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Comparison of the Estimation Efficiency of Regression Parameters Using the Bayesian Method and the Quantile Function

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Abstract: There are several classical as well as modern methods to predict model parameters. The modern methods include the Bayesian method and the quantile function method, which are both used in this study to estimate Multiple Linear Regression (MLR) parameters. Unlike the classical methods, the Bayesian method is based on the assumption that the model parameters are variable, rather than fixed, estimating the model parameters by the integration of prior information (prior distribution) with the sample information (posterior distribution) of the phenomenon under study. By contrast, the quantile function method aims to construct the error probability function using least squares and the inverse distribution function. The study investigated the efficiency of each of them, and found that the quantile function is more efficient than the Bayesian method, as the values of Theil's Coefficient U and least squares of both methods came to be $U = 0.052$ and $\sum e_i^2 = 0.011$, compared to $U = 0.556$ and $\sum e_i^2 = 1.162$, respectively.

Keywords: Bayesian Method, Prior Distribution, Posterior Distribution, Quantile Function, Prediction Model

1 Introduction

Multiple linear regression (MLR) analysis is an important statistical method for predicting the values of a *dependent variable* using the values of a set of *independent variables*. The importance of MLR stems from the estimation efficiency of its parameters, and the extent of data being free of technical problems, such as collinearity between independent variables or autocorrelation between residuals. There are many estimation methods of the model parameters. The classical methods include the least squares method and the maximum likelihood method. The modern ones include the Bayesian method and the quantile function (known also as Quantile Regression). The modern methods adopt assumptions that are different from the assumptions known in the classical methods. For example, prediction by means of the Bayesian method assumes that the parameters βs are random variables that have a probability distribution $g(\beta)$, which reflects previous information and experiences about the parameter β, and describes the reliability of the possible values of this parameter before obtaining the sample (i.e. prior distribution) [9]. In the same vein, quantile regression is based in the construction of the estimator of the dependent variable on the residues e_i using inverse distribution function $F^{-1}(p)$ whereby x_p represents the quantile value of the variable x at an aggregate probability of p, $p = P_r(X \leq x_p)$ [10].

This study attempts at a theoretical investigation of the Bayesian method and the quantile regression method to estimate MLR parameters $\underline{Y} = X\underline{\beta} + \underline{U}$. It turns out that using the Bayesian method for the dispersed prior distribution when there is no prior information – non-informative prior density function – gives estimates that do not differ from the least squares method estimates, given that, for estimator derivation, it depends on the maximum likelihood function. The sample-based prior density function with information on the parameters required to be estimated and using appropriate conjugate prior distribution shows that the prior probability distribution group goes hand in hand with the sample distribution probability, if the posterior distribution belongs to the same family of the prior distribution [9].

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Quantile regression relies on the least squares method to estimate model parameters from the study sample and calculate the value of residual $e_i = y_i - \hat{y}_i$. Then, it constructs the quantile function that gives an estimate of the dependent variable Y, where: $\underline{\hat{y}}_{mi} = X\underline{\hat{\beta}} + \hat{\sigma}\underline{N}(p_i^*)$ [2].

These two models have been discussed by many, [8] have stressed on the efficiency of the Bayesian method over other methods. Similarly, [1] emphasizes that the Bayesian method works better than the maximum likelihood method when calibrating research items with a high capacity sample and a low capacity sample. This means that the Bayesian method offers more accurate estimations than the maximum likelihood method. [2] has concluded that the use of quantile regression reduces the average sum of error squares. [11] maintains that the quantile function method has the capacity to estimate the conditional variable parameters of the model on some other parameters, while the classical method studies the conditional average of the parameter as a constant.

Broadly speaking, the main goal of this study is to estimate MLR model parameters using both the Bayesian method and the quantile function method, comparing the efficiency of each method. This study uses various ways to measure the efficiency of the estimated models, including Theils coefficient and the sum of squares.

2 Materials and Methodologies

There are several techniques that can be used for statistical inference, such as the Bayesian method, the maximum likelihood method, the moment generating function, the least squares, etc.

The Bayesian method is a modern method. When we look at the differences between the classical methods and the modern methods, we find that the former assumes that the parameter β has a unique value which can be estimated from the probability distribution $f(x,\beta)$. However, the Bayesian method considers the parameter β as a random variable that has a probability distribution $g(\beta)$ reflecting the previous information about the parameter β and describing the credibility of the possible values of such parameter or our previous experience about such parameter before obtaining the sample responses.

