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Optimal Decision Model for Sustainable Green Industry Using AHP and TOPSIS with Picture Fuzzy Soft Aczel-Alsina Operators

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Abstract: In contrast to traditional fuzzy and picture fuzzy set theories, soft set theory introduces a parameterized framework that effectively addresses uncertainty in complex data. In the context of data collection, where membership degrees require values to fall within unit intervals, the picture fuzzy soft set is especially crucial for mitigating confusion. It is also acknowledged that the Aczel-Alsina t-conorm and Aczel-Alsina t-norm are widely applicable and flexible in how they process data within these ranges. The main contributions of this paper are the introduction of multiple aggregation operations for data represented by picture fuzzy soft values, with the foundational components being AATRM and AATCRM. For PFSVs, sum and product operations are developed through operational laws based on AATRM and AATCRM. From these operational laws, two operators are proposed: Picture Fuzzy Soft Aczel-Alsina Weighted Averaging and Geometric. Additionally, the paper investigates and contrasts some of these operators' properties with those that already exist. In order to show the value of the suggested strategy it's effectiveness of this optimization method for resolving multi-criteria decision-making problems is finally provided and illustrated.

Keywords: Decision Making, Fuzzy Logic, Picture soft sets, Aczel-Alsina TN and TCN, Picture Fuzzy Soft Aczel-Alsina Weighted Averaging, and Geometric.

1 Introduction

The emergence of the "green" industry represents a paradigm shift in industrial practices, prioritizing resource efficiency, environmental stewardship, and sustainability. This transition seeks to reduce the carbon intensity of production processes, incorporate renewable energy systems, and minimize waste generation. By embracing clean technologies and advancing circular economy models, green industries not only mitigate the adverse environmental impacts of conventional manufacturing but also stimulate innovation and employment opportunities. Addressing climate change and resource depletion has become an imperative for the global community, and the development of green industries offers a viable pathway to sustainable economic growth and environmental preservation for future generations.

Green industry development underpins sustainable business practices and responsible consumption, thereby enhancing environmental protection while simultaneously reinforcing social and economic resilience. Central to this transformation is the promotion of green jobs, which contribute to the establishment of a low-carbon economy and improved quality of life. Moreover, the advancement of green industries aligns with international sustainability agendas and serves as a critical driver of a more equitable and enduring global future.

Within this context, prior scholarship has explored multiple dimensions of green industrial transformation. Keshavarz et al. [1] examined the role of green suppliers, while Yin et al. [2] introduced the notion of enhancing digital green innovation practices across industries. Haiyun et al. [3] investigated green supply chain management within the energy sector under the intuitionistic fuzzy set (IFS) framework. Banaeian et al. [4] pioneered approaches to green supplier

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selection in the food industry, and D'Angelo et al. [5] identified overlapping domains to foster applications of the green supply chain. Similarly, Dong et al. [6] proposed digital green selection mechanisms for photovoltaic systems.

1.1 Literature Review

Fuzzy tools are an effective way to address multi-criteria decision-making (MCDM) problems because of the complexity of the aforementioned situations. During the decision-making (DM) process, fuzzy tools help us manage ambiguity and uncertainty while guaranteeing a more rational and reasonable. In order to better address ambiguity and uncertainty in the DM process, Zadeh [7] established the idea of fuzzy sets (FS), which employs values between $\Lambda \in [0,1]$ to describe the membership degree (MD) of the alternatives with respect to give property. Atanassov [8] noted the deficiencies in FS and recommended an expanded version termed IFS, providing a framework for MD and non- membership degree (NMD). More definitions of interval-valued intuitionistic fuzzy sets (IVIFSs) were provided by Atanassov et al. [9]. Due to the effectiveness of IFS in modeling uncertain data and information, they have widespread scholarly interest in them. The IFS simple weighted average operators were presented by Xu and Yager [10]. When a DM's uncertainty evaluation of 0.1, the corresponding values are 0.3 for NMD and 0.2 for MD. The picture fuzzy set (PFS) theory was proposed by Omar et al [11] as a solution, employing MD, neutral degree(ND), and NMD. FS can handle fuzziness, uncertainty, and inconsistency in information. A great deal of research has also been conducted by numerous academics on the theory and implementation of the picture fuzzy (PF) environment. Sarfraz [12] generalized the Maclaurin symmetric mean AOs for PF environments employing Frank norm and conorm.

Molodtsov [13] introduction as a mathematical tool for managing uncertainties, soft set (SS) theory has shown itself to be very successful in resolving issues where conventional approaches falter due to imprecise or insufficient data. It is ideal for use in data analysis, engineering, and DM because it does not necessitate a complicated membership structure and permits flexible parameterization. Its ability to model uncertainty without requiring exact numerical values is one of its main benefits; this is especially helpful when expert opinions or subjective assessments are involved. By adding three independent MD, ND, and NMD picture fuzzy soft set (PFSS), expand on this framework and increase the expressive potential of SS. Simultaneous modeling of acceptance, hesitation, and rejection is made possible by this enhanced structure. PFSS provides a more accurate and comprehensive representation of expert opinions in MCDM problems, such as those in the development of the "green" industry, particularly when there are conflicting or unclear assessments. They are the perfect framework for intricate evaluations where traditional fuzzy models might not be sufficient due to their adaptability and descriptive power.

