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Incomplete Variable Multigranulation Rough Sets Decision

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Abstract: Recently, a multigranulation rough set (MGRS) has become a new direction in rough set theory, which is different from the Pawlake's rough set since the former takes multiple granulations on the universe into account. In this paper, by analyzing the limitations of optimistic multigranulation rough set (OMGRS) and pessimistic multigranulation rough set (PMGRS) in incomplete information system, the incomplete variable multigranulation rough set (VMGRS) is proposed, and the relationships among VMGRS, OMGRS and PMGRS are deeply explored, From the properties, it can be found that OMGRS and PMGRS are the special cases compared to our VMGRS. Furthermore, several important measurements are introduced into the VMGRS; it is shown that the measurements of the VMGRS are between the measurements of OMGRS and PMGRS. Finally, some numerical examples are employed to substantiate the conceptual arguments.

Keywords: OMGRS, PMGRS, VMGRS, Measurement, Incomplete information system

1 Introduction

Rough set [1,2], proposed by Pawlak, is a powerful tool, which can be used to deal with the inconsistency problems by separation of certain and doubtful knowledge extracted from the exemplary decisions. Though Pawlak's rough set theory has been demonstrated to be useful in the fields such as knowledge discover [3,4], decision analysis [5,6], data mining [7,8], pattern recognition [9], artificial intelligence [10], medical diagnosis [11] and so on.

It is well known that lower and upper approximation operators in Pawlak's rough set are defined by an equivalence relation (indiscernibility relation). However, due to the existence of uncertainty and complexity of particular problems, the equivalence relation is too restrictive in many practical applications. To overcome this limitation, Pawlak's rough sets has been extended to several interesting and meaningful general models by proposing other binary relations in recent years, which include similarity relation based rough set [12], tolerance relation based rough set [9,13,14,15], dominance relation based rough set [16], covering based rough set [17,18], fuzzy rough set [19,20,21,22] and others [23].

From the viewpoint of the granular computing, an equivalence relation on the universe can be regarded as a granulation, and a partition (induced by the equivalence relation) on the universe can be regarded as a granulation space, a equivalence class can be regarded as a knowledge granule [24,25,26]. So, the above expanded rough set models are based on single granulation, they also called the single granulation rough sets. However, it should be noticed that in [25,27,28], Qian et al. argued that we often need to describe concurrently a target concept through multiple granulations on the universe according to a user's requirements or targets of problem solving. Therefore, they proposed the concept of Multigranulation Rough Set (MGRS) model. In fact, the basic idea of MGRS has been also discussed by Khan et al. in [7]. Following such work, the MGRS are extended in incomplete based MGRS [25], dominance based MGRS [26], neighborhood based MGRS [29], Fuzzy MGRS [31,30], and so on [32,33,34].

The MGRS model can be classified into two parts: one is the optimistic multigranulation rough set (OMGRS) and the other is pessimistic multigranulation rough set (PMGRS). By analyzing OMGRS and PMGRS,

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we can see that OMGRS decision is too relax since if only ***one*** granulation satisfies with the inclusion condition between the knowledge granule and the target concept, then the object should belong to the lower approximation. On the other hand, PMGRS decision is too strict since if ***all*** of the granulations satisfy with the inclusion condition between the knowledge granule and the target, then the object may belongs to the lower approximation.

The purpose of this paper is study MGRS in incomplete decision information system and propose a new multigranulation rough set, which is referred to as the variable multigranulation rough set (VMGRS), In our VMGRS decision, threshold β, is used to control the number of granulations, which satisfy with the inclusion condition between the knowledge granule and the target concept. Only when the number of granulations reaches to a certain amount, the object may be belongs to the lower approximation. Such number can be controlled by a threshold β. Through the adjustment of the threshold β, the limitations of OMGRS too relax and PMGRS too strict can be overcome.

The paper is organized as following. In Section 2, the rough set and MGRS are briefly introduced. In Section 3, the VMGRS models are proposed, the immediate properties about VMGRS are also addressed. In Section 4, several important measurements are introduced into our VMGRS and then the relationships among measurements of OMGRS, PMGRS and VMGRS are discussed. In Section 5, Experiments on UCI data sets show that VMGRS is a generalization of both OMGRS and PMGRS. Results are summarized in Section 6.

