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A Note on Nonlinear Implicit Neutral Katugampola Fractional Differential Equations with Impulse Effects and Finite Delay

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Abstract: In this work, we consider nonlinear implicit neutral Katugampola fractional differential equations with impulse effects and finite delay. The existence of solution is obtained by using fixed point theorems. Also, the Ulam stability for such a class of problems will be considered. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained results.

Keywords: Katugampola fractional derivative, Implicit neutral fractional differential equations, Existence, Stability, Fixed point.

1 Introduction

In recent two decades, fractional calculus has grabbed the attention of many researchers and remarkable contributions have been made to both theory and applications of fractional differential equations. Fractional equations have incredible applications in various sciences such as material sciences, mechanics, seepage flow in porous media, in fluid dynamic traffic models, population dynamics, economics, chemical technology, medicine and many other related fields. In fact, fractional differential equations are considered as an option model for nonlinear partial differential equations. The nonlinear oscillations of earthquake can be described by the fractional differential equations. The details on the theory and its applications can be found in books [1,5,6, 14,24] and references given therein. Neutral type differential equations mainly look as models of electrical network arise in high speed computers, for example these are used to interconnect switching circuits. Benchohra et al. [7] studied the existence and stability for implicit neutral fractional differential equations with finite delay and impulses. Many physical and engineering problems, for example, fluid dynamics, electronics and kinetics can be modeled in the form of the implicit differential equations(IDEs). For more details, see the papers [2,27]. Moreover, the state of many phenomena and processes considered in optimal control theory, biology, biotechnologies, etc. are frequently subject to instantaneous perturbation and experiences sudden changes (impulses) at specific moments of time. The length of the time of these changes is very small and negligible in comparison with the total duration of the process considered. Such processes and phenomena with short-term external influences can be modeled as impulsive differential equation. For the general theory of such differential equations, we refer to the books [8,10, 20,22] and the papers [13,26]. We consider the nonlinear implicit neutral Katugampola fractional differential equations with impulse effects and finite delay,

$$\begin{cases} {}^{\rho}D^{\omega}_{x_m}[u(x)-\psi(x,u_x)] = h(x,u_x,{}^{\rho}D^{\omega}_{x_m}u(x)), \\ \text{for each } x\in(x_m,x_{m+1}],\ m=0,1,\dots,k,\ 0<\omega\le 1, \\ \Delta u|_{x=x_m} = I_m(u_{x_m^-}),\ m=1,\dots,k, \\ u(x)=\phi(x),\ x\in[-r,0], r>0, \end{cases} \tag{1}$$

where ${}^{\rho}D^{\omega}_{x_m}$ is the Katugampola fractional derivative in Caputo sense, $h:\mathfrak{J}\times\mathfrak{PC}([-r,0],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$, $\psi:\mathfrak{J}\times\mathfrak{PC}([-r,0],\mathbb{R})\to\mathbb{R}$ are given function with $\psi(0,\phi)=0$, $I_m:\mathfrak{PC}([-r,0],\mathbb{R})\to\mathbb{R}$, and $\phi\in\mathfrak{PC}([-r,0],\mathbb{R})$, $0=x_0<x_1<\dots<x_k<x_{k+1}=T$, and $\mathfrak{PC}([-r,0],\mathbb{R})$ is a Banach space. For each function u

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defined on $[-r,T]$ and for any $x \in \mathfrak{J}$, denote by u_x the element of $\mathfrak{PC}([-r,0],\mathbb{R})$ defined by $u_x(\theta) = u(x+\theta)$, $\theta \in [-r,0]$, that is $u_x(\cdot)$ means that the history of the state from time $x-r$ up to time x, and $\Delta u|_{x=x_m} = u(x_m^+) - u(x_m^-)$, where $u(x_m^+) = \lim_{l\to 0^+} u(x_m+l)$ and $u(x_m^-) = \lim_{l\to 0^-} u(x_m+l)$ represents the right and left limits of u at $x = x_m$, respectively. This paper is divided into five sections, in which second section provides some basic definitions, notations, lemmas and theorems. In third section, we obtain the existence results of the solutions of the considered problem. The fourth section is concerned with Ulam-Hyers-Rassias stable result. At the end, an example is included.

2 Prerequisites

In this section, we introduce basic notations, definitions, lemmas and theorems that are used for the main results. Let $T > 0$, $\mathfrak{J} = [0,T]$ and $\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup\{|u(x)| : x \in \mathfrak{J}\}.$$

Let $\mathfrak{J}_0 = [x_0,x_1]$ and $\mathfrak{J}_m = (x_m,x_{m+1}]$, where $m = 1,2,\dots k$, and $-r = t_0 < t_1 < \dots < t_l < t_{l+1} = 0$ with $l \le k$.
Consider the set of functions

$$\begin{aligned}\mathfrak{PC}([-r,0],\mathbb{R}) = \{u : [-r,0] \to \mathbb{R} : u \in \mathfrak{C}((t_m,t_{m+1}],\mathbb{R}),\\ m = 0,1,\dots,l, \text{ and there exist} u(t_m^-) \text{ and}\\ u(t_m^+),\ m = 1,2,\dots,l, \text{ with } u(t_m^-) = u(t_m)\}.\end{aligned}$$

$\mathfrak{PC}([-r,0],\mathbb{R})$ is a Banach space with the norm

$$\|u\|_{\mathfrak{PC}} = \sup_{x\in[-r,0]} |u(x)|.$$

$$\begin{aligned}\mathfrak{PC}([0,T],\mathbb{R}) = \{u : [0,T] \to \mathbb{R} : u \in \mathfrak{C}((x_m,x_{m+1}],\mathbb{R}),\\ m = 1,\dots,k, \text{ and there exist } u(t_m^-) \text{ and}\\ u(t_m^+),\ m = 1,2,\dots,k, \text{ with } u(x_m^-) = u(x_m)\}.\end{aligned}$$

$\mathfrak{PC}([0,T],\mathbb{R})$ is a Banach space with the norm

$$\|u\|_{\mathfrak{C}} = \sup_{x\in[0,T]} |u(x)|.$$

η is a Banach space with the norm

$$\|u\|_\eta = \sup_{x\in[-r,T]} |u(x)|.$$

$\mathfrak{L}'(\mathfrak{J},\mathbb{R})$ is the space of Lebesgue-integrable functions $u : \mathfrak{J} \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^T |u(s)|\,\mathrm{d}s.$$

$\mathfrak{AC}^n(\mathfrak{J}) = \{h : \mathfrak{J} \to \mathbb{R} : h,h',\dots h^{(n-1)} \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ and $h^{(n-1)}$ is absolutely continuous$\}$. We consider either $\omega \ge 0$, or $\omega > 0$ for the remaining sections.

Definition 1.[18, 21] *The fractional(arbitrary) order integral of the function $h \in \mathfrak{L}'([0,T],\mathbb{R}_+)$ of order $\omega \in \mathbb{R}_+$ is defined by*

$$I^\omega_{0^+} h(x) = \frac{1}{\Gamma(\omega)} \int_0^x (x-s)^{\omega-1} h(s)\mathrm{d}s,$$

where Γ is the Euler gamma function defined by $\Gamma(\omega) = \int_0^\infty x^{\omega-1} e^{-x}\mathrm{d}x$, $\omega > 0$.

