An Analogue Result of $q$-Beta Integral

Dong-Fang Liu and Qiu-Ming Luo*

Department of Mathematics, Chongqing Normal University, Chongqing Higher Education Mega Center, Huxi Campus, Chongqing 401331, People’s Republic of China

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Abstract: We give an analogue result of $q$-beta integral by applying the $q$-Chu-Vandermonde formula and get several identities which include $q$-series $_3\phi_2$ and $q$-integral.

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1 Introduction and main result

Throughout this paper we suppose $|q| < 1$, $\mathbb{N} = \{1, 2, \ldots\}$. The $q$-shifted factorial are defined by

$$ (a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \lim_{n \to \infty} \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1. $$

(1.1)

(1.2)

Clearly,

$$ (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}. $$

(1.3)

We also adopt the following compact notation for multiple $q$-shifted factorials:

$$ (a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$n$$

$$ (a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty. $$

The basic hypergeometric series $r+1\Phi_r$, or $q$-series are defined by

$$ r+1\Phi_r \left( \begin{array}{c} a_1, a_2, \ldots, a_{r+1} \end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n} z^n. $$

(1.4)

The $q$-Chu-Vandermonde convolution formula is

$$ 2\Phi_1 \left( \begin{array}{c} q^{-n}, a \end{array} ; c, q \right) = \frac{d^n(c/a; q)_n}{(c; q)_n}. $$

(1.5)

F. H. Jackson defined the $q$-integral as follows (see [4])

$$ \int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(d^n) q^n, $$

(1.6)

and

$$ \int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t. $$

(1.7)

He also defined $q$-integral on $(0, \infty)$ by

$$ \int_0^\infty f(t) d_q t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n. $$

(1.8)

on the interval $(-\infty, \infty)$ the bilateral $q$-integral is defined by

$$ \int_{-\infty}^{\infty} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)] q^n. $$

(1.9)

Askey obtained an elegant formula of the $q$-beta integral (see [2]):

$$ \int_{-\infty}^{\infty} \frac{(a,b,c,d;q)_\infty}{(c,b,a,d;q)_\infty} d_q \theta = \frac{2(1-q)(q^2 a^2 q^2 + c) c d (e a /c - d a /c - e b /d) - e}{(q a c) (q b c) q^2 c^2 d^2 (d^2 - q^2) (c^2 - q^2)} (-ab /d q). $$

(1.10)

provided that $|q| < 1$, $|ab/deq| < 1$ and there are no zero factors in the denominator of the integrals.

The $q$-beta integral is an important formula in basic hypergeometric series. For instance, Wang gave some extensions of $q$-beta integral (see [7, 8, 9, 10]).
In this paper, we give an interesting analogue result of q-beta integral using the q-Chu-Vandermonde formula. Making use of the similar method of Wang, we derive the following main result of this paper.

**Theorem 1.1** If $|q| < 1, |e/d| < 1$ and there are no zero factors in the denominator of the integrals, then for any nonnegative integers $n$, we have

\[
\int_{-\infty}^{\infty} \frac{(b/a; q)_n}{(c/a; q)_n} d_q \omega = \frac{2q^{2n}(1 - q)(q^2; q^2)_n^2}{e^{bq} (d + dq)(d', q^2'; q'/q')^n} \times \left[ \Phi_1 \left( q^{-n}, e/a - b dq, q - e/d : q, q \right) - \Phi_2 \left( q^{-n}, e/a - b dq, q - e/d : q, q \right) \right].
\]

(1.11)

2 The proof of Theorem

Recalling the q-Chu-Vandermonde convolution formula

\[
\Phi_1 \left( q^{-n}, c/a : q, q \right) = \frac{e^{n(a/c; q)_n}}{(a; q)_n}.
\]

(2.1)

By the following relation

\[
(a; q)_k = \frac{(a; q)_n}{(aq^n; q)_n},
\]

(2.2)

the formula (2.1) can be written as

\[
\sum_{k=0}^{n} \frac{(q^{-n}; c/a; q)_k}{(q; q)_k} \frac{(aq^n; q)_k}{(a; q)_k} = e^n, \frac{(a/c; q)_n}{(a; q)_n}.
\]

(2.3)

Letting $a \rightarrow a \omega$ in (2.3) and multiplying the both sides of equation (2.3) by

\[
\frac{(b/a; q)_n}{(-d/a; q)_n} = e^a, \frac{(a/c; q)_n}{(a; q)_n},
\]

we obtain that

\[
\sum_{k=0}^{n} \frac{(q^{-n}; c/a; q)_k}{(q; q)_k} \frac{(aq^n; q)_k}{(-d/q; q)_k} = e^n, \frac{(a/c; q)_n}{(a; q)_n}, \frac{(b/a; q)_n}{(-d/a; q)_n}.
\]