2 Preliminary knowledge on rough sets

2.1 Single–granulation rough set

Formally, an information system(IS) can be considered as a 4-tuple $IS = <U, AT, V, f>$, where U is a nonempty finite set of objects (called the universe), AT is a nonempty finite set of attributes, V is regard as the domain of all attributes and $V = V_{AT} = \bigcup_{a \subseteq AT} V_a$, $\forall x \in U$, $f(a,x)$ is the value the x hold on $a(a \in AT)$.

In particular, if $AT = C \cup D$ and $C \cap D = \emptyset$, where the C is called condition attribute sets and the D is called decision attribute sets, then $< U, C \cup D, V, f >$ is also regard as a decision information system(DIS).

If there exists $x \in U$ and $a \in AT$, such that $f(x,a)$ is unknown, the unknown value is denoted by "$*$" in this paper, then the$< U, C \cup D, V, f >$ is also regard as an incomplete decision information system(IDIS). We assume here that at least one of the states of x in terms of A is certain where $A \subseteq AT$, i.e. $a \in A$ such that $f(x,a)$ is known. Thus, $V = V_C \cup V_D \cup \{*\}$.

Definition 1.*Suppose that $< U, C \cup D, V, f >$ is an IDIS, $\forall A \subseteq C$, a binary relation $SIM(A)$ can be defined as [13]*

$$SIM(A) = \{(x,y) \in U^2 : f(x,a) = f(y,a) \vee f(x,a) = * \vee f(y,a) = *, \forall a \in A\}. \quad (1)$$

The binary relation $SIM(A)$ is a tolerance relation since it is reflexive and symmetric. From the relation $SIM(A)$, we can introduce a cover on the universe U, this cover can be denoted by $U/SIM(A)$. $\forall x \in U$, We denote the tolerance class including x by $SIM_A(x)$, such $SIM_A(x) = \{y \in U : (x,y) \in SIM(A)\}$.

From the viewpoint of the granular computing, the relation $SIM(A)$ can be regard as a granulation, the cover $U/SIM(A)$ is referred to as a granulation space, each tolerance class $SIM_A(x)$ may be viewed as a knowledge granule consisting of indistinguishable elements.

By tolerance relation $SIM(A)$, $\forall X \subseteq U$, we can derive the lower and upper approximations of an arbitrary subset $X$of U, They are defined as

$$\underline{A}(X) = \{x \in U : SIM_A(x) \subseteq X\}. \quad (2)$$

$$\overline{A}(X) = \{x \in U : SIM_A(x) \cap X \neq \emptyset\}. \quad (3)$$

The pair $[\underline{A}(X), \overline{A}(X)]$ is referred to as rough set of X with respect to the set of attributes A. Obviously, this rough set is constructed on the basis of one and only one granulation, it is then regarded as the single–granulation rough set model.

2.2 Multigranulation rough sets

The MGRS is different from single granulation rough set since the former is constructed on the basis of the multiple granulations instead of a single one.

2.2.1 Optimistic multigranulation rough set

In the optimistic multigranulation rough set(OMGRS), the target is approximated through the multiple granulations. In lower approximation, the word *optimistic* is used to express the idea that in multiple independent granulations, we need only at least one of the granulation to satisfy with the inclusion condition between knowledge granule and target concept. The upper approximation of OMGRS is defined by the complement of the lower approximation.

Definition 2.*Let $I = < U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, then $\forall X \subseteq U$, the OMGRS lower and upper approximations are denoted by $\underline{\sum_{i=1}^m A_i}^O(X)$ and $\overline{\sum_{i=1}^m A_i}^O(X)$, respectively [27],*

$$\underline{\sum_{i=1}^m A_i}^O(X) = \{x \in U : SIM_{A_1}(x) \subseteq X \vee SIM_{A_2}(x) \subseteq X \vee \cdots \vee SIM_{A_m}(x) \subseteq X\}. \quad (4)$$

$$\overline{\sum_{i=1}^m A_i}^O(X) = \sim \underline{\sum_{i=1}^m A_i}^O(\sim X). \quad (5)$$

where $SIM_{A_i}(x)(1 \le i \le m)$ is a knowledge granule of x in terms of the granulation $SIM(A)$, $\sim X$ is the complement of set X.