Khan et al. [14–16] utilized the PFSS and their application support system. Lu et al. [17] created a new type of PFSS on DM. Chellamani et al. [18] created the DM by the PFSS graphs. Memis et al. [19] developed PFSS and product operations with soft DM. Harl et al. [20] introduced the picture fuzzy N-soft set with the DM algorithm. For more adaptable information fusion, [21] also suggested the Aczel-Alsina t-conorm (AATCRM) and Aczel-Alsina t-norm (AATRM). To address MCDM issues, for example, [22] presented aggregation operations (AOs) based on AATRM and AATCRM for IFS. These AOs were used by Hussain et al. [23] with PyFS with AATRM and AATCRM to handle MCDM problems. Jan et al. [24] introduced the AOs operator on MAGDM with PFSWA and PFSWG. Similar to this, Mahmood et al. [25] developed AOs using AATRM and AATCRM to address MCDM problems. Naeem et al. [26] utilized the AA geometric determine the factors affecting mango crops. Serif [27] The utilization of AA AOs for "green" industry applications. Zeb et al. [28] created the AA AOs with MCDM. Lu et al. [29] introduced the Utilization of AOs in environmentally "green" supplier choice. Zulqarnain et al. [30] created the AOs with "green" supplier chain management. T. Alharbi et al. [31] developed the concepts of fuzzy algorithm. Zhang et al. [32] introduced AOs with PF information and their application to "green" supplier selection. Zhou et al. [33, 34] expanded the concept of artificial neural network based on pq-rung orthopair fuzzy. Khan et al. [35] developed the theory of soft and soft covering. Ali et al. [36] defined the concept of AOs with used the frank operator. Ali et al. [37] introduced the concept of AA operator with 2-tuple linguistic neural networks. Rehman et al. [38] used N-soft sets with lattice ordered. Hayat et al. [39] developed the concept of type-2 soft sets with new result. Raja et al. [40] expanded the concepts of N-soft set with application of solar panel evaluation.

Multi-criteria decision-making (MCDM) offers distinct advantages over conventional decision-making (DM) techniques, as it systematically incorporates the preferences and judgments of multiple stakeholders across diverse criteria. This comprehensive consideration facilitates a more objective and rigorous evaluation of available alternatives. By integrating heterogeneous perspectives, mitigating bias, and accounting for the inherent complexity of real-world problems, MCDM enhances both the robustness and reliability of decision outcomes. Accordingly, MCDM serves as a powerful methodological framework for evaluating alternatives based on their respective attributes.

Extensive research has examined the application of MCDM techniques within various fuzzy environments. A central element of MCDM is the aggregation operator (AO), which systematically integrates distributed data and thus plays a pivotal role in the decision-making process [41]. Consequently, a range of sophisticated aggregation methods has been proposed in the literature. For instance, several significant AOs have been introduced by [42–44], designed to more comprehensively synthesize alternative data compared to conventional operators. Zhao et al. [42] proposed AOs based on parametric similarity measures for MCDM, while Jan et al. [24] developed an AO operator for multi-attribute group decision-making (MAGDM) under picture fuzzy soft sets. Similarly, Tchier et al. [43] introduced a group DM approach utilizing PFSS information, Zahedi et al. [44] were the first to extend MCDM into the domain of soft set theory, and Mahmood et al. [45] explored MCDM applications through complex PFSS power aggregation operators.

The criteria's relative importance is ascertained using the Analytic Hierarchy Process (AHP), which enables a systematic pairwise comparison and consistency check. The weights are then determined, and the alternatives are ranked using the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), which calculates the alternatives' geometric distance from the ideal and anti-ideal solutions. Hajiaghaei-Keshteli et al. [46] suggested the concept of the TOPSI method for "green" supplier selection in the food industry. Paltayian et al. [47] developed an AHP tool to improve e-banking usage. Tavana et al. [48] established the concept of the AHP-TOPSIS framework for human spaceflight missions. Otay et al. [49] put forward the AHP method for transformation project selection. Gulum et al. [50] described the AHP TOPSIS Technique for evaluating fire risks following earthquakes. Gu et al. [51] devised the AHP method for internet finance companies. Tran et al. [52] defined the AHP TOPSIS for industrial robot selection. Salari et al. [53] suggested the AHP method on the PyFS environment. Hananto et al. [54] developed the AHP method for a selection decision support system. Oubahman et al. [55] described the AHP method as the application of university students' choice preference. Hussain et al. [56] devised the TOPSIS method for IFS in Dombi AOs. Mahmood et al. [57] described the TOPSIS method using PFS and their application to strategic DM. Rouyendegh et al. [58] described the TOPSIS method for the "green" supplier selection problem. Ahmed et al. [59] defined the TOPSIS method using similarity measures.

1.2 Research Gap and Motivation

The process of determining which criteria to accept or reject often presents significant challenges for decision-makers (DMs), particularly under conditions of uncertainty and ambiguity in the selection, appraisal, and evaluation of parameters. These challenges highlight the necessity of a structured decision-making framework that incorporates both supportive and critical perspectives while ensuring objectivity in the assessment of alternatives. A comprehensive review of the existing literature indicates a pressing need for a mathematical framework capable of systematically addressing these complexities. In response to this requirement, the AA AOs PFSS framework was developed, offering enhanced flexibility and adaptability in managing decision-making challenges. The principal contributions of this study are outlined below.

1. Green industry assessment has become a crucial area in policy and industrial planning due to the increased emphasis on sustainable development on a global scale. However, ambiguous, ambiguous, and contradictory information from multiple sources frequently makes DM in this area difficult.

2. Experts in complex environmental assessments frequently express hesitation or neutral opinions, which are not adequately represented by traditional DM models. This disparity emphasizes the need for a mathematical framework that is more sophisticated and expressive.

3. Support, opposition, and neutrality are three independent degrees that can be better represented by the **PFSS model**, which makes it ideal for multi-criteria problems in the green industry.

4. In contrast to classical operators, **AA AOs** offer more realistic combination mechanisms due to their flexibility and parameter tuning. By combining these with PFSS, decision analysis can become even more precise.

5. Using the picture to combine the **AHP** and **TOPSIS** methods by bridging the gap between objective ranking and subjective expert judgment, FSS not only improves the evaluation structure but also produces more fair and defensible decisions.

