Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/040104

An Algorithm for Explicit Form of Fundamental Units of Certain Real Quadratic Fields

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Received: 20 Jul. 2015, Revised: 9 Aug. 2015, Accepted: 10 AUG. 2015
Published online: 1 Jan. 2016

Abstract: Quadratic fields have applications in different areas of mathematics such as quadratic forms, algebraic geometry, diophantine equations, algebraic number theory, and even cryptography. The Unit Theorem for real quadratic fields says that every unit in the integer ring of a quadratic field is given in terms of the fundamental unit of the quadratic field. Thus determining the fundamental units of quadratic fields is of great importance. In this paper, we obtained an explicit formulation to determine the forms of continued fraction expansion and fundamental units of certain real quadratic number fields where the period in the continued fraction expansion of the quadratic irrational number of the certain real quadratic fields is equal to 7 by using a practical algorithm for special cases. Moreover, a part of this paper is generalize and complete [2].

Keywords: continued fractions for quadratic irrational numbers, fundamental unit
subjclass[2010] Primary 11A55, Secondary 11R27

1 Introduction and Notation

Determination of the fundamental units of quadratic fields has a great importance at many branches in number theory. Although the fundamental units of real quadratic fields of Richaut-Degert type are well-known, explicit form of the fundamental units are not known very well and these determinations were very limited except for these type. Therefore Tomita has described explicitly the form of the fundamental units of the real quadratic fields $Q(\sqrt{d})$ such that d is a square-free positive integer congruent to 1 modulo 4 and the period k_d in the continued fraction expansion of the quadratic irrational number $\omega_d = (\frac{1+\sqrt{d}}{2})$ in $Q(\sqrt{d})$ is equal to 3 and 4, 5 respectively in [5] and [6]. Later, explicit form of the fundamental units for all real quadratic fields $Q(\sqrt{d})$ such that the period k_d in the continued fraction expansion of the quadratic irrational number ω_d is equal to 6, has been described in [4]. The aim of this paper is to determine the general forms of continued fractions and fundamental units for special cases and also generalize and complete the some of theorems had been given in [2].

In this paper, we will deal with some real quadratic fields $Q(\sqrt{d})$ such that d is a square free positive integer not only congruent to 1 modulo 4 but also congruent to 2 modulo 4 and the period $k_d = k$ in the continued fraction expansion of the quadratic irrational number ω_d in $Q(\sqrt{d})$ is equal to 7 and describe explicitly T_d, U_d in the fundamental unit $\varepsilon_d = (\frac{T_d+U_d\sqrt{d}}{2}) > 1$ of $Q(\sqrt{d})$ and also the form of d is written by using parameters which are appearing in the continued fraction expansion of ω_d.

Let $I(d)$ be the set of all quadratic irrational numbers in $Q(\sqrt{d})$. For an element ξ of $I(d)$ if $\xi > 1$, $-1 < \xi' < 0$ then ξ is called reduced, where ξ' is the conjugate of ξ with respect to Q. More information on reduced irrational numbers may be found in [3] and [7]. We denote by $R(d)$ the set of all reduced quadratic irrational numbers in $I(d)$. It is well known that if an element ξ of $I(d)$ is in $R(d)$ then the continued fractional expansion of ξ is purely periodic. Moreover, the denominator of its modular automorphism is equal to fundamental unit ε_d of $Q(\sqrt{d})$ and the norm of ε_d is $(-1)^{k_d}$ in [1] and [7]. In this paper $[x]$ means the greatest integer less than or equal to x and continued fraction with period $k_d = k$ is generally denoted by $[a_0, \overline{a_1, a_2, \ldots.., a_k}]$.

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2 Preliminaries and Lemmas

In this section some of the important required preliminaries and lemmas are given.

Now, for any square-free positive integer d, we can put $d=a^2+b$ with $a,b\in Z$, $0<b\leq 2a$. Here, since $\sqrt{d}-1<a<\sqrt{d}$ the integers a and b are uniquely determined by d.