By the lower and upper approximations $\underline{\sum_{i=1}^{m} A_i}^{O}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{O}(X)$, the OMGRS boundary region of X is

$$BN^{O}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{O}(X) - \underline{\sum_{i=1}^{m} A_i}^{O}(X). \tag{6}$$

2.2.2 Pessimistic multigranulation rough set

In pessimistic multigranulation rough set (PMGRS) [28], the target is still approximated through the multiple granulations. However, it is different from the optimistic case. In lower approximation, the word *pessimistic* is used to express the idea that we need all of the granulations to satisfy with the inclusion condition between knowledge granule and target concept. The upper approximation of PMGRS is also defined by the complement of the lower approximation of PMGRS.

Definition 3.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, then $\forall X \subseteq U$, the PMGRS lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} A_i}^{P}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{P}(X)$, respectively,*

$$\underline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U : SIM_{A_1}(x) \subseteq X \wedge SIM_{A_1}(x) \subseteq X \wedge \cdots \wedge SIM_{A_m}(x) \subseteq X\}; \tag{7}$$

$$\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{P}(\sim X). \tag{8}$$

By the lower and upper approximations $\underline{\sum_{i=1}^{m} A_i}^{P}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{P}(X)$, the PMGRS boundary region of X is

$$BN^{P}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{P}(X) - \underline{\sum_{i=1}^{m} A_i}^{P}(X). \tag{9}$$

3 variable multigranulation rough set

By Definition 2, we can see that the OMGRS is too relax since if only *one* tolerance class(knowledge granule) satisfies with the set containment to the target concept, then the object should belong to the lower approximation. On the other hand, by Definition 3, we can see that the PMGRS is too strict since if *all* of the tolerance classes satisfy with the set containment to the target concept, then the object belongs to the lower approximation.

To solve such problem, we will propose a new multigranulation rough set approach, which is referred to as the incomplete variable multigranulation rough set(VMGRS). In our incomplete VMGRS approach, a threshold β will be used to control the number of tolerance classes, which satisfy with the set containment to the target concept.

Definition 4.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \le 1$, then $\forall X \subseteq U$, the VMGRS lower and upper approximations are denoted by $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, respectively,*

$$\underline{\sum_{i=1}^{m} A_i}^{\beta}(X) = \{x \in U : \forall A_i \in T, SIM_{A_i}(x) \subseteq X\}. \tag{10}$$

$$\overline{\sum_{i=1}^{m} A_i}^{\beta}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X). \tag{11}$$

where $T \subseteq A$ and $\frac{|T|}{m} \ge \beta$.

Remark. By Definition 4, we can see that in $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, we need the number of the granulation $SIM(A_i)$, which satisfy with the set containment to the target concept, to reach a certain amount. Such amount can be controlled by the threshold β.

By the lower and upper approximations $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$ and $\overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, the VMGRS boundary region of X is

$$BN^{\beta}_{\sum_{i=1}^{m} A_i}(X) = \overline{\sum_{i=1}^{m} A_i}^{\beta}(X) - \underline{\sum_{i=1}^{m} A_i}^{\beta}(X). \tag{12}$$

Theorem 1.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, $A = \{A_1, A_2, \cdots, A_m\}$, then $\forall X \subseteq U$, we have*

1. $\underline{\sum_{i=1}^{m} A_i}^{\frac{1}{m}}(X) = \underline{\sum_{i=1}^{m} A_i}^{O}(X)$,
$\overline{\sum_{i=1}^{m} A_i}^{\frac{1}{m}}(X) = \overline{\sum_{i=1}^{m} A_i}^{O}(X)$.
2. $\underline{\sum_{i=1}^{m} A_i}^{1}(X) = \underline{\sum_{i=1}^{m} A_i}^{P}(X)$,
$\overline{\sum_{i=1}^{m} A_i}^{1}(X) = \overline{\sum_{i=1}^{m} A_i}^{P}(X)$.

Proof. We only prove Eq. (1), the proof of Eq. (2) is similar to the proof of Eq. (1).
$\forall x \in U$, by Definition 2 and Definition 4, we have

$$x \in \underline{\sum_{i=1}^{m} A_i}^{\frac{1}{m}}(X)$$
$$\Leftrightarrow \forall A_i \in T, SIM_{A_i}(x) \subseteq X \text{ where } T \subseteq A, \frac{|T|}{m} \ge \frac{1}{m}$$
$$\Leftrightarrow \forall A_i \in T, SIM_{A_i}(x) \subseteq X \text{ where } T \subseteq A, |T| \ge 1$$
$$\Leftrightarrow \exists A_i \in A \text{ s.t. } SIM_{A_i}(x) \subseteq X$$
$$\Leftrightarrow x \in \underline{\sum_{i=1}^{m} A_i}^{O}(X)$$