6.This study offers policymakers and industry experts a useful model to evaluate green industrial strategies under uncertainty by introducing a novel hybrid approach that is tailored to current sustainability goals.

7.We developed a novel **PFSVs MCDM** technique based on the **PFSAAWA** or **PFSAAWG** operators.

Benefits of the Suggested Picture Fuzzy Soft Aczel–Alsina Operators

–The suggested operators deal with uncertainty, vagueness, and hesitation effectively factors that are prevalent in **MCGM** problems in the real world.

–Thanks to the inclusion of **AA AATCRM** and **AATRM**, which offer a non-linear and adjustable structure, they provide greater flexibility than traditional fuzzy, IF, and PyF AOs.

–The **AA operations** lead to improved stability and consistency of aggregated results, minimizing the impact of extreme or biased values.

–The model enhances the precision of weight assessment and ranking when used alongside **AHP** and **TOPSIS**, resulting in more reliable evaluations of the sustainable green industry.

–With the proposed approach, decision reliability and sensitivity are enhanced, enabling DMs to analyze multiple criteria with greater precision.
–Additionally, it upholds mathematical soundness and interpretability, which contributes to its practicality and ease of application in complex sustainability assessments.

1.3 Organization of the Study

This is how the remainder of the article is structured. An outline of the article's main concepts is given in **Section 2** for easier comprehension. The concepts of the **Picture Fuzzy Soft Aczel-Alsina Weighted Averaging (PFSAAWA)** and **Geometric (PFSAAWG)** operators are thoroughly explained and their fundamental traits are looked at in **Section 3**. The created AOs are used to solve an **MCDM problem** in **Section 4** and **AHP** and **TOPSIS** methods provided in **sub-section 4.1** and **4.2**. Following a discussion of the **PFSAAWA** and **PFSAAWG** operators' technique, an explanation of a real-world **MCDM problem** using these operators is provided. In order to verify the stability of ranking results, **Section 4.4** concentrated on a thorough **sensitivity analysis** with respect to the soft set (SS) parameter. We conduct a **comparison analysis** of the created strategy with a number of current approaches in **Section 5**. Finally, **Section 6** provides a summary of the article. **Figure 1** shows a procedural flowchart that illustrates how the article is organized.

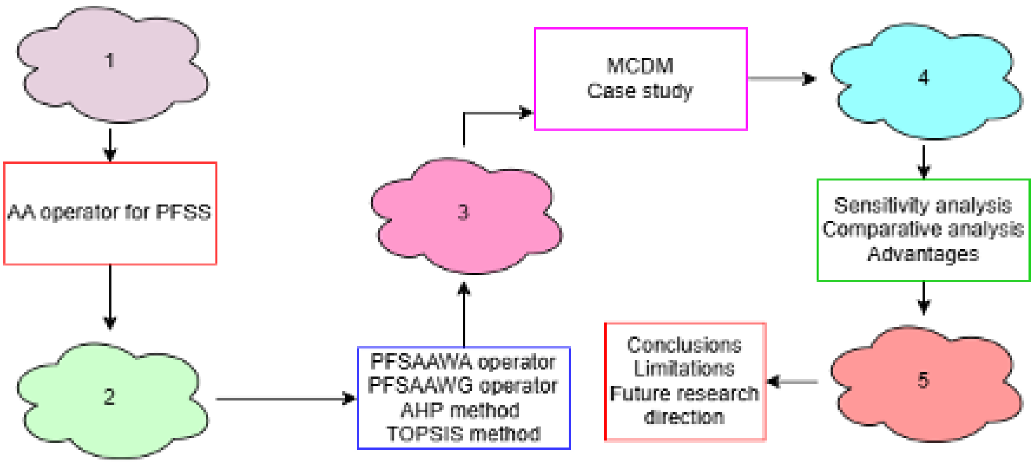

Fig. 1: Flowchart illustration of the suggested task

2 Preliminaries

In this section, let us briefly recall the fundamental concepts of SS, FSS, and PFSS. These models offer adaptable resources for managing imprecision and ambiguity in practical issues. Decision makers can more effectively convey complex information with each extension's increased descriptive power.

Definition 1 [54]: A fuzzy set $\ddot{A}$ over the universe is defined as $\ddot{X}$

$$\ddot{A} = \{\xi, \Lambda_{\xi}(\xi) \mid \xi \in \ddot{X}\} \tag{1}$$

Where $\Lambda_{\xi}(\xi) \to [0,1]$ is an MD. For each $\ddot{F} \in \ddot{X}$, $\Lambda_{\xi}(\xi)$ specifies the degree to which the element ξ belongs to the FS $\ddot{A}$.

Molodtsov [13] introduced the concept of the SS, which represents a completely new tool in science for parameterization-based uncertainty management. Consider a parameter space $\ddot{E}$ and a universal set $\ddot{X}$. If $\ddot{X}$ is infinite, then the parameter space is unrestricted. A limited set. The parameter space is mostly made up of attributes, features, or properties of the elements in the all-encompassing set.

Definition 2 [55]: Consider $\ddot{X}$ to be a universal set and $\ddot{E}$ a parameter space, the power set of $\ddot{X}$. A pair $(\ddot{F}, \ddot{A})$ is called an SS over $\ddot{X}$, where $\ddot{A} \subset \ddot{E}$ and $\ddot{F}$ is a set-valued mapping given by $\ddot{F} : \ddot{A} \to \mathscr{P}(\ddot{X})$.

P.K. Maji defined the FSS in [60], which is a hybrid model combining elements of an SS and an FS. Since the model is hybrid, each attribute should have an MD associated with it. This is because people's perceptions in real life are characterized by a certain amount of ambiguity and lack of precision. For instance, we cannot express the information in only one way when judging a woman's beauty. Two distinct numbers: 1 and 0.