This is what is known as prior distribution, which describes our previous information and experiences about a parameter β [9].

To make estimation, the Bayesian method uses prior information about the unknown parameters $\beta = \beta_1, \beta_2, \ldots, \beta_n$, which need to be estimated, considering that these parameters are random variables rather than constants. Such information is described as a primary probability function , and can be seen as a function representing all information and experiences about the parameters to be assessed and reached in advance through the analysis of these parameters. The prior information is added to the viewed sample information. Besides, estimates of the Bayesian method are unbiased because the more samples taken, the more the average value of the sample estimates will tend then to the value of the unknown parameter [6].

In unbiased estimators, the best estimate is the one that has a minimum variance. Unbiased estimators are used if they have the lowest square error average compared with other unbiased estimators [9]. The Bayesian method has uncertainty about parameter value, and so it uses probability distribution, where a probability density function is assigned to the probable and improbable values of the parameter. In the classical method, the parameter is constant in repeated samples and therefore cannot be assigned a probability density function. In other words, it turns into trivial distribution, more or less, where the probability is equal to one for the real value of the parameter and zero otherwise.

3 MLR Model Estimation Using Bayesian Method

This section presents different ways of the Bayesian method to estimate MLR model parameters by adopting the following primary density probability functions: 1) the non-informative initial density function (limited information), 2) the informative primary density function (rich information), 3) the prior sample-based density function, and 4) initial normal conjugate density function. This study relies on dispersed prior distribution in case of absence of prior information the non-informative primary density function and uses the primary density function for the parameters to be estimated based on an initial sample. Finally, it uses conjugate prior distribution [3].

3.1 Use of Dispersed Prior Distribution

Dispersed Prior distribution is used when there is no information about the parameters to be estimated (compared to uniform distribution, which is used in cased of lack of information), and therefore the prior distribution will be as follows:

$$g_1(\beta,\sigma) \propto \frac{1}{\sigma}. \tag{1}$$

The prior distribution is called incomplete distribution, because the total possibilities within their respective fields are not equal to 1, and then becomes Jefferys Prior (named after Sir Harold Jeffreys), and the maximum likelihood function becomes as follows:

$$L(\beta,\sigma|y) \propto \sigma^{-T}\exp\left\{-\frac{1}{2\sigma^2}\left[(T-k)\hat{\sigma}^2+(\underline{\beta}-\underline{b}_o)'X'X(\underline{\beta}-\underline{b}_o)\right]\right\}. \tag{2}$$

By merging the maximum likelihood function with the prior distribution function (1), the posterior distribution becomes as follows:

$$g_2(\beta,\sigma|y) \propto \sigma^{-(T+1)}\exp\left\{-\frac{1}{2\sigma^2}\left[(T-k)\hat{\sigma}^2+(\underline{\beta}-\underline{b}_o)'X'X(\underline{\beta}-\underline{b}_o)\right]\right\}. \tag{3}$$

The integration of the prior distribution leads to the posterior distribution, where $Z = X'X$ By merging the maximum likelihood function with the prior distribution function (1), the posterior distribution becomes as follows:

$$g_2(\beta|y) \propto \left[1+\frac{1}{(T-k)}(\underline{\beta}-\underline{b}_o)'\frac{Z}{\hat{\sigma}^2}(\underline{\beta}-\underline{b}_o)\right]^{\frac{T}{2}}. \tag{4}$$

Relationship (4) above shows that that the posterior marginal density function $g_2(\beta|y)$ can be considered as a weighted average of the conditional density function $g_2(\beta|\sigma,y)$ at the weights given by the marginal density function $g_2(\sigma|y)$ to σ, where integration of (4) is obtained using the Gamma function properties [4].

3.2 Use of Conjugate Prior Distribution

If we have the following regression model: $\underline{Y} = X\underline{\beta}+\underline{\varepsilon}$
Where:
$\underline{Y}$: Vector of observed random variables, which is a random normal vector of size $(T\times 1)$.
X: Matrix $(T\times K)$ of known values of explanatory variables (observed).
$\underline{\beta}$: Vector $(K\times 1)$ of parameters of the model, and
$\underline{\varepsilon}$: Random error vector $(T\times 1)$, which is a normal but non-observable distribution:

$$\underline{Y} \sim N(X\underline{\beta},\sigma^2 I_T) \qquad \& \qquad \underline{\varepsilon} \sim N(0,\sigma^2 I_T).$$