By Definition 4 and the above result, we have

$$\begin{aligned}\overline{\sum_{i=1}^{m} A_i}^{\frac{1}{m}}(X) &= \sim \underline{\sum_{i=1}^{m} A_i}^{\frac{1}{m}}(\sim X)\\ &= \sim \underline{\sum_{i=1}^{m} A_i}^{O}(\sim X)\\ &= \overline{\sum_{i=1}^{m} A_i}^{O}(X) \qquad \square\end{aligned}$$

Remark. Through Theorem 1, we can see that if $\beta = \frac{1}{m}$, then the VMGRS lower approximation and upper approximation will degenerate to be Qian et al.'s OMGRS lower approximation and upper approximation in IDIS. On the other hand, if $\beta = 1$, then the VMGRS lower approximation and upper approximation will degenerate to be Qian et al.'s PMGRS lower approximation and upper approximation in IDIS. From these point of views, the VMGRS is a generalization of both OMGRS and PMGRS.

Theorem 2. *Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, suppose that $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \le 1$, then $\forall X \subseteq U$, we have*

1. $\overline{\sum_{i=1}^{m} A_i}^{P}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^{O}(X)$.
2. $\underline{\sum_{i=1}^{m} A_i}^{O}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{P}(X)$.

Proof. It can be derived directly from Definition 4 and Theorem 1. $\square$

Theorem 3. *Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, suppose that $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \le 1$, then $\forall X \subseteq U$, the following properties hold*

1. $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq X \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$.
2. $\underline{\sum_{i=1}^{m} A_i}^{\beta}(U) = \overline{\sum_{i=1}^{m} A_i}^{\beta}(U) = U$,
$\underline{\sum_{i=1}^{m} A_i}^{\beta}(\emptyset) = \overline{\sum_{i=1}^{m} A_i}^{\beta}(\emptyset) = \emptyset$
3. $\underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X) =\sim \overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$,
$\overline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X) =\sim \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$.
4. $X \subseteq Y \Rightarrow \underline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(Y)$,
$\overline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta}(Y)$.
5. $\underline{\sum_{i=1}^{m} A_i}^{\beta}(\underline{\sum_{i=1}^{m} A_i}^{\beta}(X)) = \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$,
$\overline{\sum_{i=1}^{m} A_i}^{\beta}(\overline{\sum_{i=1}^{m} A_i}^{\beta}(X)) = \overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$.
6. $\beta_1 \le \beta_2 \Rightarrow \underline{\sum_{i=1}^{m} A_i}^{\beta_1}(X) \supseteq \underline{\sum_{i=1}^{m} A_i}^{\beta_2}(X)$,
$\overline{\sum_{i=1}^{m} A_i}^{\beta_1}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta_2}(X)$.

Proof.