Let d be a square-free positive integer then we will consider the following two special cases:

Case 1. $d\equiv 1 mod(4)$, if a is even, then $b=8\ell+5$ with $l\in Z$, $\ell\geq 0$.

Case 2. $d\equiv 2 mod(4)$, if a is odd, then $b=4m+1$ with $m\in Z$, $m\geq 0$.

Let denote by ${D_t}^k$ the set of all positive square-free integer d such that $d\equiv k(8)$ and $b\equiv t(8)$. Hence, we have ${D_t}^k=\{d\in Z\mid d\equiv k(8), b\equiv t(8)\}$. Then, we get some remarks as follows:

Remark 2.1. d can be congruent to 1 or 5 modulo 8 since d is congruent to 1 modulo 4.

In the case of $d\equiv 1(8)$, b can be congruent to 0, 1 or 5 modulo 8. Therefore, the set of all positive square-free integers congruent to 1 modulo 8 is equal ${D_0}^1\cup {D_1}^1\cup {D_5}^1$. Thus the set of all positive square free integers congruent to1 modulo 8 is the union of ${D_0}^1, {D_1}^1, {D_5}^1$.

In the case of $d\equiv 5(8)$, b can be congruent to 1, 4 or 5 modulo 8. So the set of all positive square-free integers congruent to 5 modulo 8 is equal to ${D_1}^5\cup {D_4}^5\cup {D_5}^5$.

Remark 2.2. Let d be a square-free positive integer congruent to 1 modulo 4, then,

If a is even; b can only be congruent to 1 or 5 modulo 8 since $b\equiv 1(mod 4)$ when a is even. Then, d belongs to ${D_5}^5\cup {D_5}^1$ in the Case1.

Remark 2.3. The sets ${D_0}^1, {D_1}^1, {D_5}^1, {D_1}^5, {D_4}^5$ and ${D_5}^5$ are represented as follows;

${D_0}^1=\{d\in D\mid d=a^2+8m, a\equiv 1(mod 2), 0<4m<a\}$

${D_1}^1=\{d\in D\mid d=a^2+8m+1, a\equiv 0(mod 4), 0\leq 4m<a\}$

${D_5}^1=\{d\in D\mid d=a^2+8m+5, a\equiv 2(mod 4), 0\leq 4m<a-2\}$

${D_1}^5=\{d\in D\mid d=a^2+8m+1, a\equiv 2(mod 4), 0\leq 4m<a\}$

${D_4}^5=\{d\in D\mid d=a^2+8m+4, a\equiv 1(mod 2), 0\leq 4m<a-2\}$

${D_5}^5=\{d\in D\mid d=a^2+8m+5, a\equiv 0(mod 4), 0\leq 4m<a-2\}$

Now in order to prove our theorems we need the following lemmas.

Lemma 2.4. For a square-free positive integer $d>5$ congruent to 1 modulo 4, we put $\omega_d=(\frac{1+\sqrt{d}}{2})$, $q_0=[\omega_d]$, $\omega_R=q_0-1+\omega_d$. Then $\omega_d\notin R(d)$, but $\omega_R\in R(d)$ holds. Moreover for the period k of ω_R, we get $\omega_R=[\overline{2q_0-1, q_1, ..., q_{k-1}}]$ and $\omega_d=[q_0, \overline{q_1, ..., q_{k-1}, 2q_0-1}]$. Furthermore, let $\omega_R=\frac{(P_{k-1}\omega_R+P_{k-2})}{(Q_{k-1}\omega_R+Q_{k-2})}=[2q_0-1, q_1, ..., q_{k-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ε_d of $Q(\sqrt{d})$ is given by the following formula:

$\varepsilon_d=(\frac{T_d+U_d\sqrt{d}}{2})>1$,

$T_d=(2q_0-1)Q_{k-1}+2Q_{k-2}$, $U_d=Q_{k-1}$, where Q_i is determined by $Q_{-1}=0$, $Q_0=1$, $Q_{i+1}=q_{i+1}Q_i+Q_{i-1}$, $(i\geq 0)$.