Definition 3 [57]: Consider $\ddot{X}$ to be a universal set and $\ddot{E}$ a parameter space. Consider $\mathscr{PF}(\ddot{X})$, denoted the set of all PFS of $\ddot{X}$. A pair of $(\ddot{F}, \ddot{A})$ is called a PFSS. Where $\ddot{A} \subset \ddot{E}$ and $\ddot{F}$ are mappings given by $\ddot{F} : \ddot{A} \to \mathscr{PF}(\ddot{X})$. It is clear from the definition of FSS that it is a parameterized family of picture fuzzy subsets of $\ddot{X}$ rather than a set. For any $\ddot{h} \in \ddot{A}$, $\ddot{F}(\ddot{h})$ is a PFS of $\ddot{X}$. $\ddot{F}(\ddot{h})$ can be written as a PFS such that $\ddot{F}(\ddot{h}) = \{\xi, \Lambda_{\xi}(\xi), \lambda_{\xi}(\xi), \mu_{\xi}(\xi) : \xi \in \ddot{X}\}$ where $\Lambda_{\xi}(\xi)$, $\lambda_{\xi}(\xi)$, $\mu_{\xi}(\xi)$ are the MD, ND, and NMD, respectively. Y. Yang [?] et al also defined the equality and complement of PFSS.

Example 2.1: Consider $L = \{S_1, S_2, S_3, S_4\}$ to be the set of four students who applied for a university scholarship. Suppose $D = \{d_1, d_2, d_3, d_4\}$ are the set of experts who are to select the most appropriate student for the scholarship under the set of parameters $E = \{\text{Intelligent}(e_1), \text{hardworking}(e_2), \text{moneyles}(e_3), \text{regular}(e_4)\}$. Assume that $\Xi = \{\text{hardworking}(e_2), \text{moneyles}(e_3), \text{regular}(e_4)\} \subset \ddot{E}$. Then,

1. A fuzzy SS (Θ, Ξ) by an expert says d_1 can be $(\Theta, \Xi) = \{F(e_{12}), F(e_{13}), F(e_{14})\}$ Where, $F(e_{12}) = \{S_2, S_3, S_4\}$, $F(e_{13}) = \{S_2, S_3\}$, $F(e_{14}) = \{S_1, S_3\}$ so that $(S_1, 0.1), (S_2, 0.5), (S_3, 0.4)$, etc., are fuzzy soft numbers (FSNs).

2. An IFSS can be

$$(\Theta^1, \Xi) = \left\{ \begin{array}{l} F(e_{12}) = \left\{ \frac{S_1}{(0.4,0.5)}, \frac{S_3}{(0.4,0.6)}, \frac{S_4}{(0.5,0.3)} \right\}, \\ F(e_{13}) = \left\{ \frac{S_2}{(0.3,0.5)}, \frac{S_3}{(0.1,0.5)} \right\}, \\ F(e_{14}) = \left\{ \frac{S_2}{(0.3,0.4)}, \frac{S_3}{(0.4,0.5)}, \frac{S_4}{(0.5,0.4)} \right\} \end{array} \right\}$$

The MD (M_{ij}) and NMD (N_{ij}) satisfy the condition that $0 \leq M_{ij} + N_{ij} \leq 1$.

3. A PyFSS can be

$$(\Theta^2, \Xi) = \left\{ \begin{array}{l} F(e_{12}) = \left\{ \frac{S_2}{(0.4,0.5)}, \frac{S_3}{(0.2,0.6)}, \frac{S_4}{(0.6,0.3)} \right\}, \\ F(e_{13}) = \left\{ \frac{S_2}{(0.4,0.5)}, \frac{S_3}{(0.2,0.6)} \right\}, \\ F(e_{14}) = \left\{ \frac{S_2}{(0.3,0.7)}, \frac{S_3}{(0.4,0.5)}, \frac{S_4}{(0.5,0.3)} \right\} \end{array} \right\}$$

The MD (M_{ij}) and NMD (N_{ij}) satisfy the condition that $0 \leq (M_{ij})^2 + (N_{ij})^2 \leq 1$.

4.A q-ROFSS can be

$$(\Theta^3,\Xi)=\left\{\begin{array}{l} F(e_{12})=\left\{\frac{S_2}{(0.4,0.5)},\frac{S_3}{(0.4,0.6)},\frac{S_4}{(0.5,0.3)}\right\}, \\ F(e_{13})=\left\{\frac{S_2}{(0.3,0.5)},\frac{S_3}{(0.1,0.5)}\right\}, \\ F(e_{14})=\left\{\frac{S_2}{(0.3,0.4)},\frac{S_3}{(0.4,0.5)},\frac{S_4}{(0.5,0.4)}\right\} \end{array}\right\}$$

The MD (M_{ij}) and NMD (N_{ij}) satisfy the condition that $0 \leq (M_{ij})^q + (N_{ij})^q \leq 1$.

5.A PFSS can be

$$(\Theta^4,\Xi)=\left\{\begin{array}{l} F(e_{12})=\left\{\frac{S_1}{(0.3,0.1,0.2)},\frac{S_2}{(0.3,0.2,0.4)},\frac{S_3}{(0.4,0.1,0.4)}\right\}, \\ F(e_{13})=\left\{\frac{S_2}{(0.1,0.2,0.5)},\frac{S_2}{(0.2,0.3,0.4)},\frac{S_3}{(0.5,0.1,0.2)},\frac{S_4}{(0.2,0.3,0.1)}\right\}, \\ F(e_{14})=\left\{\frac{S_2}{(0.2,0.1,0.3)},\frac{S_3}{(0.2,0.1,0.4)}\right\}, \\ F(e_{14})=\left\{\frac{S_2}{(0.3,0.3,0.4)},\frac{S_3}{(0.3,0.2,0.4)},\frac{S_4}{(0.5,0.1,0.3)}\right\} \end{array}\right\}$$

The MD (M_{ij}), Absence degree (AD) (A_{ij}), and NMD (N_{ij}) satisfy the condition that $0 \leq M_{ij}+A_{ij}+N_{ij} \leq 1$. The tabular representation of $(\ddot{F},\ddot{A})$ is shown in **Table 1**.