(2.4)

To Calculate the q-integral on both sides of (2.4) with respect to variable $\omega$, we have

\[
\sum_{k=0}^{n} \frac{(q^{-n}; c/a; q)_k}{(q; q)_k} \int_{-\infty}^{\infty} \frac{(aq^n; q)_k}{(-d/q; q)_k} d_q \omega = e^n, \int_{-\infty}^{\infty} \frac{(b/a; q)_n}{(-d/a; q)_n} d_q \omega.
\]

(2.5)

Applying the Askey’s result (1.10) to the integral on the left-hand side of (2.5), we find that

\[
\sum_{k=0}^{n} \frac{(q^{-n}; c/a; q)_k}{(q; q)_k} \frac{2(1 - q)(q^2; q^2)_n^2}{(d/a; q)_n(d/a, q^2; q)_n} \frac{(aq^n; q)_k}{(-d/q; q)_k} = e^n, \int_{-\infty}^{\infty} \frac{(b/a; q)_n}{(-d/a; q)_n} d_q \omega.
\]

(2.6)

which can be rewritten as

\[
\sum_{k=0}^{n} \frac{(q^{-n}; c/a; q)_k}{(q; q)_k} \frac{(aq^n; q)_k}{(-d/q; q)_k} \frac{2q(1 - q)(q^2; q^2)_n^2}{(d/a; q)_n(d/a, q^2; q)_n} (aq^n; q)_k = \int_{-\infty}^{\infty} \frac{(b/a; q)_n}{(-d/a; q)_n} d_q \omega.
\]

(2.7)

We therefore obtain

\[
\int_{-\infty}^{\infty} \frac{(b/a; q)_n}{(-d/a; q)_n} d_q \omega = \frac{2q^{2n}(1 - q)(q^2; q^2)_n^2}{e^{bq} (d + dq)(d', q^2'; q'/q')^n} \times \left[ \Phi_1 \left( q^{-n}, e/a - b dq, q - e/d : q, q \right) - \Phi_2 \left( q^{-n}, e/a - b dq, q - e/d : q, q \right) \right].
\]

(2.8)

Replacing $e$ by $a/e$ and interchanging $a$ and $e$ in (2.8), we obtain the formula (1.11) at once. The proof is complete.

3 Some applications

In this section, we give three identities using the formula (1.11) which include a new identity for $3\Phi_2$ and two identities of q-integral.

One of the fundamental transformations in the theory of basic hypergeometric series is the following Sears’ $3\Phi_2$ transformation (see [6]), which will be used to prove 3.1 below.

\[
3\Phi_2 \left( q^{-n}, b, c : q, dq^n/bc \right) = \frac{(a/c; q)_n}{(a; q)_n} 3\Phi_2 \left( q^{-n}, e/a - b dq^n, q - e/d : q, q \right).
\]

(3.1)

**Theorem 3.1** If $n \in \mathbb{N}$, then we have

\[
3\Phi_2 \left( q^{-n}, b, c : q, dq^n/bc \right) = e^{n(a/c; q)_n} 3\Phi_2 \left( q^{-n}, e/a - b dq^n, q - e/d : q, q \right).
\]

(3.2)

**Proof:** Letting $b = aq^n$ in (1.11), we obtain

\[
\int_{-\infty}^{\infty} \frac{(a/q^n; q)_n}{(-d/a; q)_n} d_q \omega = \frac{2q^{2n}(1 - q)(q^2; q^2)_n^2}{e^{aq^n} (d + dq)(d', q^2'; q'/q')^n} \times \left[ \Phi_1 \left( q^{-n}, e/a - a dq^n, q - e/d : q, q \right) - \Phi_2 \left( q^{-n}, e/a - a dq^n, q - e/d : q, q \right) \right].
\]

(3.3)

On the other hand, setting $b = eq^n$ in (1.10) and noting that (1.10), we obtain

\[
\int_{-\infty}^{\infty} \frac{(a/e; q)_n}{(-d/a; q)_n} d_q \omega = \int_{-\infty}^{\infty} \frac{(a/e; q)_n}{(-d/a; q)_n} d_q \omega = \frac{2q^{2n}(1 - q)(q^2; q^2)_n^2}{e^{aq^n} (d + dq)(d', q^2'; q'/q')^n} \times (-e/d; q)_n \phi \left( q^{-n}, e/a - a dq^n, q - e/d : q, q \right).
\]