1. (a) Let $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, $\forall A_i \in T$, there must be $SIM_{A_i}(x) \subseteq X$, where $T \subseteq A$ and $|T|/m \ge \beta$. Since $x \in SIM_{A_i}(x)$, then $x \in X$. So it follows that $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq X$.
(b) Let $x \notin \overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, then $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X)$, for $\forall A_i \in T$, there must be $SIM_{A_i}(x) \subseteq (\sim X)$, then $x \in (\sim X)$, hence $x \notin X$. Thus, $X \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$.
2. (a) From 1, one can get that $\underline{\sum_{i=1}^{m} A_i}^{\beta}(U) \subseteq U$, And if $x \in U$, then, for each $A_i \in T$, there must be $SIM_{A_i}(x) \subseteq U$, where $T \subseteq A$ and $|T|/m \ge \beta$. hence, $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta}(U)$ and $U \in \underline{\sum_{i=1}^{m} A_i}^{\beta}(U)$. Thus, $\underline{\sum_{i=1}^{m} A_i}^{\beta}(U) = U$.
(b) From 1, we know that $U \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta}(U)$, and $\overline{\sum_{i=1}^{m} A_i}^{\beta}(U) \subseteq U$ hold clearly. Thus, $\overline{\sum_{i=1}^{m} A_i}^{\beta}(U) = U$
(c) From 1, we know that $\underline{\sum_{i=1}^{m} A_i}^{\beta}(\emptyset) \subseteq \emptyset$. And $\emptyset \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(\emptyset)$hold clearly. Thus,$\underline{\sum_{i=1}^{m} A_i}^{\beta}(\emptyset) = \emptyset$.
(d) From Definition 4, we know that $\overline{\sum_{i=1}^{m} A_i}^{\beta}(\emptyset) =\sim \underline{\sum_{i=1}^{m} A_i}^{\beta}(U)$. From 2.(b),we know $\underline{\sum_{i=1}^{m} A_i}^{\beta}(U) = U$. Thus, $\overline{\sum_{i=1}^{m} A_i}^{\beta}(\emptyset) = \emptyset$
3. From Definition 3, we know that $\underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X) =\sim \overline{\sum_{i=1}^{m} A_i}^{\beta}(X)$. Let $X =\sim X$, then $\sim \overline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X) = \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$.
4. (a) Let $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, $\forall A_i \in T$, there must be $SIM_{A_i}(x) \subseteq X$, where $T \subseteq A$ and $|T|/m \ge \beta$. since $X \subseteq Y$, then $SIM_{A_i}(x) \subseteq Y$, hence $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(Y)$.
(b) Since $X \subseteq Y$, then $\sim Y \subseteq \sim X$ and $\underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim Y) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X)$, from (3), we know that $\overline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta}(Y)$.
5. (a) From 1, we easily know that $\underline{\sum_{i=1}^{m} A_i}^{\beta}(\underline{\sum_{i=1}^{m} A_i}^{\beta}(X)) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$. Suppose $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, then for each $A_i \in T$, there must be $SIM_{A_i}(x) \subseteq X$, where $T \subseteq A$ and $|T|/m \ge \beta$, hence $\underline{\sum_{i=1}^{m} A_i}^{\beta}(SIM_{A_i}(x)) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$. But $\underline{\sum_{i=1}^{m} A_i}^{\beta}(SIM_{A_i}(x)) = SIM_{A_i}(x)$, then we can have that $SIM_{A_i}(x) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(X)$, Hence $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta}(\underline{\sum_{i=1}^{m} A_i}^{\beta}(X))$. Then, we get that, $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\beta}(\underline{\sum_{i=1}^{m} A_i}^{\beta}(X))$. Therefore, $\underline{\sum_{i=1}^{m} A_i}^{\beta}(X) = \underline{\sum_{i=1}^{m} A_i}^{\beta}(\underline{\sum_{i=1}^{m} A_i}^{\beta}(X))$.
(b) Form 3 and 5(a), we know that $\overline{\sum_{i=1}^{m} A_i}^{\beta}(\overline{\sum_{i=1}^{m} A_i}^{\beta}(X)) = \overline{\sum_{i=1}^{m} A_i}^{\beta}(\sim \underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X)) =\sim \underline{\sum_{i=1}^{m} A_i}^{\beta}(\underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X)) =\sim \underline{\sum_{i=1}^{m} A_i}^{\beta}(\sim X)) = \overline{\sum_{i=1}^{m} A_i}^{\beta}(X))$.

6.Let $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta_2}(X)$, for each $A_i \in T$, there must be $SIM_{A_i}(x) \subseteq X$, where $T \subseteq A$ and $|T|/m \geq \beta_2$. Since $\beta_1 \leq \beta_2$, then $|T|/m \geq \beta_1$, hence $x \in \underline{\sum_{i=1}^{m} A_i}^{\beta_1}(X)$. Similarity, it is a trivial to prove $\overline{\sum_{i=1}^{m} A_i}^{\beta_1}(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^{\beta_2}(X)$

This completes the proof. □

Theorem 3 shows the basic properties about the VMGRS in IDIS. *1* says that the VMGRS lower approximation is included into the target concept and the VMGRS upper approximation includes the target concept. *2* shows the normality of the VMGRS. *3* expresses the complement properties of the VMGRS. *4* says the monotonic properties about the VMGRS in terms of the monotonic varieties of the target concepts. *5* says the idempotents of the VMGR. *6* says the monotonic properties about the VMGRS in terms of the monotonic varieties of the threshold.

*Example 1.*We then use an example to illustrate the VMGRS in incomplete decision information system. Table 1 depicts an incomplete decision information system about the emporium investment project. "Locus", "Investment", and "Population density" are the conditional attributes of the system, and "Decision" is the decision attribute (in the sequel, a_1, a_2, a_3, and d will stand for "Locus", "Investment", "Population density", and "Decision", respectively).