Therefore, the maximum likelihood function representing the sample information that follow the normal distribution takes the following form:

$$L(\beta,\sigma|y,x) = \frac{1}{(2\pi)^{\frac{T}{2}}\sigma^T}\exp\left\{-\frac{1}{2\sigma^2}\left[(\underline{Y}-X\underline{\beta})'(\underline{Y}-X\underline{\beta})\right]\right\}. \tag{5}$$

The function can be rewritten for easy analysis, taking into account that:

$$(\underline{Y}-X\underline{\beta})'(\underline{Y}-X\underline{\beta}) = \underline{Y}'\underline{Y}-\underline{Y}'X\underline{\beta}-\underline{\beta}'X'\underline{Y}+\underline{\beta}'X'X\underline{\beta}$$

$$(\underline{Y}-X\underline{\beta})'(\underline{Y}-X\underline{\beta}) = \underline{Y}'\underline{Y}-2\underline{\beta}'X'\underline{Y}+\underline{\beta}'X'X\underline{\beta}. \tag{6}$$

Assuming that $\underline{b}_o = Z^{-1}X'X\underline{Y}$ and $Z = X'X$.
By simplifying equation (6) to

$$(\underline{Y}-X\underline{\beta})'(\underline{Y}-X\underline{\beta}) = \left[SSE(\underline{b}_o)+(\underline{\beta}-\underline{b}_o)'Z(\underline{\beta}-\underline{b}_o)\right]. \tag{7}$$

The maximum likelihood function becomes as follows:

$$L(\beta,\sigma|y,x) = \frac{1}{(2\pi)^{\frac{T}{2}}\sigma^T}\exp\left\{-\frac{1}{2\sigma^2}\left[SSE(\underline{b}_o)+(\underline{\beta}-\underline{b}_o)'Z(\underline{\beta}-\underline{b}_o)\right]\right\}.$$

By deleting the constants and the $SSE(\underline{b}_o)$, which includes the parameters β and the information σ^2 in order to simplify the equation to become:

$$L(\beta) \propto \exp\left\{-\frac{1}{2\sigma^2}\left[(\underline{\beta}-\underline{b}_o)'Z(\underline{\beta}-\underline{b}_o)\right]\right\}. \tag{8}$$

If normal distribution is used as a prior distribution of β, as follows:

$$g_1(\beta) \propto \exp\left\{-\frac{1}{2\sigma^2}\left[(\underline{\beta}-\underline{\beta}_o)'\Omega^{-1}(\underline{\beta}-\underline{\beta}_o)\right]\right\} \tag{9}$$

Ω represents precision matrix, where $\Omega = \frac{X'X}{S^2}$, by applying Bayes theory and merging prior distribution function with the maximum likelihood function in (8) and (9) to obtain posterior distribution, it is found that $g_2(\beta|y) \propto g_1(\beta)\cdot L(\beta)$. Therefore

$$g_2(\beta|y) \propto \exp\left\{-\frac{1}{2\sigma^2}\left[(\underline{\beta}-\underline{b}_o)'Z(\underline{\beta}-\underline{b}_o)+(\underline{\beta}-\underline{\beta}_o)'\Omega^{-1}(\underline{\beta}-\underline{\beta}_o)\right]\right\} \tag{10}$$

Assuming that $c^* = (\underline{b}_o-\underline{\beta}_o)'Z\Omega^*\Omega^{-1}(\underline{b}_o+\underline{\beta}_o)$, $\underline{b}^* = \Omega^*(Z\underline{b}_o+\Omega^{-1}\underline{\beta}_o)$, $\Omega^* = (Z+\Omega^{-1})^{-1}$. The equation (10), representing the posterior distribution, can be reformulated as follows:

$$g_2(\beta|y) \propto \exp\left\{-\frac{1}{2\sigma^2}\left[(\underline{\beta}-\underline{b}^*)'\Omega^{*-1}(\underline{\beta}-\underline{b}^*)+c^*\right]\right\}. \tag{11}$$

This equation represents the posterior distribution, which is a normal distribution of a mean $\underline{b}^*$, and a variance and covariance matrix $\sigma^2\Omega^*$, where $\beta \sim N(\underline{b}^*, \sigma^2\Omega^*)$.