Moreover, for a square-free positive integer d congruent to $2,3$ modulo 4, we put $\omega_d=\sqrt{d}$, $q_0=[\omega_d]$, $\omega_R=q_0+\omega_d$. Then $\omega_d\notin R(d)$, but $\omega_R\in R(d)$ holds. Moreover for the period k of ω_R, we get $\omega_R=[\overline{2q_0, q_1, ..., q_{k-1}}]$ and $\omega_d=[q_0, \overline{q_1, ..., q_{k-1}, 2q_0}]$. Furthermore, let $\omega_R=\frac{(P_{k-1}\omega_R+P_{k-2})}{(Q_{k-1}\omega_R+Q_{k-2})}=[2q_0, q_1, ..., q_{k-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ε_d of $Q(\sqrt{d})$ is given by the following formula:

$\varepsilon_d=(\frac{T_d+U_d\sqrt{d}}{2})>1$,

$T_d=2q_0Q_{k-1}+2Q_{k-2}$, $U_d=2Q_{k-1}$, where Q_i is determined by $Q_{-1}=0$, $Q_0=1$, $Q_{i+1}=q_{i+1}Q_i+Q_{i-1}$, $(i\geq 0)$.

Proof. See[6, Lemma 1].

Lemma 2.5. For a square-free positive integer d, we put $d=a^2+b$ $(0<b\leq 2a)$, $a,b\in Z$. Moreover let $\omega_i=\ell_i+\frac{1}{\omega_{i+1}}$ $(\ell_i=[\omega_i], i\geq 0)$ be the continued fraction expansion of $\omega=\omega_0$ in $R(d)$. Then each ω_i is expressed in the form $\omega_i=\frac{a-r_i+\sqrt{d}}{c_i}$ $(c_i, r_i\in Z)$, and ℓ_i, c_i, r_i can be obtained from the following recurrence formula:

$\omega_0=\frac{a-r_0+\sqrt{d}}{c_0}$,

$2a-r_i=c_i\ell_i+r_{i+1}$,

$c_{i+1}=c_{i-1}+(r_{i+1}-r_i)\ell_i$ $(i\geq 0)$, where $0\leq r_{i+1}<c_i$, $c_{-1}=\frac{(b+2ar_0-{r_0}^2)}{c_0}$.

Moreover for the period $k\geq 1$ of ω_0, we get

$\ell_i=\ell_{k-i}$ $(1\leq i\leq k-1)$,

$r_i=r_{k-i+1}$, $c_i=c_{k-i}$ $(1\leq i\leq k)$.

Proof. See[1, Proposition 1] .

Lemma 2.6. For a square-free positive integer d congruent to 1 modulo 4, we put $\omega_d=(\frac{1+\sqrt{d}}{2})$, $q_0=[\omega_d]$ and $\omega_R=q_0-1+\omega_d$.

If we put $\omega=\omega_R$ in Lemma 2.5. , then we have the following recurrence formula:

$r_0=r_1=a-l_0=a-2q_0+1$,

$c_0=2$, $c_1=c_{-1}=\frac{(b+2ar_0-{r_0}^2)}{c_0}$,

$\ell_0=2q_0-1$, $\ell_i=q_i$ $(1\leq i\leq k-1)$.

For a square-free positive integer d congruent to $2,3$ modulo 4, we put $\omega_d=\sqrt{d}$, $q_0=[\omega_d]$ and $\omega_R=q_0+\omega_d$.

If we put $\omega = \omega_R$ in Lemma 2.5. , then we have the following recurrence formula:
$r_0 = r_1 = 0$, $c_0 = 1$, $c_1 = b$,
$\ell_0 = 2q_0$, $\ell_i = q_i$ $(1 \leq i \leq k-1)$.

Proof. It can be easily proved by using Lemma 2.5.