Definition 2.[18, 21] *For a function $h \in \mathfrak{AC}^n(\mathfrak{J})$, the Caputo fractional order derivative of order ω of h, is defined by*

$$({}^cD^\omega_{0^+} h)(x) = \frac{1}{\Gamma(n-\omega)} \int_0^x (x-s)^{n-\omega-1} h^n(s)\mathrm{d}s,$$

where $n = [\omega]+1$ and $[\omega]$ denotes the integer part of the real number ω.

Definition 3.[18, 21] *Let $a \in [0,T]$, $\delta > 0$ such that $a+\delta \le T$. For a function $h \in \mathfrak{AC}^n[a,T]$, the Caputo-order derivative of order ω of h, is defined by*

$$\left({}^\rho D^\omega_{a^+} h\right)(x) = \frac{1}{\Gamma(n-\omega)} \int_a^x (x-s)^{n-\omega-1} h^{(n)}(s)\mathrm{d}s,$$

where $n = [\omega]+1$ and $[\omega]$ denotes the integer part of the real number ω.

Definition 4.[15, 16, 17] *The generalized left-sided fractional integral ${}^\rho I^\omega_{0^+} h$ of order $\omega \in \mathbb{C}(Re(\omega) > 0)$ is defined by*

$$({}^\rho I^\omega_{0^+} h)(x) = \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} h(s)\mathrm{d}s, \quad (2)$$

for $x > 0$, if the integral exists.

Definition 5.[15, 16, 17] *The generalized fractional derivative, corresponding to the generalized fractional integral (2), is defined by*

$$({}^\rho D^\omega_{0^+} h)(x) = \frac{\rho^{\omega-n+1}}{\Gamma(n-\omega)} \left(x^{1-\rho}\frac{d}{dx}\right)^n \int_0^x (x^\rho - s^\rho)^{n-\omega-1} s^{\rho-1} h(s)\mathrm{d}s, \quad (3)$$

if the integral exists.

Lemma 1.[25] *Let $\omega \ge 0$ and $n = [\omega]+1$. Then*

$${}^\rho I^\omega_{0^+}\left({}^\rho D^\omega_{0^+} h(x)\right) = h(x) - \sum_{m=0}^{n-1} \frac{h^m(0)}{m!} x^m.$$

Lemma 2.[25] *Let $\omega > 0$, then the differential equation ${}^\rho D^\omega_{0^+} h(x) = 0$ has solutions*

$$h(x) = b_0 + b_1\left(\frac{x^\rho}{\rho}\right) + b_2\left(\frac{x^\rho}{\rho}\right)^2 + \dots + b_{n-1}\left(\frac{x^\rho}{\rho}\right)^{(n-1)},$$

$b_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\omega]+1$.

Lemma 3.[25] *Let $\omega > 0$, then*

$$ {}^{\rho}I^{\omega}_{0+}\left({}^{\rho}D^{\omega}_{0+}h(x)\right) = h(x) + b_0 + b_1\left(\frac{x^{\rho}}{\rho}\right) + b_2\left(\frac{x^{\rho}}{\rho}\right)^2 + \cdots + b_{n-1}\left(\frac{x^{\rho}}{\rho}\right)^{(n-1)}, $$

for some $b_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\omega]+1$.

Lemma 4.[25] *Let $w : [0,T] \to [0,+\infty)$ be a real function and $\alpha(\cdot)$ is a non-negative, locally integrable function on $[0,T]$ and there are constants $a > 0$ and $0 < \omega \le 1$ such that*

$$ w(x) \le \alpha(x) + a\int_0^x (x^{\rho} - s^{\rho})^{-\omega} s^{-\rho} w(s)\mathrm{d}s. $$

Then, there exists a constant $K = K(\omega)$ such that

$$ w(x) \le \alpha(x) + Ka\int_0^x (x^{\rho} - s^{\rho})^{-\omega} s^{-\rho} \alpha(s)\mathrm{d}s, $$

for every $x \in [0,T]$.

Now we consider the integral inequality of Gronwall type for piecewise continuous functions which was introduced by Bainov and Hristova [4].

Lemma 5.*Let for $x \ge x_0 \ge 0$, the following inequality hold,*

$$ u(x) \le a(x) + \int_{x_0}^x g(x,s)u(s)\mathrm{d}s + \sum_{x_0<x_m<x} \beta_m(x)u(x_m), $$

where $\beta_m(x)$ $(m \in \mathbb{N})$ are non-decreasing functions for $x \ge x_0$, $a \in \mathfrak{PC}([x_0,\infty),\mathbb{R}_+)$, a is non-decreasing and $g(x,s)$ is a continuous non negative function for $x,s \ge x_0$ and non decreasing with respect to x for any fixed $s \ge x_0$. Then, for $x \ge x_0$, the following inequality is valid:

$$ u(x) \le a(x) \prod_{x_0<x_m<x} (1+\beta_m(x)) \exp\left(\int_{x_0}^x g(x,s)\mathrm{d}s\right). $$

Now, we consider the concepts of Wang et al. [27] and introduce Ulam's type stability concepts for the problem (1). See the papers [3,9,19,23,25]. Let $u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$, $\varepsilon > 0$, $\varphi > 0$ and $\alpha \in \mathfrak{PC}(\mathfrak{J},\mathbb{R}_+)$ be non decreasing. We consider the set of inequalities

$$ \begin{cases} \left|{}^{\rho}D^{\omega}[u(x) - \psi(x,u_x)] - h(x,u_x,{}^{\rho}D^{\omega}_{0+}u(x))\right| \le \varepsilon, \\ x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x=x_m} - I_m(u_{x_m^-})\right| \le \varepsilon,\ m = 1,\ldots,k; \end{cases} \tag{4} $$

the set of inequalities

$$ \begin{cases} \left|{}^{\rho}D^{\omega}_{0+}[u(x) - \psi(x,u_x)] - h(x,u_x,{}^{\rho}D^{\omega}_{0+}u(x))\right| \le \alpha(x), \\ x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x=x_m} - I_m(u_{x_m^-})\right| \le \varphi,\ m = 1,\ldots,k; \end{cases} \tag{5} $$

and the set of inequalities

$$ \begin{cases} \left|{}^{\rho}D^{\omega}_{0+}[u(x) - \psi(x,u_x)] - h(x,u_x,{}^{\rho}D^{\omega}_{0+}u(x))\right| \le \varepsilon\alpha(x), \\ x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x=x_m} - I_m(u_{x_m^-})\right| \le \varepsilon\varphi,\ m = 1,\ldots,k \end{cases} \tag{6} $$