Table 1: Tabular Representation of the Picture Fuzzy Soft Set $(\ddot{F},\ddot{A})$

$\ddot{X}$	e_{11}	e_{12}	e_{13}	e_{14}
S_1	$(0.3,0.1,0.2)$	$(0.1,0.2,0.5)$	$(0.2,0.1,0.3)$	$(0,0,0)$
S_2	$(0.3,0.2,0.4)$	$(0.2,0.3,0.4)$	$(0.2,0.1,0.4)$	$(0.3,0.3,0.4)$
S_3	$(0.4,0.1,0.4)$	$(0.5,0.1,0.2)$	$(0.2,0.1,0.6)$	$(0.3,0.2,0.4)$
S_4	$(0,0,0)$	$(0.2,0.3,0.1)$	$(0,0,0)$	$(0.5,0.1,0.3)$

Definition 4 [61]. Consider $\Delta_{11}=(\Lambda_{\Delta_{11}},\lambda_{\Delta_{11}},\mu_{\Delta_{11}})$ be a PFSV score, and the accuracy function is defined as. Then

$$\operatorname{Sco}(\Delta_1)=\frac{(\Lambda_{\Delta_{11}}-\lambda_{\Delta_{11}}-\mu_{\Delta_{11}})}{2} \tag{2}$$

Be the degrees of Score of Δ_{11}.

$$\operatorname{Acc}(\Delta_1)=\frac{(\Lambda_{\Delta_{11}}+\lambda_{\Delta_{11}}+\mu_{\Delta_{11}})}{2} \tag{3}$$

Be the degrees of accuracy of Δ_{11}.

1.If $\operatorname{Sco}(\Delta_{11})<\operatorname{Sco}(\Delta_{22})$, then Δ_1 has less partiality than Δ_2.
2.If $\operatorname{Sco}(\Delta_{11})=\operatorname{Sco}(\Delta_{22})$, then Δ_1 and Δ_2 are same.
3.If $\operatorname{Acc}(\Delta_{11})<\operatorname{Acc}(\Delta_{22})$, then Δ_1 has less partiality than Δ_2.
4.If $\operatorname{Acc}(\Delta_{11})=\operatorname{Acc}(\Delta_{22})$, then Δ_{11} and Δ_{22} are same.

Definition 5 [61]: The t-norms $(L_A^Q)_{Q\in[0,\infty]}$ for AA is distinct as

$$(L_A^Q)_{(\ell,v)}=\begin{cases} L_D(\ell,v) & \text{if } Q=0 \\ \min(\ell,v) & \text{if } Q=\infty \\ e^{-((-\ln\ell)^Q+(-\ln v)^Q)^{1/Q}} & \text{otherwise} \end{cases}$$

The t-conorms $(S_A^Q)_{Q\in[0,\infty]}$ for AA, are distinct as

$$(S_A^Q)_{(\ell,v)}=\begin{cases} S_D(\ell,v) & \text{if } Q=0 \\ \max(\ell,v) & \text{if } Q=\infty \\ 1-e^{-((-\ln(1-\ell))^Q+(-\ln(1-v))^Q)^{1/Q}} & \text{otherwise} \end{cases}$$

Where $Q\in[0,\infty]$.

Definition 6: Consider $\Delta=(\Lambda_\Delta,\lambda_\Delta,\mu_\Delta)$, $\Delta_{11}=(\Lambda_{\Delta_{11}},\lambda_{\Delta_{11}},\mu_{\Delta_{11}})$ and $\Delta_{22}=(\Lambda_{\Delta_{22}},\lambda_{\Delta_{22}},\mu_{\Delta_{22}})$ be three PFSV, $\Phi\geq 0$. The definitions of the **AATRM** and **AATCRM** operations of PFSV are then as follows:

1.
$$\Delta_{11} \oplus \Delta_{22} = \begin{pmatrix} 1 - e^{-\left((-\ln(1-\Lambda_{\Delta_{11}}))^{Q} + (-\ln(1-\Lambda_{\Delta_{22}}))^{Q}\right)^{1/Q}} \\ e^{-\left((-\ln\lambda_{\Delta_{11}})^{Q} + (-\ln\lambda_{\Delta_{22}})^{Q}\right)^{1/Q}} \\ e^{-\left((-\ln\mu_{\Delta_{11}})^{Q} + (-\ln\mu_{\Delta_{22}})^{Q}\right)^{1/Q}} \end{pmatrix}$$

2.
$$\Delta_{11} \otimes \Delta_{22} = \begin{pmatrix} e^{-\left((-\ln\Lambda_{\Delta_{11}})^{Q} + (-\ln\Lambda_{\Delta_{22}})^{Q}\right)^{1/Q}} \\ 1 - e^{-\left((-\ln(1-\lambda_{\Delta_{11}}))^{Q} + (-\ln(1-\lambda_{\Delta_{22}}))^{Q}\right)^{1/Q}} \\ 1 - e^{-\left((-\ln(1-\mu_{\Delta_{11}}))^{Q} + (-\ln(1-\mu_{\Delta_{22}}))^{Q}\right)^{1/Q}} \end{pmatrix}$$

3.
$$\Phi\Delta = \begin{pmatrix} 1 - e^{-\left(\Phi(-\ln(1-\Lambda_{\Delta}))^{Q}\right)^{1/Q}} \\ e^{-\left(\Phi(-\ln\lambda_{\Delta})^{Q}\right)^{1/Q}} \\ e^{-\left(\Phi(-\ln\mu_{\Delta})^{Q}\right)^{1/Q}} \end{pmatrix}$$