3 Theorems

Theorem 3.1. Let $d = a^2 + b \equiv 1 mod(4)$ is a square free integer for positive integer a is even and b satisfying $0 < b \leq 2a$, $b \equiv 5mod(8)$, (i.e. $d \in D^1{}_5 \cup D^5{}_5$) . Let the period k_d of the integral basis element of $\omega_d = (\frac{1+\sqrt{d}}{2})$ in $Q(\sqrt{d})$ be 7. Then,

$$\omega_d = [\tfrac{a}{2}, \overline{1, \ell_2, \ell_3, \ell_3, \ell_2, 1, a-1}]$$

for the positive integers ℓ_2, ℓ_3 such that $1 \leq \ell_i \leq a$ $(i = 1, 2)$ and then

$$(T_d, U_d) = (A(AC + D) + B^2(C + E), A^2 + B^2)$$

and

$$d = C^2 + 2rF + G$$

hold, where A, B, C, D, E, F, G and the integers $r \geq 0$ and $s \geq 0$ are determined uniquely as follows:
$A = \ell_2\ell_3 + \ell_3 + 1$
$B = \ell_2 + 1$
$C = Ar + s$
$D = (A+2)\ell_2\ell_3 + \ell_2^2 + 1$
$E = \ell_3 + 1$
$F = D - AE$
$G = 2CE + (A - \ell_3)^2 + (B-2)^2 + (B-1)^2$
$a = A(r+1) + s - \ell_2, \quad \ell_2(\ell_3 - B) + 1 = rB^2 - sA.$

Proof. In the case a is even and $b \equiv 5mod(8)$, $d \equiv 1mod(4)$ can only belong to $D_5{}^1 \cup D_5{}^5$. Since $q_0 = [\omega_d] = \frac{a}{2}$, it follows from Lemma 2.6 that $r_0 = r_1 = a - 2q_0 + 1 = 1 = a - \ell_0$ then $\ell_0 = a-1$, $r_1 = 1$ and $c_0 = 2$, $c_1 = c_{-1} = a + 4m + 2$. For $i = 1$ and by Lemma 2.5 we have;
$2a - r_1 = c_1\ell_1 + r_2 \Rightarrow 2a = (a+4m+2)\ell_1 + r_2 + 1 \Rightarrow a(2-\ell_1) = (4m+2)\ell_1 + r_2 + 1 \Rightarrow \ell_1 = 1$ holds from $\ell_1 \geq 0$, $a > 0$ and $\ell_1 < 2$.
Since $\ell_1 = \ell_6$, $\ell_2 = \ell_5$, $\ell_3 = \ell_4$ then we obtain;

$$\omega_d = [\tfrac{a}{2}, \overline{1, \ell_2, \ell_3, \ell_3, \ell_2, 1, a-1}].$$

for $\ell_1 = 1$ we have

$$a = 4m + r_2 + 3. \tag{1}$$

$a = 4m + r_2 + 3 \Rightarrow r_2 = a - 4m - 3$ is an odd number because of a is even, and so $r_2 < a$ holds from (1) and $b \leq 2a$. From Lemma 2.5; $2a - r_2 = c_2\ell_2 + r_3$ and $c_2 = c_0 + (r_2 - r_1)\ell_1 \Rightarrow c_2 = a - 4m - 2$ holds , and so we have $c_2 = r_2 + 1$. Moreover, from Lemma 2.5 we get

$$2a = (r_2+1)\ell_2 + r_2 + r_3 \tag{2}$$

On the other hand, we have

$$c_3 = c_1 + (r_3 - r_2)\ell_2 \Rightarrow c_3 = (a + 4m + 2) + (r_3 - r_2)\ell_2.$$

and

$$c_4 = c_2 + (r_4 - r_3)\ell_3 \Rightarrow c_4 = (r_2 + 1) + (r_4 - r_3)\ell_3.$$