Definition 6.*The problem (1) is Ulam-Hyers stable, if there exists a real number $c_{h,k} > 0$ such that for each $\varepsilon > 0$ and for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (4), there exists a solution $u_2 \in \mathfrak{PC}([-r,0],\mathbb{R})$ of the problem (1) with*

$$ |u_1(x) - u_2(x)| \le c_{h,k}\varepsilon,\ x \in \mathfrak{J}. $$

Definition 7.*The problem (1) is generalized Ulam-Hyers stable, if there exists $\theta_{h,k} \in \mathfrak{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\theta_{h,k}(0) = 0$ such that for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (4), there exists a solution $u_2 \in \mathfrak{PC}([-r,0],\mathbb{R})$ of the problem (1) with*

$$ |u_1(x) - u_2(x)| \le \theta_{h,k}(\varepsilon),\ x \in \mathfrak{J}. $$

Definition 8.*The problem (1) is Ulam-Hyers-Rassias stable with respect to (α,φ), if there exists $c_{h,k,\alpha} > 0$ such that for each $\varepsilon > 0$ and for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (6), there exists a solution $u_2 \in \mathfrak{PC}([-r,0],\mathbb{R})$ of the problem (1) with*

$$ |u_1(x) - u_2(x)| \le c_{h,k,\alpha}\varepsilon(\alpha(x) + \varphi),\ x \in \mathfrak{J}. $$

Definition 9.*The problem (1) is generalized Ulam-Hyers-Rassias stable with respect to (α,φ), if there exists $c_{h,k,\alpha} > 0$ such that for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (5), there exists a solution $u_2 \in \mathfrak{PC}([-r,0],\mathbb{R})$ of the problem (1) with*

$$ |u_1(x) - u_2(x)| \le c_{h,k,\alpha}(\alpha(x) + \varphi),\ x \in \mathfrak{J}. $$

*Remark.*From the above definitions, we get

(i)$Definition$ 2.10 $\Rightarrow$ $Definition$ 2.11;
(ii)$Definition$ 2.12 $\Rightarrow$ $Definition$ 2.13;
(iii)$Definition$ 2.12 for $\alpha(x) = \varphi = 1 \Rightarrow$ Definition 2.10.

*Remark.*A function $u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ is a solution of the inequality (6) if and only if if there is $\sigma \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ and a sequence σ_m, $m = 1,2,\ldots k$(which depends on u) such that

(i)$|\sigma(x)| \le \varepsilon\alpha(x)$, $x \in (x_m,x_{m+1}]$, $m = 1,2,\ldots k$ and $|\sigma_m| \le \varepsilon\varphi$, $m = 1,2,\ldots k$;
(ii)${}^{\rho}D^{\omega}_{0+}[u(x) - \psi(x,u_x)] = h(x,u_x,{}^{\rho}D^{\omega}_{0+}u(x)) + \sigma(x), x \in (x_m,x_{m+1}]$, $m = 1,2,\ldots k$;
(iii)$\Delta u|_{x=x_m} = I_m(u_{x_m^-}) + \sigma_m$, $m = 1,2,\ldots k$.

Similarly, we can get remarks for inequalities (4) and (5).

Theorem 1.[12](*Ascoli-Arzela's Theorem*) *Let $E \subset \mathfrak{PC}(\mathfrak{J},\mathbb{R})$, E is relatively compact(i.e,$\bar{E}$ is compact), if:*

(1)E is uniformly bounded, that is there exists $N > 0$ such that
$$|h(x)| < N,$$
for every $h \in E$ and $x \in (x_m, x_{m+1})$, $m = 1,2,\dots k$.
(2)E is equicontinuous on (x_m, x_{m+1}), that is for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x_1, x_2 \in (x_m, x_{m+1})$, $|x_1 - x_2| \leq \delta$ implies $|h(x_1) - h(x_2)| \leq \varepsilon$, for every $h \in E$.

Theorem 2.*[11](Banach's fixed point theorem) Let $\mathscr{C}$ be a non empty closed subset of a Banach space $\mathscr{X}$, then any contraction mapping T of $\mathscr{C}$ into itself has a unique fixed point.*

Theorem 3.*[11](Schaefer's fixed point theorem) Let $\mathscr{X}$ be a Banach space, and $M : \mathscr{X} \to \mathscr{X}$ a completely continuous operator. If the set*
$$S = \{u \in \mathscr{X} : u = \mu M u, \text{ for some } \mu \in (0,1)\},$$
is bounded, then M has at least one fixed points.

3 Existence of solutions

Definition 10.*A function $u \in \eta$ whose ω-derivative exists on $\mathfrak{J}_m$ is said to be a solution of (1), if u satisfies the equation*
$$^{\rho}D^{\omega}_{x_m}[u(x) - \psi(x,u_x)] = h(x, u_x, {}^{\rho}D^{\omega}_{x_m} u(x)),$$
on $\mathfrak{J}_m$, and satisfy the conditions $\Delta u|_{x=x_m} = I_m(u_{x_m^-})$, $m = 1,\dots,k$ and $u(x) = \phi(x)$, $x \in [-r,0]$.

We need the following lemma to prove the existence of solutions to (1).

Lemma 6.*Let $0 < \omega \leq 1$ and let $\sigma : \mathfrak{J} \to \mathbb{R}$ be continuous. A function u is a solution of the Katugampola fractional integral equation*
$$u(x) = \begin{cases} \phi(0) + \psi(x,u_x) \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds, \\ \text{if } x \in [0, x_1], \\ \\ \phi(0) + \psi(x,u_x) + \sum_{i=1}^m I_i(u_{x_i^-}) \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^m \int_{x_{i-1}}^{x_i} (x_i^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds, \\ \text{if } x \in (x_m, x_{m+1}], \\ \\ \phi(x), \text{ if } x \in [-r,0], \end{cases} \tag{7}$$
where $m = 1,2,\dots k$, if and only if u is a solution of the following fractional problem
$$\begin{cases} ^{\rho}D^{\omega}_{x_m}[u(x) - \psi(x,u_x)] = \sigma(x),\ x \in \mathfrak{J}_m, \\ \Delta u|_{x=x_m} = I_m(u_{x_m^-}),\ m = 1,2,\dots k, \\ u(x) = \phi(x),\ x \in [-r,0]. \end{cases} \tag{8}$$

*Proof.*Assume that u satisfies (8). If $x \in [0, x_1]$, then ${}^{\rho}D^{\omega}_{0+}[u(x) - \psi(x,u_x)] = \sigma(x)$. From Lemma 3, we get
$$\begin{aligned} u(x) - \psi(x,u_x) &= \phi(0) + {}^{\rho}I^{\omega}_{0+}\sigma(x) \\ &= \phi(0) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds. \end{aligned}$$
If $x \in (x_1, x_2]$, then from Lemma 3, we get
$$\begin{aligned} u(x) - \psi(x,u_x) &= u(x_1^+) - \psi(x_1, u_{x_1}) \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds \\ &= \Delta u|_{x=x_1} + u(x_1^-) - \psi(x_1, u_{x_1}) \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds \\ &= \phi(0) + I_1(u_{x_1^-}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^{x_1} (x_1^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds \end{aligned}$$
If $x \in (x_2, x_3]$, then from Lemma 3, we have,
$$\begin{aligned} u(x) - \psi(x,u_x) &= u(x_2^+) - \psi(x_2, u_{x_2}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds \\ &= \Delta u|_{x=x_2} + u(x_2^-) - \psi(x_2, u_{x_2}) \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds \\ &= I_2(u_{x_2^-}) \\ &+ [\phi(0) + I_1(u_{x_1^-}) \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^{x_1} (x_1^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^{x_2} (x_2^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds] \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds. \\ &= \phi(0) + \left[I_1(u_{x_1^-}) + I_2(u_{x_2^-}) \right] \\ &+ [\frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^{x_1} (x_1^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^{x_2} (x_2^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds] \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s) ds. \end{aligned}$$
In this way, continuing the process, we get the solution $u(x)$ for $x \in (x_m, x_{m+1}]$, where $m = 1,2,\dots k$. Therefore,
$$\begin{aligned} u(x) &= \phi(0) + \psi(x,u_x) + \sum_{i=1}^m I_i(u_{x_i^-}) \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^m \int_{x_{i-1}}^{x_i} (x_i^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s) ds. \end{aligned}$$
Conversely, let us assume that u satisfies the equation (7). If $x \in [0, x_1]$, then $u(0) = \phi(0)$ and using the concept that ${}^{\rho}D^{\omega}_{0+}$ is the left inverse of ${}^{\rho}I^{\omega}_{0+}$, we get