4.
$$\Delta^{\Phi} = \begin{pmatrix} e^{-\left(\Phi(-\ln\Lambda_{\Delta})^{Q}\right)^{1/Q}} \\ 1 - e^{-\left(\Phi(-\ln(1-\lambda_{\Delta}))^{Q}\right)^{1/Q}} \\ 1 - e^{-\left(\Phi(-\ln(1-\mu_{\Delta}))^{Q}\right)^{1/Q}} \end{pmatrix}$$

3 Proposed Aggregation Operators PFSAAWA and PFSAAWG Operator

The development of the suggested AOs based on the **AATRM** and **AATCRM** is covered in this section of the article. This section presents the **PFSAAWA** and **PFSAAWG** operators for the aggregation of the family of different **PFSVs**.

Definition 7. Consider of PFSVs $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$; $i = 1,2,...,n,\ j = 1,2,...,m$. After that, the **PFSAAWA** operator is made as shown below.

$$\text{PFSAAWA}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) = \bigoplus_{j=1}^{m} \zeta_j \left(\bigoplus_{i=1}^{n} \rho_i \Delta_{ij} \right)$$

Where the weight vectors are denoted by $\rho_i > 0$, $\sum_{j=1}^{m} \zeta_j = 1$, and $\sum_{i=1}^{n} \rho_i = 1$. we prove the following theorem on PFVs by using AA operations.

Theorem 1. The aggregated value of the group of PFSVs, which is still a PFSV, is produced by the PFSAAWA operator and is given by:

$$\text{PFSAAWA}(\Delta_{11}, \Delta_{12}, \ldots, \Delta_{nm}) = \begin{pmatrix} 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(1-\Lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}} \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(\lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}} \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(\mu_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}} \end{pmatrix} \tag{4}$$

Proof. Consider $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$ remain an assembly of PFSVs. Next, applying the operating laws of PFSV, we have

$$\rho_i\Delta = \left(1 - e^{-\left(\rho_i(-\ln(1-\Lambda))^{Q}\right)^{1/Q}}, e^{-\left(\rho_i(-\ln(\lambda))^{Q}\right)^{1/Q}}, e^{-\left(\rho_i(-\ln(\mu))^{Q}\right)^{1/Q}} \right)$$

$$\bigoplus_{i=1}^{n} \rho_i \Delta_{ij} = \begin{pmatrix} 1 - e^{-\left(\sum_{i=1}^{n} \rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}} \\ e^{-\left(\sum_{i=1}^{n} \rho_i(-\ln(\lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}} \\ e^{-\left(\sum_{i=1}^{n} \rho_i(-\ln(\mu_{\Delta_{ij}}))^{Q}\right)^{1/Q}} \end{pmatrix}$$

Moreover, we have

$$\bigoplus_{j=1}^{m}\zeta_j\left(\bigoplus_{i=1}^{n}\rho_i\Delta_{ij}\right)=\begin{pmatrix}1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\Lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(\lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(\mu_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\end{pmatrix}$$

Thus,

$$\text{PFSAAWA}(\Delta_{11},\Delta_{12},\ldots,\Delta_{nm})=\begin{pmatrix}1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\Lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(\lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(\mu_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\end{pmatrix}$$

The following are some of the fundamental characteristics of the suggested **PFSAAWA** operator.

Theorem 2. *Idempotency*: Consider Δ_{ij},δ_{ij}; $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, are PFSVs. Next, we have the characteristics. If every PFSV is the same, i.e., $\Delta_{ij}=\Delta$ for all i,j, then

$$\text{PFSAAWA}(\Delta_{11},\Delta_{12},\ldots,\Delta_{nm})=\Delta$$

Proof: *Idempotency*: Consider all Δ_{ij} are the same, i.e., $\Delta_{ij}=\Delta=(\Lambda,\lambda,\mu),\forall i,j$. Then by Eq. 4, we have

$$\begin{aligned}\text{PFSAAWA}(\Delta_{11},\Delta_{12},\ldots,\Delta_{nm})&=\begin{pmatrix}1-e^{-\left(\sum_{j=1}^{m}\zeta_j(\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda))^{Q})\right)^{1/Q}}\\ e^{-\left(\sum_{j=1}^{m}\zeta_j(\sum_{i=1}^{n}\rho_i(-\ln(\lambda))^{Q})\right)^{1/Q}}\\ e^{-\left(\sum_{j=1}^{m}\zeta_j(\sum_{i=1}^{n}\rho_i(-\ln(\mu))^{Q})\right)^{1/Q}}\end{pmatrix}\\
&=\begin{pmatrix}1-e^{-\left((-\ln(1-\Lambda))^{Q}\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\right)^{1/Q}}\\ e^{-\left((-\ln(\lambda))^{Q}\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\right)^{1/Q}}\\ e^{-\left((-\ln(\mu))^{Q}\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\right)^{1/Q}}\end{pmatrix}\\
&=\begin{pmatrix}1-e^{-\left((-\ln(1-\Lambda))^{Q}\right)^{1/Q}}\\ e^{-\left((-\ln(\lambda))^{Q}\right)^{1/Q}}\\ e^{-\left((-\ln(\mu))^{Q}\right)^{1/Q}}\end{pmatrix}\\
&=\left(1-e^{-(-\ln(1-\Lambda))},e^{-(-\ln(\lambda))},e^{-(-\ln(\mu))}\right)\\
&=(\Lambda,\lambda,\mu)=\Delta\end{aligned}$$

