treatment, the infected population $I(t)$, decreases considerably from the beginning to practically ending on day 40. However the number of susceptible population $S(t)$, remains constant around 3 in the 10th day. Regarding $I(t)$, it can be observed that the numerical approximation of the solution to the problem with the ordinary derivative practically coincides with the approximation found using the fractional derivative with $\alpha = 0,95$.

5 Conclusion

In this work we extend results concerning existence and uniqueness of solutions to systems of differential equations, to the case of Caputo Fabrizio fractional derivative. We proved continuity of the solution with respect to data. Afterwards we numerically approximate the solutions to an epidemiological model described by this derivative. We comment on the advantages as well as the restrictions in the use of this fractional derivate.

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