${}^{\rho}D^{\omega}_{0^+}[u(x)-\psi(x,u_x)]=\sigma(x)$, for each $x\in[0,x_1]$. If $x\in(x_m,x_{m+1}]$, $m=1,2,\dots k$ and using the fact that ${}^{\rho}D^{\omega}_{0^+}L=0$, where L is a constant, we get

$$ {}^{\rho}D^{\omega}_{x_m}[u(x)-\psi(x,u_x)]=\sigma(x),\ \text{for each}\ x\in(x_m,x_{m+1}]. $$

Also, we can show that $\Delta u|_{x=x_m}=I_m(u_{x_m^-})$, $m=1,2,\dots k$.

Now we state and prove the existence results for the problem (1), based on Banach's fixed point theorem.

Theorem 4. *Assume that*

(A1) The function $h:\mathfrak{J}\times\mathfrak{PC}([-r,0],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is continuous.

(A2) There exist constants $c_1>0$, $0<c_2<1$ and $c_3>0$ such that

$$ |h(x,z_1,z_2)-h(x,\bar z_1,\bar z_2)|\le c_1\|z_1-\bar z_1\|_{\mathfrak{PC}}+c_2|z_2-\bar z_2|, $$

and

$$ |\psi(x,z_1)-\psi(x,\bar z_1)|\le c_3\|z_1-\bar z_1\|_{\mathfrak{PC}}, $$

for any $z_1,\bar z_1\in\mathfrak{PC}([-r,0],\mathbb{R})$, $z_2,\bar z_2\in\mathbb{R}$ and $x\in\mathfrak{J}$.

(A3) There exists a constant $c_4>0$ such that

$$ |I_m(z_1)-I_m(\bar z_1)|\le c_4\|z_1-\bar z_1\|_{\mathfrak{PC}}, $$

for each $z_1,\bar z_1\in\mathfrak{PC}([-r,0],\mathbb{R})$ and $m=1,2,\dots k$.

If

$$ kc_4+c_3+\frac{(k+1)c_1T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}<1, \tag{9} $$

then, there exists a unique solution for the problem (1) on $\mathfrak{J}$.

Proof. Transform the problem (1) into a fixed point problem. Consider the operator $M:\eta\to\eta$ defined by

$$ Mu(x)=\begin{cases}\phi(0)+\psi(x,u_x)+\sum_{0<x_m<x}I_m(u_{x_m^-})\\ +\frac{\rho^{1-\omega}}{\Gamma(\omega)}\\ \left[\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}\left(x_m^{\rho}-s^{\rho}\right)^{\omega-1}s^{\rho-1}\,g(s)\mathrm{d}s\right]\\ +\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}\left(x^{\rho}-s^{\rho}\right)^{\omega-1}s^{\rho-1}\,g(s)\mathrm{d}s,\\ x\in[0,T],\\ \phi(x),\ x\in[-r,0],\end{cases} \tag{10} $$

where $g\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that

$$ g(x)=h(x,u_x,g(x)). $$

Clearly, the fixed points of operator M are solutions of the problem (1). Let $y,z\in\eta$. If $x\in[-r,0]$, then

$$ |M(y)(x)-M(z)(x)|=0. $$

For $x\in\mathfrak{J}$, we get

$$ \begin{aligned}&|M(y)(x)-M(z)(x)|\\ &\le\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}\left(x_m^{\rho}-s^{\rho}\right)^{\omega-1}s^{\rho-1}\,|g_1(s)-g_2(s)|\mathrm{d}s\\ &+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}\left(x^{\rho}-s^{\rho}\right)^{\omega-1}s^{\rho-1}\\ &[|g_1(s)-g_2(s)|\mathrm{d}s+|\psi(x,y_x)-\psi(x,z_x)|]\mathrm{d}s\\ &+\sum_{0<x_m<x}\left|I_m(y_{x_m^-})-I_m(z_{x_m^-})\right|,\end{aligned} $$

where $g_1,g_2\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that

$$ g_1(x)=h(x,y_x,g_1(x)), $$

and

$$ g_2(x)=h(x,z_x,g_2(x)). $$

By $(A2)$, we get

$$ \begin{aligned}|g_1(x)-g_2(x)|&=|h(x,y_x,g_1(x))-h(x,z_x,g_2(x))|\\ &\le c_1\|y_x-z_x\|_{\mathfrak{PC}}+c_2|g_1(x)-g_2(x)|\end{aligned} $$

This implies,

$$ |g_1(x)-g_2(x)|\le\frac{c_1}{1-c_2}\|y_x-z_x\|_{\mathfrak{PC}}. $$

Therefore, for each $x\in\mathfrak{J}$,

$$ \begin{aligned}&|M(y)(x)-M(z)(x)|\\ &\le\frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\sum_{m=1}^{k}\int_{x_{m-1}}^{x_m}\left(x_m^{\rho}-s^{\rho}\right)^{\omega-1}s^{\rho-1}\,\|y_s-z_s\|_{\mathfrak{PC}}\mathrm{d}s\\ &+\frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_{x_m}^{x}\left(x^{\rho}-s^{\rho}\right)^{\omega-1}s^{\rho-1}\,\|y_s-z_s\|_{\mathfrak{PC}}\mathrm{d}s\\ &+\sum_{m=1}^{k}c_4\|y_{x_m^-}-z_{x_m^-}\|_{\mathfrak{PC}}+c_3\|y_x-z_x\|_{\mathfrak{PC}}.\\ &\le\left[kc_4+c_3+\frac{kc_1T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}+\frac{c_1T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}\right]\times\\ &\|y-z\|_{\eta}.\end{aligned} $$

Thus,

$$ \|M(y)-M(z)\|_{\eta}\le\left[kc_4+c_3+\frac{(k+1)c_1T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}\right]\|y-z\|_{\eta}. $$

By (9), the operator M is a contraction. Hence, by the Banach's contraction principle, M has a unique fixed point which is a unique solution of the problem (1).

Now, Schaefer's fixed point theorem is used to prove the second result.

Theorem 5. *Assume that $(A1),(A2)$ and*

(A4) There exists $p_1,p_2,p_3\in\mathfrak{C}(\mathfrak{J},\mathbb{R}_+)$ with $p_3^*=\sup_{x\in\mathfrak{J}}p_3(x)<1$ such that

$$ |h(x,y,z)|\le p_1(x)+p_2(x)\|y\|_{\mathfrak{PC}}+p_3(x)|z|, $$

for $x\in\mathfrak{J}$, $y\in\mathfrak{PC}([-r,0],\mathbb{R})$ and $z\in\mathbb{R}$.

