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Some Simple Conditions for Univalence

Mamoru Nunokawa[1] *and Janusz Sokół*[2,*]

[1] University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan
[2] Faculty of Mathematics and Natural Sciences, University of Rzeszów, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland

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Abstract: The aim of this paper is to determine the sufficient conditions for function $f(z) = z + a_2 z^2 + \dots$ to be starlike of order $1/2$ which shows also the convexity of $f(z)$.

Keywords: analytic, univalent, convex, starlike, close-to-convex, differential subordination

1 Introduction

Let $\mathscr{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. We will use the following notations:

$$\begin{cases} J_{CV}(f;z) := 1 + \frac{zf''(z)}{f'(z)}, \\ J_{ST}(f;z) := \frac{zf'(z)}{f(z)}. \end{cases} \tag{1}$$

Let the function $f \in \mathscr{H}$ be univalent in the unit disc $\mathbb{D}$ with the normalization $f(0) = 0$. Then f maps $\mathbb{D}$ onto a starlike domain with respect to $w_0 = 0$ if and only if [4]

$$\mathfrak{Re}\{J_{ST}(f;z)\} > 0 \quad \text{for all } z \in \mathbb{D}. \tag{2}$$

Such function f is said to be starlike in $\mathbb{D}$ with respect to $w_0 = 0$ (or briefly starlike). Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. A function f maps $\mathbb{D}$ onto a convex domain E if and only if [16]

$$\mathfrak{Re}\{J_{CV}(f;z)\} > 0 \quad \text{for all } z \in \mathbb{D} \tag{3}$$

and then f is said to be convex in $\mathbb{D}$ (or briefly convex). It is well known that if an analytic function f satisfies (2) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{D}$. Let $\mathscr{A}$ denote the subclass of $\mathscr{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathscr{A}$ that are starlike univalent in $\mathbb{D}$ will be denoted by $\mathscr{S}^*$. The set of all functions $f \in \mathscr{A}$ that are convex univalent in $\mathbb{D}$ by $\mathscr{K}$. It is known that for $f(z) \in \mathscr{A}$ condition (3) is sufficient for starlikeness of $f(z)$. Also the condition

$$|J_{CV}(f;z) - 1| < 2 \quad z \in \mathbb{D}$$

is sufficient for starlikeness of $f(z)$. In this paper we shall consider certain sufficient conditions for starlikeness of order $1/2$. The class $\mathscr{S}^*_\alpha$ of starlike functions of order $\alpha < 1$ may be defined as

$$\mathscr{S}^*_\alpha := \{f \in \mathscr{A} : \mathfrak{Re}\{J_{ST}(f;z)\} > \alpha,\ z \in \mathbb{D}\}.$$

The class $\mathscr{S}^*_\alpha$ and the class $\mathscr{K}_\alpha$ of convex functions of order $\alpha < 1$

$$\begin{aligned} \mathscr{K}_\alpha &:= \{f \in \mathscr{A} : \mathfrak{Re}\{J_{CV}(f;z)\} > \alpha,\ z \in \mathbb{D}\} \\ &= \{f \in \mathbb{D} : zf' \in \mathscr{S}^*_\alpha\} \end{aligned}$$

were introduced by Robertson in [11]. It is known the old Strohhäcker result [13] that $\mathscr{K}_0 \subset \mathscr{S}^*_{1/2} \subset \mathscr{S}^*_0$. Furthermore, note that if $f \in \mathscr{K}_\alpha$ then $f \in \mathscr{S}^*_{\delta(\alpha)}$, see [14], where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for } \alpha \neq 1/2, \\ \frac{1}{2\log 2} & \text{for } \alpha = 1/2. \end{cases} \tag{4}$$

Robertson [12] proved that if $f(z) \in \mathscr{A}$ with $f(z)/z \neq 0$ and if there exists a k, $0 < k \leq 2$, such that

$$|J_{CV}(f;z) - 1| \leq k|J_{ST}(f;z)| \quad z \in \mathbb{D},$$

* Corresponding author e-mail: jsokol@ur.edu.pl

then $f(z)\in\mathscr{S}^*_{2/(2+k)}$. In [3] it was proved that for $f(z)\in\mathscr{A}$ with $f(z)f'(z)/z\neq 0$

$$|J_{CV}(f;z)|\leq\sqrt{2}|J_{ST}(f;z)+1|\quad z\in\mathbb{D},$$

then $f(z)\in\mathscr{S}^*$. Several more complicated sufficient conditions for starlikeness and for convexity are collected in the book [2], Chap. 5.

In this paper we shall determine the new sufficient conditions for the starlikeness of order $1/2$. The key in proving is Nunokawa's lemma[5] and the following lemma which generalizes it, [5], [6], see also [1].