By using equalities $c_3 = c_4$ and $a = 4m + r_2 + 3$ we obtain

$$8m + 4 = (r_2 - r_3)\ell_2 + (r_4 - r_3)\ell_3. \tag{3}$$

Since $2a = c_3\ell_3 + r_3 + r_4$ from Lemma 2.5 then we have

$$r_4 = 2a - [(a + 4m + 2) + (r_3 - r_2)\ell_2]\ell_3 - r_3. \tag{4}$$

and

$$8m + 6 = (r_2 + 1)\ell_2 + r_3 - r_2. \tag{5}$$

It follows from (3) and (5) we get immediately

$$r_2 = (r_3 - r_4)\ell_3 + (r_3 + 1)\ell_2 + r_3 - 2. \tag{6}$$

By taking $a = 4m + r_2 + 3$ and by using equalities (1), (3) and (4) we can make an explication as follows:
$d \in D^1{}_5 \Rightarrow r_2 \equiv 3mod(4), r_3 \equiv 1mod(4)$ holds for $a \equiv 2mod(4)$,
$d \in D^5{}_5 \Rightarrow r_2 \equiv 1mod(4)$, $r_3 \equiv 1mod(4)$ or $r_3 \equiv 3mod(4)$ holds for $a \equiv 0mod(4)$.
If $d \in D^5{}_5 \bigcup D^1{}_5$ then we have $r_3 = 2r + 1 \equiv 1mod(2)$, $r \geq 0$ and $r_4 = 2s + 1 \ni s \geq 0$. Furthermore we can easily see that

$$r_2 = 2(r - s)\ell_3 + 2(r+1)\ell_2 + 2r - 1 \tag{7}$$

from the Lemma 2.5 and from (4), (6).
We know that $c_3 = c_4 = (r_2 + 1) + (r_3 - r_4)\ell_3$ and so if we put $r_2 = 2\ell_3(r - s) + 2(r+1)\ell_2 + 2r - 1$ in $2a = [(r_2 + 1) + (r_3 + r_4)\ell_3]\ell_3 + r_3 + r_4$ then we can obtain

$$a = r(\ell_2\ell_3 + \ell_3 + 1) + s + \ell_2\ell_3 + 1. \tag{8}$$

In this equation, if we take $\ell_2\ell_3 + \ell_3 + 1 = A$ then we can also write $a = A(r+1) + s - \ell_3$. By using equalities (1), (3) and (7) we get $2a = (r_2 - r_3)\ell_2 + (r_4 - r_3)\ell_3 + 4\ell_2(r + 1) + 4\ell_3(r - s) + 4r - 2$ and by taking in this equation $r_2 = 2(r-s)\ell_3 + 2(r+1)\ell_2 + 2r - 1$, $r_3 = 2r+1$ and $r_4 = 2s+1$ we have $r(\ell_2 + 1)^2 - s(\ell_2\ell_3 + \ell_3 + 1) - \ell_2[\ell_3 - \ell_2 - 1] - 1 = 0$. Since $A = \ell_2\ell_3 + \ell_3 + 1$, $B = \ell_2 + 1$ then $\ell_2[\ell_3 - B] + 1 = rB^2 - sA$ holds. We can immediately that r and s uniquely-defined from the equalities $a = (r+1)A + s - \ell_3$ and $\ell_2[\ell_3 - B] + 1 = rB^2 - sA$.
Now, let's determine the coefficients T_d and U_d of the fundamental unit ε_d by using Lemma 2.4. Since
$Q_-1 = 0$
$Q_0 = 1$
$q_i = \ell_i, (1 \leq i \leq k_d - 1)$
$Q_{i+1} = q_{i+1}.Q_i + Q_{i-1} \qquad (i \geq 0)$
$Q_1 = \ell_1 = 1$
$Q_2 = \ell_2 + 1 = B$

$Q_3 = \ell_2\ell_3 + \ell_3 + 1 = A$
$Q_4 = A\ell_3 + B$
$Q_5 = \ell_2(A\ell_3 + B) + A = A(\ell_2\ell_3 + 1) + B\ell_2$
$Q_6 = A(\ell_2\ell_3 + 1) + B\ell_2 + A\ell_3 + B = A^2 + B^2$
then we have $T_d = (Ar+s)(A^2+B^2) + A[\ell_2\ell_3(A+2)+2] + \ell_2[(A+1)+B]$ and $U_d = A^2+B^2$ for taking the following equalities $2q_0 - 1 = a - 1 = Ar + s + \ell_2\ell_3$, $T_d = (2q_0 - 1)Q_6 + 2Q_5$, $C = Ar + s$, $D = (A+2)\ell_2\ell_3 + \ell_2^2 + 1$, $E = \ell_3 + 1$ and so $T_d = A(AC+D) + B^2(C+E)$, $U_d = A^2 + B^2$ hold.