(A5)The functions $I_m : \mathfrak{PC}([-r,0],\mathbb{R}) \to \mathbb{R}$ are continuous and there exists constants $M_1^*, M_2^* > 0$ such that

$$|I_m(y)| \leq M_1^* \|y\|_{\mathfrak{PC}} + M_2^*,$$

for each $y \in \mathfrak{PC}([-r,0],\mathbb{R})$, $m = 1,2,\dots k$.

(A6)The function ψ is completely continuous, and for each bounded set B_{Ω^*} in η, the set $\{x \to \psi(x,u_x) : u \in B_{\Omega^*}$ is equicontinuous in $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ and there exist constants $d_1 > 0$, $d_2 > 0$ with $kM_1^* + d_1 < 1$ such that

$$|\psi(x,y)| \leq d_1 \|y\|_{\mathfrak{PC}} + d_2,\ x \in \mathfrak{J},\ y \in \mathfrak{PC}([-r,0],\mathbb{R}).$$

Then, the problem (1) has at least one solution.

*Proof.*We assume the operator $M_1 : \eta \to \eta$ defined by

$$M_1u(x) = \begin{cases} \phi(0) + \sum_{0<x_m<x} I_m(u_{x_m^-}) \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ g(s)\mathrm{d}s \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ g(s)\mathrm{d}s,\ x \in [0,T], \\ \\ \phi(x),\ x \in [-r,0]. \end{cases}$$

The operator M defined in (10) can be written as

$$Mu(x) = \psi(x,u_x) + M_1u(x),\ \text{for each } x \in \mathfrak{J}.$$

The Schaefer's fixed point theorem is used to prove that M has a fixed point. That is, we want to show that M is completely continuous by $(A6)$, we shall prove that M_1 is completely continuous. The proof contains four steps.

Step 1: *M_1 is continuous*. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in η. If $x \in [-r,0]$, then

$$|M_1(y_n)(x) - M_1(y)(x)| = 0.$$

For $x \in \mathfrak{J}$, we have
$|M_1(y_n)(x) - M_1(y)(x)|$

$$\begin{aligned}
&\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ |g_n(s) - g(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ |g_n(s) - g(s)|\,\mathrm{d}s \\
&+ \sum_{0<x_m<x} \left| I_m(y_{n_{x_m^-}}) - I_m(y_{x_m^-}) \right|, \\
&\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ |g_n(s) - g(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ |g_n(s) - g(s)|\,\mathrm{d}s \\
&+ \sum_{0<x_m<x} c_4 \left\| y_{n_{x_m^-}} - y_{x_m^-} \right\|_{\mathfrak{PC}}.
\end{aligned}$$

and then
$|M_1(y_n)(x) - M_1(y)(x)|$

$$\begin{aligned}
&\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ |g_n(s) - g(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\ |g_n(s) - g(s)|\,\mathrm{d}s \\
&+ kc_4 \|y_n - y\|_\eta . \qquad (11)
\end{aligned}$$

where $g_n, g \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ such that

$$g_n(x) = h(x, y_{n_x}, g_n(x)),$$

and

$$g(x) = h(x, y_x, g(x)).$$

By $(A2)$, we have

$$\begin{aligned}
|g_n(x) - g(x)| &= |h(x, y_{n_x}, g_n(x)) - h(x, y_x, g(x))| \\
&\leq c_1 \|y_{n_x} - y_x\|_{\mathfrak{PC}} + c_2 |g_n(x) - g(x)| .
\end{aligned}$$

Then,

$$|g_n(x) - g(x)| \leq \left(\frac{c_1}{1 - c_2} \right) \|y_{n_x} - y_x\|_{\mathfrak{PC}} .$$

Since $y_n \to y$, then we have $g_n(x) \to g(x)$ as $n \to \infty$ for each $x \in \mathfrak{J}$. And let $\Omega > 0$ be such that, for each $x \in \mathfrak{J}$, we have $|g_n(x)| \leq \Omega$ and $|g(x)| \leq \Omega$. Then, we get

$$\begin{aligned}
(x^\rho - s^\rho)^{\omega-1} |g_n(s) - g(s)| &\leq (x^\rho - s^\rho)^{\omega-1} [|g_n(s)| + |g(s)|] \\
&\leq 2\Omega (x^\rho - s^\rho)^{\omega-1},
\end{aligned}$$

and

$$\begin{aligned}
(x_m^\rho - s^\rho)^{\omega-1} |g_n(s) - g(s)| &\leq (x_m^\rho - s^\rho)^{\omega-1} [|g_n(s)| + |g(s)|] \\
&\leq 2\Omega (x_m^\rho - s^\rho)^{\omega-1}.
\end{aligned}$$

For each $x \in \mathfrak{J}$, the functions $s \to 2\Omega(x^\rho - s^\rho)^{\omega-1}$ and $s \to 2\Omega(x_m^\rho - s^\rho)^{\omega-1}$ are integrable on $[0,x]$, then by the Lebesgue Dominated Convergence Theorem and (11) implies that

$$|M_1(y_n)(x) - M_1(y)(x)| \to 0,\ \text{as } n \to \infty.$$

and hence,

$$\|M_1(y_n) - M(y)\|_\eta \to 0,\ \text{as } n \to \infty,$$

Consequently, M_1 is continuous.

Step 2: *M_1 maps bounded sets into bounded sets in η*. To prove this, it is enough to show that for any $\Omega^* > 0$, there exists a positive constant $\tilde{\ell}$ such that for each $y \in B_{\Omega^*} = \{y \in \Omega : \|y\|_\Omega \leq \Omega^*\}$, we have $\|M_1(y)\|_\eta \leq \tilde{\ell}$. We have for each $x \in \mathfrak{J}$,

$$\begin{aligned}
M_1(y)(x) = \phi(0) &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} g(s)\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} g(s)\mathrm{d}s \\
&+ \sum_{0<x_m<x} I_m(y_{x_m^-}), \qquad (12)
\end{aligned}$$

where $g \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ is such that

$$g(x) = h(x, y_x, g(x)).$$

By $(A4)$, for each $x \in \mathfrak{J}$, we get

$$\begin{aligned}|g(x)| &= |h(x,y_x,g(x))|\\ &\le p_1(x) + p_2(x)\|y_x\|_{\mathfrak{PC}} + p_3(x)|g(x)|\\ &\le p_1(x) + p_2(x)\|y\|_\eta + p_3(x)|g(x)|\\ &\le p_1(x) + p_2(x)\Omega^* + p_3(x)|g(x)|\\ &\le p_1^* + p_2^*\Omega^* + p_3^*|g(x)|,\end{aligned}$$

where $p_1^* = \sup_{x\in\mathfrak{J}} p_1(x)$, and $p_2^* = \sup_{x\in\mathfrak{J}} p_2(x)$. Then,

$$|g(x)| \le \frac{p_1^* + p_2^*\Omega^*}{1-p_3^*} := N.$$

Thus (12) implies

$$\begin{aligned}|M_1(y)(x)| &\le |\phi(0)| + \frac{kNT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + \frac{NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\\ &+ \sum_{m=1}^{k}\left(M_1^*\|y_{x_m^-}\|_{\mathfrak{PC}} + M_2^*\right)\\ &\le |\phi(0)| + \frac{(k+1)NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + k\left(M_1^*\|y_{x_m^-}\|_\eta + M_2^*\right)\\ &\le |\phi(0)| + \frac{(k+1)NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + k(M_1^*\Omega^* + M_2^*) := \tilde{R}.\end{aligned}$$