Lemma 1.*[7] Let $p(z)$ be of the form*

$$p(z)=1+\sum_{n=m\geq 1}^{\infty}a_nz^n,\qquad a_m\neq 0,\quad (|z|<1),$$

with $p(z)\neq 0$ in $|z|<1$. If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}|<\pi\alpha/2\quad \text{in}\quad |z|<|z_0|$$

and

$$|\arg\{p(z_0)\}|=\pi\alpha/2$$

for some $\alpha>0$, then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\alpha,$$

where

$$k\geq m\left(a^2+1\right)/(2a)\quad \text{when}\ \arg\{p(z_0)\}=\pi\alpha/2$$

and

$$k\leq -m\left(a^2+1\right)/(2a)\quad \text{when}\quad \arg\{p(z_0)\}=-\pi\alpha/2,$$

where

$$\{p(z_0)\}^{1/\alpha}=\pm ia,\ a>0.$$

2 Main result

Theorem 1.*Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic and $J_{CV}(f;z)\neq 1/2$ in $\mathbb{D}$. Suppose also that*

$$\left|\arg\left\{J_{CV}(f;z)-\frac{1}{2}\right\}\right| < \pi-\tan^{-1}\{4|\mathfrak{Im}\{J_{ST}(f;z)\}|\}\quad (z\in\mathbb{D}),\tag{5}$$

then

$$\mathfrak{Re}\{J_{ST}(f;z)\}>\frac{1}{2}\quad (z\in\mathbb{D}),\tag{6}$$

or f is starlike of order $1/2$, $f(z)\in\mathscr{S}^*_{1/2}$.

*Proof.*Let us put

$$p(z)=\frac{2zf'(z)}{f(z)}-1=1+2a_2z+\cdots.\tag{7}$$

If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}|<\frac{\pi}{2}\quad |z|<|z_0|$$

and

$$|\arg\{p(z_0)\}|=\frac{\pi}{2},$$

then, from Nunokawa's Lemma 1 with $\alpha=1$, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik,$$

where

$$k\geq(a^2+1)/(2a)\ \text{ when }\ \arg\{p(z_0)\}=\pi/2$$

and

$$k\geq-(a^2+1)/(2a)\ \text{ when }\ \arg\{p(z_0)\}=-\pi/2,$$

$p(z_0)=\pm ia$, $a>0$.

For the case $p(z_0)=ia$, $a>0$, we have from (7)

$$\begin{aligned}&\left|\arg\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{1}{2}\right\}\right|\\ =&\left|\arg\left\{\frac{z_0p'(z_0)}{1+p(z_0)}+\frac{p(z_0)}{2}\right\}\right|\\ =&\left|\arg\left\{\frac{-ka}{ia+1}+\frac{ia}{2}\right\}\right|\\ =&\left|\arg\left\{\frac{2ak}{2(a^2+1)}\left(-1+i\frac{1+a^2+2ak}{2k}\right)\right\}\right|\\ =&\left|\arg\left\{-1+i\frac{1+a^2+2ak}{2k}\right\}\right|.\end{aligned}$$

In considered case we have $a>0$, $k\geq(a^2+1)/(2a)$ and so, we have

$$\frac{\pi}{2}<\arg\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{1}{2}\right\}<\pi$$

and we can write

$$\begin{aligned}&\left|\arg\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{1}{2}\right\}\right|\\ =&\ \pi-\tan^{-1}\left\{\frac{1+a^2+2ak}{2k}\right\}.\end{aligned}$$

Observe that for $a>0$, $k\geq(a^2+1)/(2a)$ we also have

$$\begin{aligned}\frac{1+a^2+2ak}{2k}&=a+\frac{1+a^2}{2k}\\ &\leq a+\frac{1+a^2}{2(1+a^2)/(2a)}\\ &=a+a=2a.\end{aligned}\tag{8}$$

Moreover, applying (7) with $p(z_0) = ia$, we have

$$a = \mathfrak{Im}\left\{\frac{2z_0 f'(z_0)}{f(z_0)}\right\}.$$

Therefore, we obtain

$$\begin{aligned}&\left|\arg\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{1}{2}\right\}\right|\\ &\geq \pi - \tan^{-1}\{2a\}\\ &= \pi - \tan^{-1}\left\{\mathfrak{Im}\left\{\frac{4z_0 f'(z_0)}{f(z_0)}\right\}\right\}.\end{aligned}$$

This is the contradiction with (5) and for the case $\arg\{p(z_0)\} = -\pi/2$, applying the same method as the above, we can also get the contradiction.

Since $\mathscr{K}_0 \subset \mathscr{S}^*_{1/2} \subset \mathscr{S}^*_0$, then (5) is a sufficient condition for $f(z)$ to be convex in $\mathbb{D}$.

Corollary 1.*Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic and $J_{CV}(f;z) \neq 1/2$ in $\mathbb{D}$. Suppose also that*

$$|\mathfrak{Im}\{J_{ST}(f;z)\}| < \frac{1}{4}$$

and that

$$\left|\arg\left\{J_{CV}(f;z)-\frac{1}{2}\right\}\right| < \frac{3\pi}{4} \quad (z \in \mathbb{D}),$$

then $\mathfrak{Re}\{J_{ST}(f;z)\} > 1/2$ in $\mathbb{D}$ or $f(z) \in \mathscr{S}^*_{1/2}$.