Let $A = \{A_1, A_2, A_3\} = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}\}$, since the decision attribute determines a partition on the universe such that $U/\{d\} = \{D_1, D_2\} = \{\{x_3, x_4, x_5, x_7, x_9\}, \{x_1, x_2, x_6, x_8, x_{10}\}\}$, then by Definition 2, we have

$\underline{\sum_{i=1}^{3} A_i}^{O}(D_1) = \{x_3, x_5, x_7, x_9\}$.

$\underline{\sum_{i=1}^{3} A_i}^{O}(D_2) = \{x_1, x_2, x_8, x_{10}\}$.

$\overline{\sum_{i=1}^{3} A_i}^{O}(D_1) = \{x_3, x_4, x_5, x_6, x_7, x_9\}$.

$\overline{\sum_{i=1}^{3} A_i}^{O}(D_2) = \{x_1, x_2, x_4, x_6, x_8, x_{10}\}$.

By Definition 3, we have

$\underline{\sum_{i=1}^{3} A_i}^{P}(D_1) = \emptyset$.

$\underline{\sum_{i=1}^{3} A_i}^{P}(D_2) = \{x_{10}\}$.

$\overline{\sum_{i=1}^{3} A_i}^{P}(D_1) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

$\overline{\sum_{i=1}^{3} A_i}^{P}(D_2) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$.

Suppose $\beta = 0.5$, then by Definition 4, we have

$\underline{\sum_{i=1}^{3} A_i}^{\beta}(D_1) = \{x_3, x_5, x_9\}$.

$\underline{\sum_{i=1}^{3} A_i}^{\beta}(D_2) = \{x_1, x_8, x_{10}\}$.

$\overline{\sum_{i=1}^{3} A_i}^{\beta}(D_1) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_9\}$.

$\overline{\sum_{i=1}^{3} A_i}^{\beta}(D_2) = \{x_1, x_2, x_4, x_6, x_7, x_8, x_{10}\}$.

Through the above results, we have

$\underline{\sum_{i=1}^{3} A_i}^{P}(D_1) \subseteq \underline{\sum_{i=1}^{3} A_i}^{\beta}(D_1) \subseteq \underline{\sum_{i=1}^{3} A_i}^{O}(D_1)$.

$\underline{\sum_{i=1}^{3} A_i}^{P}(D_2) \subseteq \underline{\sum_{i=1}^{3} A_i}^{\beta}(D_2) \subseteq \underline{\sum_{i=1}^{3} A_i}^{O}(D_2)$.

Table 1: An incomplete decision information table of the emporium investment project

Project	Locus	Investment	Population density	Decision
x_1	*Common*	*High*	*High*	*Yes*
x_2	*Bad*	*High*	*High*	*Yes*
x_3	*Bad*	$*$	*Small*	*No*
x_4	*Bad*	*Low*	$*$	*No*
x_5	*Bad*	*Low*	*Small*	*No*
x_6	*Bad*	$*$	*Medium*	*Yes*
x_7	*Common*	*High*	*Medium*	*No*
x_8	*Good*	$*$	*Medium*	*Yes*
x_9	*Bad*	*Low*	*Bad*	*Yes*
x_{10}	*Good*	*High*	*High*	*Yes*

$\overline{\sum_{i=1}^{3} A_i}^{O}(D_1) \subseteq \overline{\sum_{i=1}^{3} A_i}^{\beta}(D_1) \subseteq \overline{\sum_{i=1}^{3} A_i}^{P}(D_1)$.

$\overline{\sum_{i=1}^{3} A_i}^{O}(D_2) \subseteq \overline{\sum_{i=1}^{3} A_i}^{\beta}(D_2) \subseteq \overline{\sum_{i=1}^{3} A_i}^{P}(D_2)$.

these results show the correctness of Theorem 2.

4 Measurements

Uncertainty of a set (category) is due to the existence of a borderline region. The greater the borderline region of a set, the lower is the accuracy of the set. To more precisely express this idea, we introduce some accuracy measurements as following.