And if $x \in [-r,0]$, then

$$|M_1(y)(x)| \le \|\phi\|_{\mathfrak{PC}},$$

thus

$$\|M_1(y)\|_\eta \le \max\{\tilde{R}, \|\phi\|_{\mathfrak{PC}}\} := \tilde{\ell}.$$

Step 3: *M_1 maps bounded sets into equicontinuous sets of η*. Let $t_1,t_2 \in (0,T]$, $t_1 < t_2$, B_{Ω^*} be a bounded set of η as in Step 2, and let $y \in B_{\Omega^*}$. Then
$|M_1(y)(t_2) - M_1(y)(t_1)|$

$$\begin{aligned}&\le \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^{t_1}\left|(t_2^\rho - s^\rho)^{\omega-1} - (t_1^\rho - s^\rho)^{\omega-1}\right|\left|s^{\rho-1}\right||g(s)|\,ds\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{t_1}^{t_2}\left|(t_2^\rho - s^\rho)^{\omega-1}\right|\left|s^{\rho-1}\right||g(s)|\,ds\\ &+ \sum_{0<x_m<t_2-t_1}\left|I_m(y_{x_m^-})\right|\\ &\le \frac{N}{\rho^\omega\Gamma(\omega+1)}[2(t_2^\rho - t_1^\rho)^\omega + (t_2^{\rho\omega} - t_1^{\rho\omega})]\\ &+ (t_2^\rho - t_1^\rho)\left(M_1^*\left\|y_{x_m^-}\right\|_\eta + M_2^*\right)\\ &\le \frac{N}{\rho^\omega\Gamma(\omega+1)}[2(t_2^\rho - t_1^\rho)^\omega + (t_2^{\rho\omega} - t_1^{\rho\omega})]\\ &+ (t_2^\rho - t_1^\rho)(M_1^*\Omega^* + M_2^*).\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. From Step 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $M_1 : \eta \to \eta$ is completely continuous.

Step 4: *A priori bounds*. Now, we shall prove that the set

$$G = \{y \in \eta : y = \mu M(y),\ \text{for some } 0 < \mu < 1\},$$

is bounded. Let $y \in G$, then $y = \mu M(y)$, for some $0 < \mu < 1$. Thus, for each $x \in \mathfrak{J}$, we get

$$\begin{aligned}y(x) &= \mu\phi(0) + \mu\psi(x,u_x)\\ &+ \frac{\mu\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-1}s^{\rho-1}g(s)ds\\ &+ \frac{\mu\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}g(s)ds\\ &+ \mu\sum_{0<x_m<x} I_m(y_{x_m^-}).\end{aligned} \qquad (13)$$

And, by $(A4)$, for each $x \in \mathfrak{J}$, we get,

$$\begin{aligned}|g(x)| &= |h(x,y_x,g(x))|\\ &\le p_1(x) + p_2(x)\|y_x\|_{\mathfrak{PC}} + p_3(x)|g(x)|\\ &\le p_1^* + p_2^*\|y_x\|_{\mathfrak{PC}} + p_3^*|g(x)|.\end{aligned}$$

Thus,

$$|g(x)| \le \frac{p_1^* + p_2^*\|y_x\|_{\mathfrak{PC}}}{1-p_3^*}.$$

This implies, by (13), $(A5)$ and $(A6)$, for each $x \in \mathfrak{J}$, we get

$$\begin{aligned}|y(x)| &\le |\phi(0)| + d_1\|y_x\|_{\mathfrak{PC}} + d_2\\ &+ \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\\ &\left[\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-1}s^{\rho-1}(p_1^* + p_2^*\|y_s\|_{\mathfrak{PC}})ds\right]\\ &+ \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}(p_1^* + p_2^*\|y_s\|_{\mathfrak{PC}})ds\\ &+ k(M_1^*\|y_{x_m^-}\|_{\mathfrak{PC}} + M_2^*).\end{aligned}$$

Now, we consider the function q defined by

$$q(x) = \sup\{|y(s)| : -r \le s \le x\},\ 0 \le x \le T,$$

then there exists $x^* \in [-r,T]$ such that $q(x) = |y(x^*)|$. If $x \in [0,T]$, then by the previous inequality, for $x \in \mathfrak{J}$, we get

$$\begin{aligned}q(x) &\le |\phi(0)| + \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\\ &\left[\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho - s^\rho)^{\omega-1}s^{\rho-1}(p_1^* + p_2^*q(s))ds\right]\\ &+ \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}(p_1^* + p_2^*q(s))ds\\ &+ (kM_1^* + d_1)q(x) + (kM_2^* + d_2).\end{aligned}$$

Thus,

$$\begin{aligned}q(x) &\leq \frac{|\phi(0)|+kM_2^*+d_2}{1-(kM_1^*+d_1)}\\&+\frac{\rho^{1-\omega}}{(1-(kM_1^*+d_1))(1-p_3^*)\Gamma(\omega)}\\&\left[\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*q(s))\mathrm{d}s\right]\\&+\frac{\rho^{1-\omega}}{(1-(kM_1^*+d_1))(1-p_3^*)\Gamma(\omega)}\\&\left[\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*q(s))\mathrm{d}s\right]\\&\leq\frac{|\phi(0)|+kM_2^*+d_2}{1-(kM_1^*+d_1)}\\&+\frac{(k+1)p_1^*T^{\rho\omega}}{(1-(kM_1^*+d_1))(1-p_3^*)\rho^\omega\Gamma(\omega+1)}\\&+\frac{(k+1)p_3^*\rho^{1-\omega}}{(1-(kM_1^*+d_1))(1-p_3^*)\Gamma(\omega)}\\&\left[\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}q(s)ds\right].\end{aligned}$$

Applying Lemma 4, we get

$$\begin{aligned}q(x)&\leq [\frac{|\phi(0)|+kM_2^*+d_2}{1-(kM_1^*+d_1)}\\&+\frac{(k+1)p_1^*T^{\rho\omega}}{(1-(kM_1^*+d_1))(1-p_3^*)\rho^\omega\Gamma(\omega+1)}]\\&\times\left[1+\frac{\lambda(k+1)p_2^*T^{\rho\omega}}{(1-(kM_1^*+d_1)(1-p_3^*)\rho^\omega\Gamma(\omega+1)}\right]:=\tilde{A},\end{aligned}$$

where $\lambda=\lambda(\omega)$ a constant. If $x^*\in[-r,0]$, then $q(x)=\|\phi\|_{\mathfrak{PC}}$, thus for any $x\in\mathfrak{J}$, $\|y\|_\eta\leq q(x)$, we get

$$\|y\|_\eta\leq\max\{\|\phi\|_{\mathfrak{PC}},\tilde{A}\}.$$

From this, the set G is bounded. From Schaefer's fixed point theorem, we conclude that M has at least one fixed point which is a solution of the problem (1).