Theorem 2.*Let $f(z) = z + \sum_{n=m\geq 2}^{\infty} a_n z^n$, $a_m \neq 0$, be analytic and $J_{CV}(f;z) \neq 1/2$ in $\mathbb{D}$. Suppose also that*

$$\begin{aligned}&\left|\arg\left\{J_{CV}(f;z)-\frac{1}{2}\right\}\right|\\ &< \pi - \tan^{-1}\left\{2\left(1+\frac{1}{m-1}\right)|\mathfrak{Im}\{J_{ST}(f;z)\}|\right\}\end{aligned}$$

in the unit disc $\mathbb{D}$, then $\mathfrak{Re}\{J_{ST}(f;z)\} > 1/2$ in $\mathbb{D}$ or $f(z) \in \mathscr{S}^*_{1/2}$.

*Proof.*If we put

$$p(z) = \frac{2zf'(z)}{f(z)} - 1,$$

then

$$p(z) = 1 + 2a_m(m-1)z^{m-1} + \cdots, \quad a_m \neq 0, \quad (|z| < 1).$$

Now the proof runs in the same way as the proof of Theorem 1. By Lemma 1, instead of (8), we have here

$$\begin{aligned}\frac{1+a^2+2ak}{2k} &= a + \frac{1+a^2}{2k} \leq a + a/(m-1)\\ &= a\left(1+\frac{1}{m-1}\right).\end{aligned}$$

Corollary 2.*Let $f(z) = z + \sum_{n=3}^{\infty} a_n z^n$ be analytic and $J_{CV}(f;z) \neq 1/2$ in $\mathbb{D}$. Suppose also that*

$$\begin{aligned}&\left|\arg\left\{J_{CV}(f;z)-\frac{1}{2}\right\}\right|\\ &< \pi - \tan^{-1}\{3|\mathfrak{Im}\{J_{ST}(f;z)\}|\} \quad (z \in \mathbb{D}),\end{aligned}$$

then $\mathfrak{Re}\{J_{ST}(f;z)\} > 1/2$ in $\mathbb{D}$ or $f(z) \in \mathscr{S}^*_{1/2}$.

3 Applications

Since $\mathscr{K}_0 \subset \mathscr{S}^*_{1/2} \subset \mathscr{S}^*_0$, then each of the above proved sufficient conditions for $f(z) \in \mathscr{S}^*_{1/2}$ is also sufficient condition for $f(z)$ to be convex in $\mathbb{D}$. Recall the convolution:

$$\sum_{n=0}^{\infty} a_n z^n * \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} a_n b_n z^n$$

It is known that $f(z) \in \mathscr{K}_0$, $g(z) \in \mathscr{K}_0$ implies $f(z) * g(z) \in \mathscr{K}_0$. Such implication does't hold for starlike functions. However, if $f(z) \in \mathscr{S}^*_{1/2}$, $g(z) \in \mathscr{S}^*_{1/2}$, then $f(z) * g(z) \in \mathscr{S}^*_{1/2}$. Moreover, if $F(z) \in \mathscr{H}$, then

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \subset \overline{co}F(\mathbb{D}),$$

where $\overline{co}F(\mathbb{D})$ denotes the closed convex hull of $F(\mathbb{D})$. For these reasons new sufficient conditions for the starlikeness of order $1/2$ may be valuable.

For other results on the several problems connected to the starlikeness we refer also to the recent papers [8], [9] and [15].

4 Conclusion

We have presented some sufficient conditions for the starlikeness of order $1/2$ and applications of these conditions. The class $\mathscr{S}^*_{1/2}$ of starlike functions of order $1/2$ is an important class. It is known that $\mathscr{K}_0 \subset \mathscr{S}^*_{1/2} \subset \mathscr{S}^*_0$. Furthermore, note that if $f \in \mathscr{K}_\alpha$ then $f \in \mathscr{S}^*_{\delta(\alpha)}$, see (4). For some new conditions for starlikeness and strongly starlikeness of order alpha we refer to the recent paper [10].

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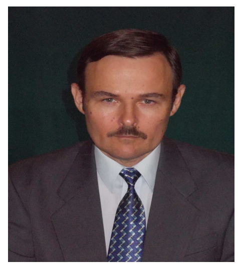
Mamoru Nunokawa graduated in mathematics from the University of Gunma(Japan), then pursued graduate studies in Tokyo Kyoiku Univrsity, Tsukubuniversity and University of Mississipi. He was the professor of mathematics at Gunma Univesity since 1968. He is currently an ameritius professor. His work has focused on several aspects of complex analysis. Well known is the Nunokawas Lemma.



Janusz Sokół is an professor of mathematics at Rzeszow University (Poland) since 2016. He recived his doctorate under the supervision of Jan Stankiewicz in University of d, while he received his habilitation in Mariae Curie-Skodowska University in Lublin. Long time he worked at Rzeszw University of Technology. His research focuses on geometric function theory.