Definition 5.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \leq 1$, then $\forall X \subseteq U (X \neq \emptyset)$. The accuracies degrees of X in terms of OMGRS, PMGRS and VMGRS in incomplete decision information system are defined respectively as*

$$\alpha_O\left(\sum_{i=1}^{m} A_i, X\right) = \frac{\left|\underline{\sum_{i=1}^{m} A_i}^{O}(X)\right|}{\left|\overline{\sum_{i=1}^{m} A_i}^{O}(X)\right|} \tag{13}$$

$$\alpha_P\left(\sum_{i=1}^{m} A_i, X\right) = \frac{\left|\underline{\sum_{i=1}^{m} A_i}^{P}(X)\right|}{\left|\overline{\sum_{i=1}^{m} A_i}^{P}(X)\right|} \tag{14}$$

$$\alpha_\beta\left(\sum_{i=1}^{m} A_i, X\right) = \frac{\left|\underline{\sum_{i=1}^{m} A_i}^{\beta}(X)\right|}{\left|\overline{\sum_{i=1}^{m} A_i}^{\beta}(X)\right|} \tag{15}$$

Definition 6.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \leq 1$, $U/D = \{D_1, D_2, ...D_r\}$ be the partition induced by the decision attribute sets D. The qualities of approximations of D by A, also called the degrees of dependencies in terms of the OMGRS, PMGRS and VMGRS in incomplete decision information system are* *defined respectively as*

$$\gamma_O(\sum_{i=1}^{m} A_i, D) = \frac{\sum\{|\underline{\sum_{i=1}^{m} A_i^O}(D_r)| : D_r \in D\}}{|U|} \quad (16)$$

$$\gamma_P(\sum_{i=1}^{m} A_i, D) = \frac{\sum\{|\underline{\sum_{i=1}^{m} A_i^P}(D_r)| : D_r \in D\}}{|U|} \quad (17)$$

$$\gamma_\beta(\sum_{i=1}^{m} A_i, D) = \frac{\sum\{|\underline{\sum_{i=1}^{m} A_i^\beta}(D_r)| : D_r \in D\}}{|U|} \quad (18)$$

Definition 7.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \leq 1$, then $\forall X \subseteq U (X \neq \emptyset)$, the approximated degrees of X in terms of OMGRS, PMGRS and VMGRS in incomplete decision information system are defined respectively as*

$$\pi_O(\sum_{i=1}^{m} A_i, X) = \frac{|\underline{\sum_{i=1}^{m} A_i}^O(X)|}{|X|} \quad (19)$$

$$\pi_P(\sum_{i=1}^{m} A_i, X) = \frac{|\underline{\sum_{i=1}^{m} A_i}^P(X)|}{|X|} \quad (20)$$

$$\pi_\beta(\sum_{i=1}^{m} A_i, X) = \frac{|\underline{\sum_{i=1}^{m} A_i}^\beta(X)|}{|X|} \quad (21)$$

Theorem 4.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \leq 1$, then $\forall X \subseteq U (X \neq \emptyset)$. The accuracies degrees and the approximated degrees of X have the following properties:*

$$\alpha_P(\sum_{i=1}^{m} A_i, X) \leq \alpha_\beta(\sum_{i=1}^{m} A_i, X) \leq \alpha_O(\sum_{i=1}^{m} A_i, X) \quad (22)$$

$$\pi_P(\sum_{i=1}^{m} A_i, X) \leq \pi_\beta(\sum_{i=1}^{m} A_i, X) \leq \pi_O(\sum_{i=1}^{m} A_i, X) \quad (23)$$

Proof. From Theorem 2, we know that $\overline{\sum_{i=1}^{m} A_i}^P(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^\beta(X) \subseteq \overline{\sum_{i=1}^{m} A_i}^O(X)$ and $\underline{\sum_{i=1}^{m} A_i}^O(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^\beta(X) \subseteq \underline{\sum_{i=1}^{m} A_i}^P(X)$. then $\frac{|\underline{\sum_{i=1}^{m} A_i}^P(X)|}{|\overline{\sum_{i=1}^{m} A_i}^P(X)|} \leq \frac{|\underline{\sum_{i=1}^{m} A_i}^\beta(X)|}{|\overline{\sum_{i=1}^{m} A_i}^\beta(X)|} \leq \frac{|\underline{\sum_{i=1}^{m} A_i}^O(X)|}{|\overline{\sum_{i=1}^{m} A_i}^O(X)|}$. thus, $\alpha_P(\sum_{i=1}^{m} A_i, X) \leq \alpha_\beta(\sum_{i=1}^{m} A_i, X) \leq \alpha_O(\sum_{i=1}^{m} A_i, X)$. Similarity, it is a trivial to prove $\pi_P(\sum_{i=1}^{m} A_i, X) \leq \pi_\beta(\sum_{i=1}^{m} A_i, X) \leq \pi_O(\sum_{i=1}^{m} A_i, X)$. □