Theorem 3: *Monotonicity*: Consider $\Delta_{ij}=(\Lambda_{\Delta_{ij}},\lambda_{\Delta_{ij}},\mu_{\Delta_{ij}})$ and $\delta_{ij}=(\tau_{ij},\upsilon_{ij},g_{ij})$ be two assemblages of the PFSVs such that $\Lambda_{\Delta_{ij}}\leq\tau_{\Delta_{ij}}$, $\lambda_{\Delta_{ij}}\geq\upsilon_{\Delta_{ij}}$ and $\mu_{\Delta_{ij}}\geq g_{\Delta_{ij}}$ then

$$\text{PFSAAWA}(\Delta_{11},\Delta_{12},\ldots,\Delta_{nm})\leq\text{PFSAAWA}(\delta_{11},\delta_{12},\ldots,\delta_{nm})$$

Proof: *Monotonicity*: Then we have $\Lambda_{ij}\leq\tau_{ij}$; consequently, for all i,j, we have

$$-\ln(1-\Lambda_{\Delta_{ij}})\leq-\ln(1-\tau_{\Delta_{ij}})$$

Since $\rho_i>0$ and $\zeta_j>0$, and $\sum\rho_i=\sum\zeta_j=1$,

$$\sum_{i=1}^{n}\rho_i\left(-\ln(1-\Lambda_{\Delta_{ij}})\right)^{Q}\leq\sum_{i=1}^{n}\rho_i\left(-\ln(1-\tau_{\Delta_{ij}})\right)^{Q}$$

Applying the outer summation and power:

$$\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\Lambda_{\Delta_{ij}})\right)^{Q}\right)\leq\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\tau_{\Delta_{ij}})\right)^{Q}\right)$$

Since e^{-x} is a decreasing function of x:

$$e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\Lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\geq e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\tau_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}$$

Multiplying by -1 and adding 1 reverses the inequality:

$$1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\Lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}\leq 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\tau_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}}$$

This proves the membership degree part. Similarly, since $\lambda_{\Delta_{ij}}\geq\upsilon_{\Delta_{ij}}$, we have $-\ln(\lambda_{\Delta_{ij}})\leq-\ln(\upsilon_{\Delta_{ij}})$, which leads to the second component (neutral degree) of PFSAAWA(Δ) being greater than or equal to that of PFSAAWA(δ). Since $\mu_{\Delta_{ij}}\geq g_{\Delta_{ij}}$, we have $-\ln(\mu_{\Delta_{ij}})\leq-\ln(g_{\Delta_{ij}})$, which leads to the third component (non-membership degree) of PFSAAWA(Δ) being greater than or equal to that of PFSAAWA(δ). By Definition 4, PFSAAWA$(\Delta_{11},\Delta_{12},\ldots,\Delta_{nm})\leq$ PFSAAWA$(\delta_{11},\delta_{12},\ldots,\delta_{nm})$. Hence, the proof is completed.

Theorem 4: *Boundedness*: Consider

$$\Delta^{-}=\left(\min_j\min_i\left\{\Lambda_{\Delta_{ij}}\right\},\max_j\max_i\left\{\lambda_{\Delta_{ij}}\right\},\max_j\max_i\left\{\mu_{\Delta_{ij}}\right\}\right)$$

and

$$\Delta^{+}=\left(\max_j\max_i\left\{\Lambda_{\Delta_{ij}}\right\},\min_j\min_i\left\{\lambda_{\Delta_{ij}}\right\},\min_j\min_i\left\{\mu_{\Delta_{ij}}\right\}\right)$$

Then $\Delta^{-}\leq$ PFSAAWA$(\Delta_{11},\Delta_{12},\ldots,\Delta_{nm})\leq\Delta^{+}$.

Proof. *Boundedness*: Since for all i,j we have $\Delta^{-}\leq\Delta_{ij}\leq\Delta^{+}$, therefore by using the monotonicity property (Theorem 3), we get $\Delta^{-}\leq$ PFSAAWA$(\Delta_{11},\Delta_{12},\ldots,\Delta_{nm})\leq\Delta^{+}$.

Theorem 5: *Shift Invariance*: Consider $\Delta_{ij}=(\Lambda_{\Delta_{ij}},\lambda_{\Delta_{ij}},\mu_{\Delta_{ij}})$ be a set of PFSV. Consider $\alpha=(\Lambda_\alpha,\lambda_\alpha,\mu_\alpha)$ is PFSV then:

$$\text{PFSAAWA}(\Delta_{11}\oplus\alpha,\Delta_{22}\oplus\alpha,\ldots,\Delta_{nm}\oplus\alpha)=\text{PFSAAWA}(\Delta_{11},\Delta_{22},\ldots,\Delta_{nm})\oplus\alpha$$

Proof of **Theorem 5** is provided **Appendix-1**.

Theorem 6: *Homogeneity*: Consider $\Delta_{ij}=(\Lambda_{\Delta_{ij}},\lambda_{\Delta_{ij}},\mu_{\Delta_{ij}})$ be a set of PFSV. If $\Phi>0$, then:

$$\text{PFSAAWA}(\Phi\Delta_{11},\Phi\Delta_{22},\ldots,\Phi\Delta_{nm})=\Phi\text{PFSAAWA}(\Delta_{11},\Delta_{22},\ldots,\Delta_{nm})$$

Proof of **Theorem 6** is provided **Appendix-2**.

Theorem 7: *Generalized Shift Invariance*: Consider $\Delta_{ij}=(\Lambda_{\Delta_{ij}},\lambda_{\Delta_{ij}},\mu_{\Delta_{ij}})$ be a set of PFSV, if $\Phi>0$, $\alpha=(\Lambda_\alpha,\lambda_\alpha,\mu_\alpha)$ be a PFSV. Then

$$\text{PFSAAWA}(\Phi\Delta_{11}\oplus\alpha,\Phi\Delta_{22}\oplus\alpha,\ldots,\Phi\Delta_{nm}\oplus\alpha)=\Phi\text{PFSAAWA}(\Delta_{11},\Delta_{22},\ldots,\Delta_{nm})\oplus\alpha$$

Proof of **Theorem 7** is provided **Appendix-3**.