(3.4)

Comparing the equations (3.3) and (3.4), we find that

\[
3\Phi_2 \left( q^{-n}, e/a - a dq^n, q - e/d : q, q \right) = e^{n(a/c; q)_n} 3\Phi_2 \left( q^{-n}, e/a - a dq^n, q - e/d : q, q \right).
\]

(3.5)

Applying (3.1) in (3.5), we get

\[
\phi \left( q^{-n}, e/a - a dq^n, q - e/d : q, q \right) = \frac{(a/c; q)_n}{(a; q)_n} 3\Phi_2 \left( q^{-n}, e/a - a dq^n, q - e/d : q, q \right).
\]

(3.6)

Replacing $(e/a, -a/d, -e/d)$ by $(a, b, c)$ in (3.6), we obtain (3.2) immediately.
Theorem 3.2 We have
\[ \int_{-\infty}^{\infty} \frac{(b; q)_n}{(a; q)_n} \frac{d\alpha}{(a^{\infty}; q^2)_n} = 2d \omega^{n+1} \left[ -1-q(q^2; q^2)_n^2 \right] \]
\[ \times \left[ \phi_4 \left( q^{-\alpha}, -q^{-\alpha}, -b/dq, \omega, q, q, a/d \right) - \phi_4 \left( q^{-\alpha}, -q^{-\alpha}, -b/dq, \omega, q, q, a/d \right) \right] \]
provided that no zero factors in the denominator of the integrals.

Proof. Using the formula
\[ (a; q)_n = (a; q)_n(aq^n; q)_k, \quad (a^2; q^2)_n = (a; q)_n (-a; q)_n \]
we easily get
\[ \frac{(a\alpha; q)_n}{(a^{2\alpha}; q^2)_n} = \frac{(a\alpha; q)_n(aq^n; q)_n}{(a^n; q)_n} = (a^n; q)_n \quad (3.8) \]
Replacing \( a \) by \( aq^n \) and setting \( \epsilon = -a \) in (1.11), noting that (3.8), we obtain that
\[ \int_{-\infty}^{\infty} \frac{(b; q)_n}{(a; q)_n} \frac{d\alpha}{(a^{\infty}; q^2)_n} = 2d \omega^{n+1} \left[ -1-q(q^2; q^2)_n^2 \right] \]
\[ \times \left[ \phi_4 \left( q^{-\alpha}, -q^{-\alpha}, -b/dq, \omega, q, q, a/d \right) - \phi_4 \left( q^{-\alpha}, -q^{-\alpha}, -b/dq, \omega, q, q, a/d \right) \right] \]
This proof is complete.

Theorem 3.3 We have
\[ \int_{-\infty}^{\infty} \frac{(b; q)_n}{(a; q)_n} \frac{d\alpha}{(a^{\infty}; q^2)_n} = 0 \quad (3.9) \]
provided that no zero factors in the denominator of the integrals.

Proof. We recall the Ramanujan’s bilateral summation formula (see [5])
\[ \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} q^{\alpha} = \frac{(q; q)_n(b/a; q)_n(ax; q)_n(q/ax; q)_n}{(b; q)_n(q/a; q)_n(x; q)_n(b/ax; q)_n} \quad (3.10) \]
Rewriting the above formula as
\[ \sum_{n=-\infty}^{\infty} \frac{(bq^n; q)_n}{(aq^n; q)_n} x^{\alpha} = \frac{(q; q)_n(b/a; q)_n(ax; q)_n(q/ax; q)_n}{(a; q)_n(q/a; q)_n(x; q)_n(b/ax; q)_n} \quad (3.11) \]
Applying the formula (3.11) and noting that definition (1.9), we get
\[ \int_{-\infty}^{\infty} \frac{(bq^n; q)_n}{(aq^n; q)_n} d\alpha(1-q) \sum_{n=-\infty}^{\infty} \frac{(bq^n; q)_n}{(aq^n; q)_n} q^{\alpha} \]
\[ = (1-q) \sum_{n=-\infty}^{\infty} \left[ \frac{(b; q)_n}{(a; q)_n}(dq^n; q)_n \right] q^{\alpha} \]
\[ = (1-q) \sum_{n=-\infty}^{\infty} \left[ -dq^n(-1-dq^2)(-b/dq)_{q; q} \right] \]
\[ = (1-q) \left[ 1+1/d(1+b/dq) \right] \]
\[ = (1-q) \frac{1}{dq+b} \]
\[ = 0. \]
This proof is complete.

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