Based on the above, we find that the prior distribution and posterior distribution both follow normal distribution. That is why the prior distribution is called the conjugate prior distribution, confirming that the prior probability distribution group is associated with the probability distribution for the withdrawn sample if posterior distribution belongs to the same family of the prior distribution. The posterior distribution is used in the estimation of the parameters of the regression model. Estimators are given to the parameters of the regression model β, as follows:

$$\Omega^* = (Z+\Omega^{-1})^{-1}, \qquad \underline{b}^*_{\text{bayes}} = \Omega^*(Z\underline{b}_o+\Omega^{-1}\underline{\beta}_o).$$

Therefore

$$\underline{b}^*_{\text{bayes}} = (Z+\Omega^{-1})^{-1}Z\underline{b}_o+(Z+\Omega^{-1})^{-1}\Omega^{-1}\underline{\beta}_o. \tag{12}$$

$$\underline{b}^*_{\text{bayes}} = (H_1\underline{b}_o)+H_2\underline{\beta}_o. \tag{13}$$

where: $H_1 = (Z+\Omega^{-1})^{-1}Z \quad H_2 = (Z+\Omega^{-1})^{-1}\Omega^{-1}$

Equation (13) represents the Bayesian estimate for the regression parameters based on the estimate of the maximum likelihood weighted by the weights matrix and the prior distribution average [4].

4 Quantile function & Quantile Regression

It is known that the cumulative distribution function – symbolized by $F(x)$ – represents the probability that the variable X is less than or equal to a given value x, i.e., $F(x) = P(X \leq x)$. If $F(x) = p$, the value of $F(x)$ indicates the cumulative probability p [10].

Quantile function is considered to be one of the most important statistical concepts associated with the cumulative distribution function (CDF). It serves as an approach to a modern prediction method known as Quantile Regression, which depends on the Ordinary Least Squares method to obtain the least value for the sum of squares compared to other methods.

4.1 Quantile Function (QF)

Quantile Function – which is symbolized by $Q(p)$ – is the inverse of the accumulated probability value p in the cumulative distribution function, where $Q(p) = F^{-1}(p)$ and $Q(x) = F^{-1}(x)$. This shows that these two functions are mutually inverse, provided that both always increase simultaneously during the extent of x. This is a condition of the cumulative function, where x_p represents the p-quantile value of the accumulated probability$P_r(X \leq x_p) = p$. As shown in Figure 1 [10].

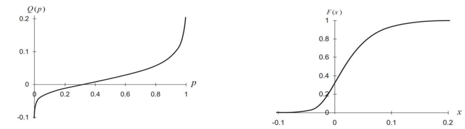

Fig. 1: Curves of probability distribution function $F(x)$ and quantile function $Q(p)$.

4.2 Quantile Function Modeling

The idea of Quantile Functions Modeling is derived from the concept of quantile function $Q(p) = F^{-1}(p)$, which represents the inverse cumulative function of any probability function. To obtain the middle value of the distribution, we can put $p = 0.5$, which represents $0.5 = \int_{-\infty}^{M} f(x)dx$, where M is, for example, the Median. If we have an exponential distribution probability function $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$, the cumulative distribution function of this function is: $F(x) = 1 - e^{-\lambda x}$, and the inverse function of the cumulative function is $F^{-1}(p) = \frac{-1}{\lambda}\ln(1-p)$. The quantile function becomes as follows: $Q(p) = -\eta\ln(1-p)$, where $\eta = 1/\lambda$. To obtain the mean middle value of the distribution, we can put $p = 0.5$, which represents the median, which becomes: $M = -\eta\ln(1-0.5) = -\eta\ln(0.5) = \eta\ln(2)$, assuming that $Q(p) = \lambda - \eta\ln(1-p)$ and that $\lambda = 0$, $\eta = 1$, so the error term becomes $S(p) = -\ln(1-p)$ at median of $M = \ln(2)$, where η scale for the measurement parameter [10].

4.3 Approaches to Regression Modeling

The MLR model comes in the form $\underline{Y} = X\underline{\beta} + \underline{U}$. Using the least squares method in the estimation the models parameters (getting as least sum of error squares as possible):

$$\hat{\beta} = \frac{\sum(x_i y_i)}{\sum x_i^2}; \quad Min\sum e_i^2 = Min\sum(y_i - \hat{y}_i)^2 = Min\sum(y_i - \hat{\beta}x_i)^2 \tag{14}$$

by deriving the function $\sum e_i^2$ in terms of $\hat{\beta}$, and having derivation equal to zero, we can obtain the estimators of the parameters of simple or multiple linear regression or multiple model [7]. By contrast, the quantile regression model becomes

$$\underline{Q}_y(p|x) = X\underline{\beta} + \eta\underline{S}(p). \tag{15}$$

It is called the quantile regression function of Y on X, and sometimes the quantile conditional function. It is possible to obtain $\hat{\beta}$ and $\hat{\sigma}$ in Equation (14) of the sample data or previous experience. The previous equation includes both the model components X, the error $\underline{S}(p)$, the parameters β.