Theorem 5.*Let $I =< U, C \cup D, V, f >$ be an IDIS in which $A_1, A_2, \cdots, A_m \subseteq C$, $A = \{A_1, A_2, \cdots, A_m\}$, $0 < \beta \leq 1$, $U/D = \{D_1, D_2, ...D_r\}$ be the partition induced by the decision attribute sets D. the dependencies degrees of D have the following properties:*

$$\gamma_\beta(\sum_{i=1}^{m} A_i, D) \leq \gamma_\beta(\sum_{i=1}^{m} A_i, D) \leq \gamma_O(\sum_{i=1}^{m} A_i, D) \quad (24)$$

Proof. It can be derived directly from Theorem 2 and Definition 6. □

Table 2: Data sets descrition

Date sets	Samples	Data type	Conditional density	Decision calss
Breast-cancer -wisconsin	699	Incomplete	9	2
Agaricus -lepiota	8124	Incomplete	22	2

Table 3: Comparisons of lower approximation in breast-cancer-wisconsin data set

Lower Approximations	Descision classes	
	D_1	D_2
OMGRS	434	239
VMGRS($\beta = 0.2$)	**434**	**239**
VMGRS($\beta = 0.5$)	**231**	**228**
VMGRS($\beta = 1$)	**43**	**163**
PMGRS	43	163

5 Experimental Analysis

In the following, through experimental analysis, we will illustrate the relations among VMGRS, OMGRS and PMGRS in IDIS. We have downloaded two public data sets from UCI Repository of machine learning databases, which are described in Table 2.

Here, we compare the lower and upper Approximations in our VMGRS with that in OMGRS and PMGRS on these two practical data sets. The comparisons of the lower and upper approximations in Breast-cancer-wisconsin data set are shown in Table 3 and Table 4.

Table 3 and Table 4 show the number of elements of lower and upper approximations in terms of three different MGRS approaches. We can see that if $\beta = 0.2$, then the lower and upper approximations in VMGRS are same to that in OMGRS; if $\beta = 1$, then the lower and upper approximations in VMGRS are same to that in PMGRS. Such results are consistent to Theorem 1.

In particular, if $\beta = 0.5$, then the Lower and upper approximations in VMGRS are between that in OMGRS and PMGRS. Such results are consistent to Theorem 2.

Similarity, it is not difficult to draw the same conclusions from Table 5 and Table 6.

So, we can conclude that VMGRS is a generalization of both OMGRS and PMGRS, OMGRS and PMGRS are two type especial instances of VMGRS

6 Conclusions

In this paper, the VMGRS is proposed. In our VMGRS approach, a threshold is used to control the number of granulations, which satisfy with the inclusion condition

Table 4: Comparisons of upper approximation in breast-cancer-wisconsin data set

Upper Approximations	Descision classes D_1	D_2
OMGRS	460	265
VMGRS($\beta = 0.2$)	**460**	**265**
VMGRS($\beta = 0.5$)	**471**	**468**
VMGRS($\beta = 1$)	**536**	**656**
PMGRS	536	656

Table 5: Comparisons of lower approximation in agaricus-lepiota data set

Lower Approximations	Descision classes D_1	D_2
OMGRS	4196	3916
VMGRS($\beta = 0.2$)	**4196**	**3916**
VMGRS($\beta = 0.5$)	**3956**	**3756**
VMGRS($\beta = 1$)	**1228**	**2160**
PMGRS	1228	2160

Table 6: Comparisons of upper approximation in agaricus-lepiota data set

Upper Approximations	Descision classes D_1	D_2
OMGRS	4208	3928
VMGRS($\beta = 0.2$)	**4208**	**3928**
VMGRS($\beta = 0.5$)	**4368**	**4168**
VMGRS($\beta = 1$)	**5964**	**6896**
PMGRS	5964	6896

between the knowledge granules and the target. Not only the theoretical discussions, but also the experimental analyses show that our VMGRS is a generalization of the OMGRS and PMGRS in IDIS. In our further researching, the reduction in terms of the proposed incomplete VMGRS is an interesting topic to be addressed.

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