Certain Properties of Modified Laguerre Polynomials Via Lie Algebra

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Abstract: The aim of present paper is to discuss some operators defined on a Lie algebra for the purpose of deriving some properties of modified Laguerre polynomials.

Keywords: Lie algebra, Modified Laguerre polynomials and Differential equation.

1 Introduction


The Modified Laguerre polynomials (McBride [8]), defined as

\[ f^β_n(x) = \frac{(β)_n}{n!} F_1 \left[ \begin{array}{c} -n ; \\ 1 - β - n, x \end{array} \right] = (-1)^n L_n^{-β-n}(x) \quad (1) \]

Then \( f^β_n(x) \) satisfies the two independent differential recurrence relations

\[ \frac{d}{dx}(f^β_n(x)) = f^β_{n-1}(x) \quad (2) \]

and

\[ x \frac{d}{dx}(f^β_n(x)) = (x+n+β)f^β_n(x) - (n+1)f^β_{n+1}(x) \quad (3) \]

Also (5) and (6) determine the ordinary differential equation

\[ x \frac{d^2}{dx^2}(f^β_n(x)) + (1 - β - n - x) \frac{d}{dx}f^β_n(x) + nf^β_n(x) = 0 \quad (4) \]

2 Main Result

Let End \( V \) be the Lie algebra of endomorphisms of a vector space \( V \), endowed with the Lie bracket \([,\cdot,]\) defined by \([A,B] = AB - BA\), for every \( A,B \in \text{End} \ V \). The main result of the paper is as follows.

Theorem 1. Let \( A,B \in \text{End} \ V \) be such that \([A,B]y_n = -y_n\), where the sequence \((y_n)_n \subset V\) is defined as follows: \(Ay_0 = 0\) and \(By_n = -(n+1)y_{n+1}\), for every \( n \geq 1 \). Then \( Ay_n = y_{n-1} \) and \( y_n \) is an eigenvector of eigenvalue \(-n\) for \( BA \), for every \( n \geq 1 \).

Proof: First, we shall prove

\[ Ay_n = y_{n-1}, \text{ for every } n \geq 1. \]

For \( n = 1 \), this equality is evident, because

\[ [A,B]y_0 = -y_0. \]
\[ A(By_0) - B(Ay_0) = -y_0, \]
also \( Ay_0 = 0 \) and \( By_0 = -y_1 \) and therefore,
\[ Ay_1 = y_0. \]

Now, suppose that \( Ay_n = y_{n-1} \), then we have
\[ [A, B]y_n = -y_n, \]
i.e. \( A(By_n) - B(Ay_n) = -y_n, \)
i.e. \( A(-(n+1)y_{n-1}) - B(y_{n-1}) = -y_n, \)
i.e. \( -(n+1)A(y_{n-1}) + ny_n = -y_n, \)
i.e. \( A(y_{n-1}) = -\frac{(n+1)}{(n+1)}y_n, \)
i.e. \( A(y_{n+1}) = y_n. \)

Therefore by mathematical induction \( Ay_n = y_{n-1} \), for every \( n \geq 1 \). It immediately follows that \( BAy_n = -ny_n \). Hence, \( y_n \) is an eigenvector of eigenvalue \(-n\) for \( BA \), for every \( n \geq 1 \).

### 3 A Concrete Application

Let \( V = C^\infty(R \times R) \), we define the operators \( A, B \in \text{End} \ V \) as
\[ A(u, y) = y^{-1} \frac{\partial u}{\partial x}, \]
\[ B(u, y) = xy \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} - (x + \beta)yu \]
for \( (x, y) \in R \times R \).

We claim that the operators (5) and (6) obey the commutation relation \([A, B]y_n = -y_n\).

Indeed,
\[ [A, B]u(x, y) = A(Bu(x, y)) - B(Au(x, y)) \]
which gives
\[ [A, B]u(x, y) = \left( y^{-1} \frac{\partial}{\partial x} \right) \left( xy \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} - (x + \beta)y \right) \]
\[ - \left( xy \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} - (x + \beta)y \right) \left( y^{-1} \frac{\partial u}{\partial x} \right) \]
\[ = -u, \]
i.e.
\[ [A, B]u(x, y) = -u(x, y). \]

Now, if \( u(x, y) \) assumes the form \( y_n(x, y) = f_n(x)y^n \in C^\infty(R \times R) \), then we have
\[ [A, B](f_n(x)y^n) = -f_n(x)y^n, \]
and our claim is justified.

Now, the relation \( By_n = -(n+1)y_{n+1} \) gives
\[ \left( xy \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} - (x + \beta)y \right) \left( f_n(x)y^n \right) = -(n+1)f_{n+1}(x)y^{n+1} \]
i.e.
\[ xy \frac{\partial}{\partial x} \left( f_n(x) \right) = (x + \beta)f_n(x) - (n+1)f_{n+1}(x) \]
(9)
Again, the relation \( Ay_n = y_{n-1} \) gives
\[ \left( y^{-1} \frac{\partial}{\partial x} \right) \left( f_n(x)y^n \right) = f_{n-1}(x)y^{n-1} \]
i.e.
\[ \frac{\partial}{\partial x} \left( f_n(x) \right) = f_{n-1}(x) \]
(10)
Finally, the relation \( BAy_n = -ny_n \) gives
\[ \left( xy \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} - (x + \beta)y \right) \left( y^{-1} \frac{\partial}{\partial x} \right) \left( f_n(x)y^n \right) = -nf_n(x)y^n \]
i.e.
\[ xy \frac{\partial^2}{\partial x^2} \left( f_n(x) \right) + \left( 1 - \beta - n - x \right) \frac{\partial}{\partial x} \left( f_n(x) \right) + nf_n(x) = 0 \]
(11)
Now, we observe that modified Laguerre polynomials \( f_n^\beta(x) \) is a solution of the differential equation (11).

Further we note that the relations (9) and (10) are differential recurrence relations satisfied by modified Laguerre polynomials \( f_n^\beta(x) \).

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### References


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