This method provides us with the possibility of estimating regression parameters as well as probability distribution parameters, so that the model in hand becomes as follows [10]:

$$\underline{Q}_y(p|x) = X\underline{\beta} + \sigma\underline{N}(p) \tag{16}$$

where $\underline{\hat{Y}} = X\underline{\hat{\beta}}$. We can find the estimated values $\underline{\hat{Y}}$ and then find the error values by the following formula:

$$\underline{e} = \underline{Y} - X\underline{\beta}. \tag{17}$$

After that, we sort the errors down by rank or rank function $r_i = RANK(z_i; z_1, z_3, \ldots, z_n; n)$, which is the figure given to any value within the group ranked from smallest to largest or vice versa [10]. Then, we find the probability value for each rank using the inverse Beta function, as follows:

$$P^* = BETAINV(0.5, r_i, n+1-r_i). \tag{18}$$

After that, the probability values for errors in (17) are converted to a standard normal distribution function using the following formula:

$$N(p_i^*) = (p_i^*; 0, 1). \tag{19}$$

Excel program can be used to obtain the results of formulas (17) and (18), which lead to [2]:

$$\underline{\hat{Y}}_{Mi} = X\underline{\hat{\beta}} + \hat{\sigma}\underline{N}(p_i^*) \tag{20}$$

where $\underline{\hat{Y}}_{Mi}$ refers to the vector of new expected value of the dependent variable;
$X\underline{\hat{\beta}}$: The multiplication value/result of the matrix of the independent variables (information matrix) and the vector of the estimated parameters in the least squares;
$\hat{\sigma}$: The standard deviation of the original data (scale parameter); $\underline{N}(p_i^*)$: The vector of the quantile function of the ordinary standard distribution of errors.

Using operations on matrices, the model parameters of the quantile function model can be obtained in terms of

$$\underline{\hat{\beta}}_q = (X'X)^{-1}X'\underline{\hat{Y}}. \tag{21}$$

4.4 Scales of Estimation Model Efficiency

Theil's Coefficient can be used to study the estimation efficiency of the prediction model. It is a statistical measure which can determine the prediction efficiency of any model, revealing the difference between the original values and the predictive values by the proposed model, and can be given through the following relationship:

$$U = \sqrt{\frac{\sum_{i=1}^{n-1}(P_i - A_i)^2}{\sum_{i=1}^{n-1}A_i^2}} \tag{22}$$

where: $P_i = (\hat{Y}_{i+1} - Y_i)/Y_i$ and $A_i = (Y_{i+1} - Y_i)/Y_i$ [5].

The closer to zero the value of the Theils coefficient the more efficient the model is. And the residual sum of squares is used to help you decide if a statistical model is a good fit for your data. It measures the overall difference between your data and the values predicted by your estimation model (a residual is a measure of the distance from a data point to a regression line). $\sum e_i^2 = \sum(Y_i - \hat{Y}_i)^2$.

5 Results and Discussion

This section displays the results of the data analysis of the study period, representing the data of the electricity sector in the Kingdom of Saudi Arabia for the period 2000–2016, [12] as shown in Table (1). Y is the dependent variable and represents the number of employees in the electricity sector, and independent variables X_1, X_2, X_3, X_4, X_5, which stand for Energy consumption in the industrial sector, total energy consumption, energy produced, number of subscribers, average subscribers share respectively. Excel program is used to estimate the parameters of the model by least squares, and MATLAB is also used to find the following matrices:

Table 1: Electricity sector data for the period (2016–2000)*

Year	Y	X_1	X_2	X_3	X_4	X_5
	(1000)	MWH 10^6	MWH 10^6	MWH 10^6	10^6	10^4
2000	30.2	27.7	114.2	154.5	3.6	31.5
2001	29.9	28.2	123	164.6	3.8	32.4
2002	30	29.3	128.6	150.6	4	31.9
2003	29.5	33.4	142.2	162.1	4.3	33.5
2004	29.1	32	144.4	166.7	4.5	32.4
2005	29	33.8	153.3	180.1	4.7	32.4
2006	28.7	32.6	163.2	187.6	5	32.9
2007	28.5	30.6	169.8	195	5.2	32.8
2008	28.6	32.4	181.1	207.4	5.4	33.4
2009	28.5	34.7	193.5	213.2	5.7	33.9
2010	29	38.6	212.3	214.1	6	35.4
2011	28.7	42.1	219.7	218	6.3	34.6
2012	29	41.7	240.3	235.2	6.7	35.7
2013	28.9	51.1	256.7	238.2	7.1	35.9
2014	29	51.5	274.5	243.2	7.6	36.1
2015	28.5	45.1	286.1	252.7	8.1	35.4
2016	27	46.5	287.4	246.5	8.6	33.5

5.1 Estimation of Parameters of MLR Model Using Bayesian Method

The vector of parameters estimated for prior distribution is as follows:

$$\underline{\beta}'_{\circ} = (68.4015 \quad -0.3331 \quad 0.6154 \quad -0.0598 \quad -9.4915 \quad -1.8233)'$$

for the data of the period (1986–1999) on basis of least squares, with $S^2 = 0.151$.

Precision Matrix: $\Omega = \frac{X'X}{0.151}$, where the matrix X represents to the data of the period (1986–1999), and: exp(5)

$$\Omega = \frac{X'X}{0.151} = 1.0\exp{+5}\begin{pmatrix} 0.0010 & 0.0191 & 0.0747 & 0.0866 & 0.0027 & 0.0272 \\ 0.0191 & 0.3903 & 1.5337 & 1.7706 & 0.0542 & 0.5345 \\ 0.0747 & 1.5337 & 6.0396 & 6.9716 & 0.2126 & 2.0952 \\ 0.0866 & 1.7706 & 6.9716 & 8.0558 & 0.2457 & 2.4252 \\ 0.0027 & 0.0542 & 0.2126 & 0.2457 & 0.0075 & 0.0749 \\ 0.0272 & 0.5345 & 2.0952 & 2.4252 & 0.0749 & 0.7490 \end{pmatrix}$$

Independent variables matrix L for the study period shown in Table (1). So, $Z = L'L$:

$$Z = L'L = \exp(+5)\begin{pmatrix} 0.0002 & 0.0063 & 0.0329 & 0.0343 & 0.0010 & 0.0057 \\ 0.0063 & 0.2442 & 1.2904 & 1.3118 & 0.0376 & 0.2147 \\ 0.0329 & 1.2904 & 6.9177 & 6.9544 & 0.2012 & 1.1222 \\ 0.0343 & 1.3118 & 6.9544 & 7.1102 & 0.2031 & 1.1645 \\ 0.0010 & 0.0376 & 0.2012 & 0.2031 & 0.0059 & 0.0329 \\ 0.0057 & 0.2147 & 1.1222 & 1.1645 & 0.0329 & 0.1940 \end{pmatrix}$$

Therefore, it is possible to calculate: $H_1 = (Z+\Omega^{-1})^{-1}Z$ and $H_2 = (Z+\Omega^{-1})^{-1}\Omega^{-1}$ as follows:

$$H_1 = (Z+\Omega^{-1})^{-1}Z = \begin{pmatrix} 0.0094 & 0.0074 & -0.0098 & 0.0005 & 0.0520 & 0.0511 \\ -0.0029 & 1.0020 & -0.0000 & 0.0001 & -0.0283 & 0.0014 \\ -0.0007 & -0.0020 & 1.0001 & -0.0001 & 0.0268 & -0.0011 \\ -0.0004 & 0.0002 & 0.0000 & 1.0000 & -0.0031 & 0.0002 \\ 0.0231 & 0.0676 & -0.0034 & 0.0036 & 0.1477 & 0.0339 \\ 0.0352 & -0.0030 & 0.0004 & -0.0001 & 0.0380 & 0.9965 \end{pmatrix}$$

$$H_2 = (Z+\Omega^{-1})^{-1}\Omega^{-1} = \begin{pmatrix} 0.9906 & -0.0074 & 0.0098 & -0.0005 & -0.0520 & -0.0511 \\ 0.0029 & -0.0020 & 0.0000 & -0.0001 & 0.0283 & -0.0014 \\ 0.0007 & 0.0020 & -0.0001 & 0.0001 & -0.0268 & 0.0011 \\ 0.0004 & -0.0002 & -0.0000 & 0.0000 & 0.0031 & -0.0002 \\ -0.0231 & -0.0676 & 0.0034 & -0.0036 & 0.8523 & -0.0339 \\ -0.0352 & 0.0030 & -0.0004 & 0.0001 & -0.0380 & 0.0035 \end{pmatrix}$$