4 Ulam-Hyers-Rassias Stability

Now, we present the following Ulam-Hyers-Rassias stability result.

Theorem 6. *Assume that* (A1)-(A3), *(9) and*

(A7) There exists a nondecreasing function $\alpha\in\mathfrak{PC}(\mathfrak{J},\mathbb{R}_+)$ and there exists $\mu_\alpha>0$ such that for any $x\in\mathfrak{J}$:

$${}^\rho I_{0^+}^\omega\alpha(x)\leq\mu_\alpha\alpha(x),$$

are satisfied, and if $c_3<1$, then the problem (1) is Ulam-Hyers-Rassias stable with respect to (α,φ).

Proof. Let $v\in\eta$ be a solution of the inequality (6). Denote by u the unique solution of the problem

$$\begin{cases}{}^\rho D_{x_m}^\omega[u(x)-\psi(x,u_x)]=h(x,u_x,{}^\rho D_{x_m}^\omega u(x)),\\ \text{for each } x\in(x_m,x_{m+1}],\ m=1,\dots k;\\ \Delta u|_{x=x_m}=I_m(u_{x_m^-}),\ m=1,\dots,k;\\ u(x)=v(x)=\phi(x),\ x\in[-r,0].\end{cases}$$

From Lemma 6 and for each $x\in(x_m,x_{m+1}]$, we get

$$\begin{aligned}u(x)&=\phi(0)+\psi(x,u_x)+\sum_{i=1}^m I_i(u_{x_i^-})\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s,\ x\in(x_m,x_{m+1}],\end{aligned}$$

where $g\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that

$$g(x)=h(x,u_x,g(x)).$$

Since v is a solution of the inequality (6) and by Remark 2 , we get

$$\begin{cases}{}^\rho D_{x_m}^\omega[v(x)-\psi(x,v_x)]=h(x,v_x,{}^\rho D_{x_m}^\omega v(x))+\sigma(x),\\ x\in(x_m,x_{m+1}],\ m=1,\dots k;\\ \Delta v|_{x=x_m}=I_m(v_{x_m^-})+\sigma_m,\ m=1,\dots,k.\end{cases}\tag{14}$$

Clearly, the solution of (14) is given by,

$$\begin{aligned}v(x)&=\phi(0)+\psi(x,v_x)+\sum_{i=1}^m I_i(v_{x_i^-})+\sum_{i=1}^m\sigma_i\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}f(s)\mathrm{d}s\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}\sigma(s)\mathrm{d}s\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}f(s)\mathrm{d}s\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\sigma(s)\mathrm{d}s,\ x\in(x_m,x_{m+1}],\end{aligned}$$

where $f\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that $f(x)=h(x,v_x,f(x))$. Hence, for each $x\in(x_m,x_{m+1}]$, we get,

$$\begin{aligned}|v(x)-u(x)|&\leq\sum_{i=1}^m|\sigma_i|+|\psi(x,v_x)-\psi(x,u_x)|+\sum_{i=1}^m\left|I_i(v_{x_i^-})-I_i(u_{x_i^-})\right|\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\mathrm{d}s\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}|\sigma(s)|\mathrm{d}s\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\mathrm{d}s\\&+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}|\sigma(s)|\mathrm{d}s.\end{aligned}$$

Hence, we get,

$$|v(x)-u(x)| \le k\varepsilon\varphi+(k+1)\varepsilon\mu_\alpha\alpha(x)+c_3\|v_x-u_x\|_{\mathfrak{PC}} + \sum_{i=1}^{m} c_4 \left\| v_{x_i^-}-u_{x_i^-}\right\|_{\mathfrak{PC}} + \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,ds + \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,ds.$$

From $(A2)$, we get

$$|f(x)-g(x)| = |h(x,v_x,f(x))-h(x,u_x,g(x))| \le c_1\|v_x-u_x\|_{\mathfrak{PC}}+c_2|f(x)-g(x)|.$$

Then,

$$|f(x)-g(x)| \le \frac{c_1}{1-c_2}\|v_x-u_x\|_{\mathfrak{PC}}.$$

Therefore, for each $x\in\mathfrak{J}$,
$|v(x)-u(x)|$

$$\le k\varepsilon\varphi+(k+1)\varepsilon\mu_\alpha\alpha(x)+c_3\|v_x-u_x\|_{\mathfrak{PC}} + \sum_{i=1}^{m} c_4\left\|v_{x_i^-}-u_{x_i^-}\right\|_{\mathfrak{PC}} + \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}\|v_s-u_s\|_{\mathfrak{PC}}\,ds + \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\|v_s-u_s\|_{\mathfrak{PC}}\,ds.$$

Thus,
$|v(x)-u(x)|$

$$\le \varepsilon(\varphi+\alpha(x))(k+(k+1)\mu_\alpha) + \sum_{0<x_i<x} c_4\left\|v_{x_i^-}-u_{x_i^-}\right\|_{\mathfrak{PC}}+c_3\|v_x-u_x\|_{\mathfrak{PC}} + \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\|v_s-u_s\|_{\mathfrak{PC}}\,ds. \tag{15}$$

Now, we consider the function q_1 defined by

$$q_1(x)=\sup\{|v(s)-u(s)| : -r\le s\le x\},\ 0\le x\le T,$$

then, there exists $x^*\in[-r,T]$ such that $q_1(x)=|v(x^*)-u(x^*)|$. If $x^*\in[-r,0]$, then $q_1(x)=0$. If $x^*\in[0,T]$, then by the equation (15), we get

$$q_1(x)\le\frac{\varepsilon(\varphi+\alpha(x))(k+(k+1)\mu_\alpha)}{1-c_3}+\sum_{0<x_i<x}\frac{c_4}{1-c_3}q_1(x_i^-) + \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)(1-c_3)\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}q_1(s)ds.$$

Applying Lemma 5, we have,

$$q_1(x)\le\left[\frac{\varepsilon(\varphi+\alpha(x))(k+(k+1)\mu_\alpha)}{1-c_3}\right] \times\Big[\prod_{0<x_i<x}\left(1+\frac{c_4}{1-c_3}\right) \exp\left(\int_0^x\frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)(1-c_3)\Gamma(\omega)}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}ds\right)\Big] \le l_\alpha\varepsilon(\varphi+\alpha(x)),$$

where

$$l_\alpha=\frac{(k+(k+1)\mu_\alpha)}{1-c_3}\times \left[\prod_{i=1}^{k}\left(1+\frac{c_4}{1-c_3}\right)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)(1-c_3)\rho^\omega\Gamma(\omega+1)}\right)\right] =\frac{(k+(k+1)\mu_\alpha)}{1-c_3}\times \left[\left(1+\frac{c_4}{1-c_3}\right)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)(1-c_3)\rho^\omega\Gamma(\omega+1)}\right)\right]^k.$$

Therefore, the problem (1) is Ulam-Hyers-Rassias stable with respect to (α,φ). Hence the proof is complete.

Now, we present the following Ulam-Hyers stable result.