Now, we write $d = a^2 + b$ depends on the parameters ℓ_2, ℓ_3, r and s. For this if we put $r_2 = 2(r-s)\ell_3 + 2(r+1)\ell_2 + 2r - 1$, $r_3 = 2r+1$, $r_4 = 2s+1$ instead of r_2, r_3 and r_4 in (4) then we obtain $8m+4 = [2\ell_3(r-s)\ell_2 + 2\ell_2^2(r+1) - 2\ell_2 + 2(s-r)]\ell_3$ and $b = 8m+5 = 2\ell_2^2(r+1) + 2(r-s)(\ell_2 - 1)\ell_3 - 2\ell_2 + 1$. By putting the values $a = A(r+1) + s - \ell_3$ and b in $d = a^2 + b$ we have $d = a^2 + b = (Ar+s)^2 + 2r(D - AE) + 2CE + (A-\ell_3)^2 + (B-2)^2 + (B-1)^2$. Where, if we take $D - AE = F$ and $2CE + (A-\ell_3)^2 + (B-2)^2 + (B-1)^2 = G$ then $d = C^2 + 2rF + G$ holds. Thus, the theorem is proved completely.

Theorem 3.2. Let $d = a^2 + b \equiv 2 mod(4)$ is a square free integer such that a is odd integer and the period k_d of the integral basis element of $\omega_d = \sqrt{d}$ in $Q(\sqrt{d})$ be 7. If $b \equiv 1 mod(4)$ then,

$$\omega_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$$

for the positive integers ℓ_1, ℓ_2, ℓ_3 such that $\ell_i \geq 1$ $(i = 1,2,3)$
and then

$$(T_d, U_d) = (2[a(A^2+B^2) + BC + A\ell_2], 2(A^2+B^2))$$

and

$$d = A^2r^2 - 2rD + E$$

hold, where A, B, C, D, E, $r \geq 0$, $e \geq 0$ are integers and these are determined uniquely as follows:
$A = \ell_1\ell_2 + 1$
$B = \ell_1 + A\ell_3$
$C = \ell_2\ell_3 + 1$
$D = Ae\ell_1 - \ell_2$
$E = \ell_1^2 e^2 - 2e + 1$
$a = Ar - \ell_1 e$,
$A^2 + B^2 - C^2 - \ell_2^2 = 2rB + 2e(A + B\ell_3)$.

Proof. Since $d \equiv 2$ (mod4) and $b \equiv 1(mod4)$ then we have $b = 4m+1$ for the positive integers a, b, m with $a < b \leq 2a$. From the Lemma 2.6. it is clear that $w_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, 2a}]$ for $q_0 = a$ and $k_d = 7$. Besides from the Lemma2.6 we obtain $r_0 = r_1 = 0$, $c_0 = 1$, $c_1 = b = 4m+1$ $\ell_0 = 2q_0 = 2a$. By using Lemma 2.5 and Lemma 2.6 $w_d = [a, \overline{\ell_1, \ell_2, \ell_3, \ell_3, \ell_2, \ell_1, 2a}]$ for $\ell_1 = \ell_6$, $\ell_2 = \ell_5$, $\ell_3 = \ell_4$ and $\ell_i \geq 1 \ni \forall i = 1,2,3$.
If we use the equality $2a - r_i = c_i\ell_i + r_{i+1}$ for $i \geq 0$ in Lemma 2.5 then we write $2a = (4m+1)\ell_1 + r_2$. Therefore $(4m+1)\ell_1 + r_2 \equiv 0(mod2)$ and $r_2 = 2r - \ell_1$ hold for the convenient integer $r \geq 0$. If we consider these equalities then $a = 2m\ell_1 + r$ holds, where a is an odd number and it is clear that r should be an odd number. Furthermore we obtain $c_2 = c_0 + (r_2 - r_1)\ell_1 = 1 + r_2\ell_1$ from the equality $c_{i+1} = c_{i-1} + (r_{i+1} - r_i)\ell_i \quad (i \geq 0)$. Therefore if we use this equality and $2a - r_2 = c_2\ell_2 + r_3$ then we obtain $2a = (1 + r_2\ell_1)\ell_2 + r_2 + r_3$.