The estimated prior distribution parameters are:

$$\underline{\beta}'_{\circ} = (68.4015 \quad -0.3331 \quad 0.6154 \quad -0.0598 \quad -9.4915 \quad -1.8233)'$$

and the parameters estimated from the sample are:

$$\underline{b}'_{\circ} = (53.9688 \quad -0.0446 \quad 0.1755 \quad -0.0300 \quad -5.7720 \quad -0.5475)'$$

Using Equation No. (13),

$$\underline{b}^{*\prime}_{\text{bayes}} = H_1\underline{b}_{\circ} + H_2\underline{\beta}_{\circ},$$

the Bayes estimator for MLR model parameters is:

$$\underline{b}^{*\prime}_{\text{bayes}} = (68.5315 \quad -0.1049 \quad 0.2836 \quad -0.0357 \quad -9.2109 \quad -0.9197)'$$

and the estimated regression model is:

$$\hat{Y}_{\text{bayes}} = 68.53 - 0.11X_1 + 0.28X_2 - 0.04X_3 - 9.21X_4 - 0.92X_5$$

5.2 Estimation of Parameters of MLR Model Using Quantile Analysis Method

The estimate of the dependent variable can be calculated by using quantile analysis, where

$$\underline{\hat{Y}}_{Mi} = X\underline{\hat{\beta}} + \hat{\sigma}\underline{N}(p_i^*),$$

using Excel and MATLAB (as shown in Table (2)).

Table 2: Estimates of dependent variable using quantile analysis*

Y	$\hat{Y}_{OLS}$	e_{i-OLS}	Rank	P_i^*	$N(P_i^*)$	$\hat{Y}_{Qu}$
30.2	30.11503	0.08497	15	0.845782	1.018511	30.2294
29.9	29.68698	0.21302	17	0.960047	1.751228	29.8836
30	30.16007	-0.16007	1	0.039953	-1.75123	29.9635
29.5	29.41141	0.08859	16	0.903218	1.30011	29.5574
29.1	29.1698	-0.0698	5	0.2694	-0.61463	29.1008
29	29.09507	-0.09507	3	0.154218	-1.01851	28.9807
28.7	28.65569	0.04431	12	0.672962	0.448107	28.7060
28.5	28.58154	-0.08154	4	0.211785	-0.80024	28.4917
28.6	28.62951	-0.02951	8	0.442342	-0.14503	28.6132
28.5	28.52378	-0.02378	9	0.5	0	28.5238
29	29.06939	-0.06939	6	0.327038	-0.44811	29.0191
28.7	28.80139	-0.10139	2	0.096782	-1.30011	28.6554
29	29.00748	-0.00748	10	0.557658	0.145033	29.0238
28.9	28.95814	-0.05814	7	0.384687	-0.29319	28.9252
29	28.9187	0.0813	14	0.788215	0.800243	29.0086
28.5	28.45219	0.04781	13	0.7306	0.614627	28.5212
27	26.95815	0.04185	11	0.615313	0.293193	26.9911

and estimated model parameters using quantile analysis where $\underline{\beta}^{*\prime}_{qu} = (X'X)^{-1}X'\underline{\hat{Y}}$

$$\underline{\beta}^{*\prime}_{qu} = (53.1649 \quad -0.0456 \quad 0.1735 \quad -0.0303 \quad -5.7055 \quad -0.5207)'$$

The estimated regression model is

$$\hat{Y}_{quantile} = 53.17 - 0.05X_1 + 0.17X_2 - 0.03X_3 - 5.71X_4 - 0.52X_5.$$

5.3 Comparison of the Estimation Efficiency of the Two Models

The predictive efficiency of the two models estimated by means of the Bayesian method and the quantile analysis method can be compared on basis of sum of squares $\sum e_i^2$ and the Theils coefficient. Based on Tables (3) and (4), it is possible to create a comparison table for the efficiency of both Bayesian method and the quantile analysis, and calculate Theils coefficient:

Table 3: Error squares for both methods, the Bayesian and the quantile analysis*

Y	$\hat{Y}_{bayes}$	$\hat{Y}_{Qu}$	$e^2_{i\,bayes}$	$e_{i\,quantile}$
30.2	30.3675	30.2294	0.02804	0.000864
29.9	29.7802	29.8836	0.014352	0.000269
30	30.3704	29.9635	0.137226	0.001336
29.5	29.152	29.5574	0.121125	0.003293
29.1	28.928	29.1008	0.029577	6.25E-07
29	28.9427	28.9807	0.003286	0.000372
28.7	28.3853	28.706	0.099017	3.6E-05
28.5	28.4525	28.4917	0.002256	6.91E-05
28.6	28.6317	28.6132	0.001004	0.000175
28.5	28.4769	28.5238	0.000535	0.000565
29	29.2245	29.0191	0.050396	0.000364
28.7	28.7892	28.6554	0.007964	0.001988
29	29.3633	29.0238	0.13198	0.000565
28.9	29.0529	28.9252	0.023369	0.000636
29	29.0911	29.0086	0.008299	7.31E-05
28.5	28.7514	28.5212	0.063207	0.000449
27	26.3366	26.9911	0.440166	7.98E-05
			1.161797	0.011134

Table 4: Errors squares and Theils coefficient data*

Y	$\hat{Y}_{bayes}$	$\hat{Y}_{Qu}$	Y_{i+1}	$\hat{Y}_{(i+1)bayes}$	$\hat{Y}_{(i+1)quant}$	$(P_i-A_i)^2_{bayes}$	$(P_i-A_i)^2_{quant}$	A_i^2
30.2	30.3675	30.2294	29.9	29.7802	29.8836	1.57362E-05	2.94899E-07	9.86799E-05
29.9	29.7802	29.8836	30	30.3704	29.9635	0.000153495	1.49428E-06	1.11856E-05
30	30.3704	29.9635	29.5	29.152	29.5574	0.000134583	3.65829E-06	0.00027778
29.5	29.152	29.5574	29.1	28.928	29.1008	3.39869E-05	7.1715E-10	0.00018386
29.1	28.928	29.1008	29	28.9427	28.9807	3.87995E-06	4.39419E-07	1.1809E-05
29	28.9427	28.9807	28.7	28.3853	28.706	0.000117737	4.28062E-08	0.00010702
28.7	28.3853	28.706	28.5	28.4525	28.4917	2.7392E-06	8.38375E-08	4.8562E-05
28.5	28.4525	28.4917	28.6	28.6317	28.6132	1.23561E-06	2.15491E-07	1.23115E-05
28.6	28.6317	28.6132	28.5	28.4769	28.5238	6.54062E-07	6.9134E-07	1.22255E-05
28.5	28.4769	28.5238	29	29.2245	29.0191	6.20446E-05	4.48195E-07	0.00030779
29	29.2245	29.0191	28.7	28.7892	28.6554	9.46941E-06	2.36311E-06	0.00010702
28.7	28.7892	28.6554	29	29.3633	29.0238	0.00016023	6.85376E-07	0.00010926
29	29.3633	29.0238	28.9	29.0529	28.9252	2.77874E-05	7.563E-07	1.18906E-05
28.9	29.0529	28.9252	29	29.0911	29.0086	9.93667E-06	8.75259E-08	1.1973E-05
29	29.0911	29.0086	28.5	28.7514	28.5212	7.51569E-05	5.34411E-07	0.00029727
28.5	28.7514	28.5212	27	26.3366	26.9911	0.000541909	9.81778E-08	0.00277008
27	26.3366	26.9911				0	0	0
						0.001350582	1.18942E-05	0.00437870

Table (5) below shows the result of the comparison of the efficiency of the Bayesian analysis and the quantile analysis.

Through the results in Table (5), it can be found that the estimation of the parameters of MLR model by means of the quantile analysis is more efficient than by means of the Bayesian method, as the values of Theil coefficient and the sum

Table 5: Comparison of the efficiency of Bayesian analysis and quantile analysis*

Type of Analysis	Sum square of error $\sum e_i^2$	Theil's Coefficient U
Bayesian	1.161797	0.55538
Quantile	0.011134	0.05212

of squares for the former is much less than those for the latter. This can be interpreted as an indicator of the increased efficiency of the quantile method, which this study recommends for use to predict the values of the variable of economic and other studies.

6 Conclusion:

The quantile function is more efficient than the Bayesian method, as the values of Theil's Coefficient U and least squares of both methods came to be $U = 0.052$ and $\sum e_i^2 = 0.011$, compared to $U = 0.556$ and $\sum e_i^2 = 1.162$, respectively.

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