Theorem 7.*Assume that* $(A1)$-$(A3)$ *and (9) are satisfied, if* $c_3<1$, *then the problem (1) is Ulam-Hyers stable.*

*Proof.*Let $v\in\eta$ be a solution of (4). Denote by u the unique solution of the problem.

$$\begin{cases} {}^\rho D^\omega_{x_m}[u(x)-\psi(x,u_x)] = h(x,u_x,{}^\rho D^\omega_{x_m}u(x)),\\ x\in(x_m,x_{m+1}],\ m=1,\dots k;\\ \Delta u|_{x=x_m}=I_m(u_{x_m^-}),\ m=1,\dots,k;\\ u(x)=v(x)=\phi(x),\ x\in[-r,0]. \end{cases}$$

From the proof of the Theorem 6, we get

$$q_1(x)\le\sum_{0<x_i<x}\frac{c_4}{1-c_3}q_1(x_i^-)+\frac{k\varepsilon}{1-c_3} + \frac{\varepsilon(k+1)T^{\rho\omega}}{(1-c_3)\rho^\omega\Gamma(\omega+1)} + \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)(1-c_3)\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}q_1(s)ds.$$

Applying Lemma 5, we have

$$q_1(x)\le\varepsilon\left(\frac{k\rho^\omega\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{(1-c_3)\rho^\omega\Gamma(\omega+1)}\right) \times\left[\prod_{0<x_i<x}\left(1+\frac{c_4}{1-c_3}\right) \exp\left(\int_0^x\frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)(1-c_3)\Gamma(\omega)}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}ds\right)\right] \le\bar{l}\varepsilon,$$

where,

$$\bar{l} = \left(\frac{k\rho^{\omega}\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{(1-c_3)\rho^{\omega}\Gamma(\omega+1)}\right)\times \left[\prod_{i=1}^{k}\left(1+\frac{c_4}{1-c_3}\right)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)(1-c_3)\rho^{\omega}\Gamma(\omega+1)}\right)\right]$$
$$= \left(\frac{k\rho^{\omega}\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{(1-c_3)\rho^{\omega}\Gamma(\omega+1)}\right)\times \left[\left(1+\frac{c_4}{1-c_3}\right)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)(1-c_3)\rho^{\omega}\Gamma(\omega+1)}\right)\right]^k,$$

which completes the proof of the theorem.Moreover, if we set $\gamma(\varepsilon)=\bar{l}\varepsilon$; $\gamma(0)=0$, then the problem (1) is generalized Ulam-Hyers stable.

5 Examples

*Example 1.*Consider the nonlinear implicit neutral Katugampola fractional differential equations,

$$\begin{cases} {}^{\rho}D^{\frac{1}{2}}_{x_m}\left[u(x)-\frac{xe^{-x}|u_x|}{(19+e^x)(1+|u_x|)}\right] = \\ \frac{e^{-x}}{(21+e^x)}\left[\frac{|u_x|}{1+|u_x|}-\frac{\left|{}^{\rho}D^{\frac{1}{2}}_{x_m}u(x)\right|}{1+\left|{}^{\rho}D^{\frac{1}{2}}_{x_m}u(x)\right|}\right], \\ \Delta u|_{x=\frac{1}{3}} = \frac{\left|u(\frac{1}{3}^-)\right|}{20+\left|u(\frac{1}{3}^-)\right|}, \\ u(x)=\phi(x),\ x\in[-r,0], r>0, \end{cases} \tag{16}$$

where $\phi\in\mathfrak{PC}([-r,0],\mathbb{R})$, $\mathfrak{J}_0=[0,\frac{1}{3}]$, $\mathfrak{J}_1=(\frac{1}{3},1]$, $x_0=0$ and $x_1=\frac{1}{3}$.

For $x\in[0,1]$, $y\in\mathfrak{PC}([-r,0],\mathbb{R})$ and $z\in\mathbb{R}$. Let us assume,

$$h(x,y,z)=\frac{e^{-x}}{21+e^x}\left[\frac{|y|}{1+|y|}-\frac{|z|}{1+|z|}\right],$$

and

$$\psi(x,y)=\frac{xe^{-x}|y|}{(19+e^x)(1+|y|)}.$$

Here $\psi(0,\phi)=0$, for any $\phi\in\mathfrak{PC}([-r,0],\mathbb{R})$. Clearly, the function h is jointly continuous. For each $y,\bar{y}\in\mathfrak{PC}([-r,0],\mathbb{R})$, $z,\bar{z}\in\mathbb{R}$ and $x\in[0,1]$, we get,

$$\begin{aligned}|h(x,y,z)-h(x,\bar{y},\bar{z})| &\le \frac{e^{-x}}{21+e^x}(\|y-\bar{y}\|_{\mathfrak{PC}}+|z-\bar{z}|)\\ &\le \frac{1}{22}\|y-\bar{y}\|_{\mathfrak{PC}}+\frac{1}{22}|z-\bar{z}|,\end{aligned}$$

and

$$|\psi(x,y)-\psi(x,\bar{y})|\le\frac{1}{20}\|y-\bar{y}\|_{\mathfrak{PC}}.$$

Hence the condition $(A2)$ is satisfied with $c_1=c_2=\frac{1}{22}$, $c_3=\frac{1}{20}$. Let $I_1(y)=\frac{|y|}{20+|y|}$, $y\in\mathfrak{PC}([-r,0],\mathbb{R})$. For each $y,z\in\mathfrak{PC}([-r,0],\mathbb{R})$, we get

$$\begin{aligned}|I_1(y)-I_1(z)| &= \left|\frac{|y|}{20+|y|}-\frac{|z|}{20+|z|}\right|\\ &\le\frac{1}{20}\|y-z\|_{\mathfrak{PC}}.\end{aligned}$$

Let us assume $T=1$, $k=1$, $c_4=\frac{1}{20}$, $\omega=\frac{1}{2}$, $\rho=1$, then

$$kc_4+c_3+\frac{(k+1)c_1T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}=\frac{1}{10}+\frac{4}{21\sqrt{\pi}}<1.$$

Therefore, the condition (9) is satisfied. From Theorem 4, the given problem (16) has a unique solution on $\mathfrak{J}$. Set for any $x\in[0,1]$, $\alpha(x)=x$, and $\varphi=1$, $\rho=1$, $\omega=\frac{1}{2}$. Since

$$\begin{aligned}{}^{\rho}I^{\omega}\alpha(x) &= \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}ds\\ &= \frac{2}{\pi}\int_0^x(x^{\rho}-s^{\rho})^{\frac{-1}{2}}ds\\ &\le\frac{2x}{\sqrt{\pi}},\end{aligned}$$

Thus, the condition $(A7)$ is satisfied with $\mu_{\alpha}=\frac{2}{\sqrt{\pi}}$ and since $c_3<1$, from the above, the problem (16) is Ulam-Hyers-Rassias stable with respect to (α,φ).

6 Conclusion

In this article, with the help of standard fixed point theorems of Schaefer's and Banach, we successfully developed the existence of solutions of nonlinear implicit neutral Katugampola fractional differential equations with impulse effects and finite delay. The obtained conditions ensure that the existence of at least one solution to the proposed problem. Further, Ulam-Hyers and Ulam-Hyers-Rassias stability have been investigated.

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