Since $2a = (4m+1)\ell_1 + r_2$ and $2a = (1 + r_2\ell_1)\ell_2 + r_2 + r_3$ then we have

$(4m+1)\ell_1 = (1 + r_2\ell_1)\ell_2 + r_3$. If we get $(mod\ell_1)$ then $\ell_2 + r_3 \equiv 0(mod\ell_1)$ and $r_3 = \ell_1 t - \ell_2$ hold for the convenient integer $t \geq 0$. If $r_3 = \ell_1 t - \ell_2$ then it is easily seen that $4m = t + 2r\ell_2 - \ell_1\ell_2 - 1$. Moreover if we take $A = \ell_1\ell_2 + 1$ then $t - A = 4m - 2r\ell_2$ holds and if $t < A$ then there is an integer $s < 0$ such that $t - A = 2s$. (if $t > A$ then look in [2].) If it is taken $s < 0$, $s = -e$ and $e > 0$ then it is obtained $2e = A - t = 2r\ell_2 - 4m$, $e = r\ell_2 - 2m$ and $2m = r\ell_2 - e$ By putting $2m = r\ell_2 - e$ in $a = 2m\ell_1 + r$ then we have $a = (r\ell_2 - e)\ell_1 + r = Ar - \ell_1 e$.
Since $c_3 = c_1 + (r_3 - r_2)\ell_2 = 4m + 1 + (r_3 - r_2)\ell_2$
$r_2 = 2r - \ell_1$ ve $r_3 = \ell_1 t - \ell_2$ from the Lemma 2.5 then $c_3 = At - \ell_2^2$ holds. If we put the value c_3 in $2a = c_3\ell_3 + r_3 + r_4$ then we have $2a = (At - \ell_2^2)\ell_3 + r_3 + r_4$. We know that $c_3 = c_4$ therefore if we take the equalities $At - \ell_2^2 = c_2 + (r_4 - r_3)\ell_3$, $c_2 = 1 + r_2\ell_1$, $r_2 = 2r - \ell_1$,$r_3 = \ell_1 t - \ell_2$ ve $r_4 = (2r - \ell_1 - t\ell_3)A + \ell_2(\ell_2\ell_3 + 1)$ then we obtain $At - \ell_2^2 = r_2 + (r_4 - r_3)\ell_3$ $= 1 + r_2\ell_1 + r_4\ell_3 - r_3\ell_3 = 1 + (2r - \ell_1)\ell_1 + [(2r - \ell_1 - t\ell_3)A + \ell_2(\ell_2\ell_3 + 1)]\ell_3 - (\ell_1 t - \ell_2)\ell_3 = (1 + \ell_2\ell_3)^2 + 2r(\ell_1 + A\ell_3) - t\ell_3(\ell_1 + A\ell_3) - \ell_1(\ell_1 + A\ell_3)$.

If it is taken $\ell_1 + A\ell_3 = B$, $t = A - 2e$ and $1 + \ell_2\ell_3 = C$ then $A^2 + B^2 - C^2 - \ell_2^2 = 2rB + 2e(A + B\ell_3)$ holds from $At - \ell_2^2 = C^2 + B(2r - t\ell_3 - \ell_1)$ and $t = A - 2e$.

Now we will show that the integers r and e are uniquely determined with the inequalities $a = Ar - \ell_1 e$ and $A^2 + B^2 - C^2 - \ell_2^2 = 2rB + 2e(A + B\ell_3)$. If we assume that the integers r and s is not determined uniquely then we have $A^2 + B^2 = 0$ which is a contradiction because of $A, B > 0$. Therefore, the integers r and e are uniquely determined.

Then, we can calculate , $Q_{i+1} = q_{i+1}Q_i + Q_{i-1}$, $(i \geq 0)$ where $Q_{-1} = 0$ $Q_0 = 1$ as follows
$Q_-1 = 0$
$Q_0 = 1$
$Q_1 = \ell_1$
$Q_2 = A$
$Q_3 = B$
$Q_4 = \ell_3 B + A$
$Q_5 = C(A\ell_3 + \ell_1) + A\ell_2 = BC + A\ell_2$ and $Q_6 = A(\ell_1\ell_2 + 1) + \ell_3(A\ell_3 + \ell_1) + BC\ell_1 = A^2 + B^2$ hold by Lemma 2.4, we obtain that

$$T_d = 2[a(A^2+B^2) + BC + A\ell_2] \text{ and } U_d = 2(A^2+B^2).$$

4 An Application

In this section, we will give numerical example by using the algorithm of our Theorem 3.1. and Theorem 3.2. This provides us to determine ω_d and ε_d rapidly.

As an application of Theorem 3.1. we can practically determine the continued fraction expansion of ω_d where $d = 113 = 10^2 + 13 \equiv 1 mod(4)$ for $a = 10 \equiv 2 mod(4) \equiv 0 mod(2)$ and $b = 13 \equiv 5 mod(8)$. We easily see that $\ell_1 = 1$, $c_0 = 2$, $r_0 = r_1 = 1$, $c_1 = a+4m+2 = 16$, $r_2 = 3$, for $a = 4m+3+r_2$ and $c_2 = 4$ for $c_2 = r_2+1$. Moreover

$2a = (r_2+1)\ell_2 + r_2 + r_3 \Rightarrow r_3 = 1$ holds for $\ell_2 = 4$, $r_2 = 3$ $a = 10$,

$c_3 = c_1 + (r_3 - r_2)\ell_2 \Rightarrow c_3 = 8$ and $8m+4 = (r_2 - r_3)\ell_2 + (r_4 - r_3)\ell_3 \Rightarrow r_4 = 3$ hold for $\ell_1 = 1$, $\ell_2 = 4$, $\ell_3 = 2$, $m = 1$, $r_2 = 3$, $r_3 = 1$.

Hence ω_d can be determined rapidly as follows;

$$\omega_d = [5, \overline{1,4,2,2,4,1,9}]$$

Moreover, the fundamental unit of $Q(\sqrt{113})$ is easily determined as

$$\varepsilon_d = \frac{1552+146\sqrt{113}}{2}$$

since $A = 11$, $B = 5$, $C = 1$, $D = 121$, $E = 3$, $F = 88$ and $G = 112$.

In the same way, we can give an application for theorem 3.2 by using the algorithm has been expressed in this theorem and so if we take

$d = 538 = 23^2 + 9 \equiv 2 mod(4)$ for $a = 23 \equiv 1 mod(2)$ and $b = 9 \equiv 1 mod(4)$. We can easily get that $\ell_1 = 5$, $c_0 = 1$, $r_0 = r_1 = 0$, $c_1 = b = 9$,$m = 2$, $r_2 = 1$,$r_3 = 3$ $r = 3$, and $c_2 = 6$. Furthermore, we can calculate $c_3 = c_1 + (r_3 - r_2)\ell_2$, $\Rightarrow c_3 = 23$,$r_4 = 55$ $\ell_2 = 7$, $\ell_3 = 1$,$t = 2$, $s = -17$, $e = 17$, $r_4 = 55$.

Hence ω_d can be determined rapidly as follows;

$$\omega_d = [23, \overline{5,7,1,1,7,5,46}]$$

Moreover, the fundamental unit of $Q(\sqrt{538})$ is obtained that

$$\varepsilon_d = \frac{138102+5954\sqrt{538}}{2}$$

since $A = 36$, $B = 41$, $C = 8$, $D = 3053$, $E = 7192$.

Acknowledgement.

This project was supported by the research scientific project of Kirklareli Universitesi with the number KLUBAP-044.

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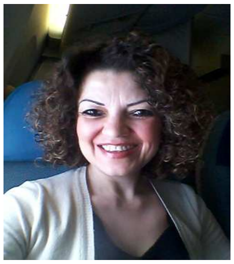

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