Theorem 8: *Additivity*: Consider $\Delta_{ij}=(\Lambda_{\Delta_{ij}},\lambda_{\Delta_{ij}},\mu_{\Delta_{ij}})$ and $\alpha_{ij}=(\Lambda_{\alpha_{ij}},\lambda_{\alpha_{ij}},\mu_{\alpha_{ij}})$ be are two sets of PFSV, Then

$$\text{PFSAAWA}(\Delta_{11}\oplus\alpha_{11},\Delta_{22}\oplus\alpha_{22},\ldots,\Delta_{nm}\oplus\alpha_{nm})=\text{PFSAAWA}(\Delta_{11},\Delta_{22},\ldots,\Delta_{nm})\oplus\text{PFSAAWA}(\alpha_{11},\alpha_{22},\ldots,\alpha_{nm})$$

Proof of **Theorem 8** is provided **Appendix-4**.

Definition 8: Consider of PFSVs $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$; $i = 1,2,...,n,\ j = 1,2,...,m$. Then **PFSAAWG** operator is created as follows.

$$\text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) = \bigotimes_{j=1}^{m} (\Delta_{ij})^{\zeta_j} \left(\bigotimes_{i=1}^{n} \Delta_{ij}^{\rho_i} \right)$$

Where the weight vectors are denoted by $\rho_i > 0$, $\sum_{j=1}^{m} \zeta_j = 1$, and $\sum_{i=1}^{n} \rho_i = 1$. We use AA operations on PFVs to prove the following theorem.

Theorem 9. The aggregated value of PFSVs, which is still a PFSV, is produced by the PFSAAWG operator and is given by:

$$\text{PFSAAWG}(\Delta_{11}, \Delta_{12}, \ldots, \Delta_{nm}) = \begin{pmatrix} e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(\Lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}} \\ 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(1-\lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}} \\ 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(1-\mu_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}} \end{pmatrix}$$

Theorem 9 can also be proved using the same method as **Theorem 1**.

Theorem 10: *Idempotency*: Consider that all $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$ where PFVs are collected. If for every $\Delta_{ij} = \Delta$. So $\text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) = \Delta$. **Proof**: Since $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$, $\Delta = (\Lambda_\Delta, \lambda_\Delta, \mu_\Delta)$. We have:

$$\begin{aligned} \text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) &= \begin{pmatrix} e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i (-\ln(\Lambda_\Delta))^{Q}\right)\right)^{1/Q}} \\ 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i (-\ln(1-\lambda_\Delta))^{Q}\right)\right)^{1/Q}} \\ 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i (-\ln(1-\mu_\Delta))^{Q}\right)\right)^{1/Q}} \end{pmatrix} \\ &= \begin{pmatrix} e^{-\left((-\ln(\Lambda_\Delta))^{Q} \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i\right)^{1/Q}} \\ 1 - e^{-\left((-\ln(1-\lambda_\Delta))^{Q} \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i\right)^{1/Q}} \\ 1 - e^{-\left((-\ln(1-\mu_\Delta))^{Q} \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i\right)^{1/Q}} \end{pmatrix} \\ &= \begin{pmatrix} e^{-\left((-\ln(\Lambda_\Delta))^{Q}\right)^{1/Q}} \\ 1 - e^{-\left((-\ln(1-\lambda_\Delta))^{Q}\right)^{1/Q}} \\ 1 - e^{-\left((-\ln(1-\mu_\Delta))^{Q}\right)^{1/Q}} \end{pmatrix} \\ &= \left(e^{-\ln(\Lambda_\Delta)}, 1 - e^{-\ln(1-\lambda_\Delta)}, 1 - e^{-\ln(1-\mu_\Delta)}\right) \\ &= (\Lambda_\Delta, \lambda_\Delta, \mu_\Delta) = \Delta \end{aligned}$$

Hence $\text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) = \Delta$ holds.

Theorem 11: *Boundedness*: Consider that $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$ is a set of PFSV. Consider $\Delta^- = \min(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm})$ and $\Delta^+ = \max(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm})$. Therefore $\Delta^- \leq \text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) \leq \Delta^+$.
Proof: Consider that $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$ characterizes a set of PFSV. Consider that $\Delta^+ = \max(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) = (\Lambda_\Delta^+, \lambda_\Delta^+, \mu_\Delta^+)$ and $\Delta^- = \min(\Delta_1, \Delta_2, \ldots, \Delta_{nm}) = (\Lambda_\Delta^-, \lambda_\Delta^-, \mu_\Delta^-)$. Therefore, the subsequent differences exist: Since $\Lambda_\Delta^- \leq \Lambda_{\Delta_{ij}} \leq \Lambda_\Delta^+$, we have $-\ln(\Lambda_\Delta^-) \geq -\ln(\Lambda_{\Delta_{ij}}) \geq -\ln(\Lambda_\Delta^+)$. Applying the aggregation and exponential functions (where e^{-x} is decreasing):

$$e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(\Lambda_\Delta^-)\right)^{Q}\right)\right)^{1/Q}} \leq e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(\Lambda_{\Delta_{ij}})\right)^{Q}\right)\right)^{1/Q}} \leq e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \rho_i \left(-\ln(\Lambda_\Delta^+)\right)^{Q}\right)\right)^{1/Q}}$$

This simplifies to $\Lambda_\Delta^- \leq \Lambda_{\text{PFSAAWG}} \leq \Lambda_\Delta^+$.

Similarly, for the neutral and non-membership degrees (since $1-x$ is decreasing and $\ln(x)$ is increasing):

$$1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\lambda_\Delta^{+})\right)^{Q}\right)\right)^{1/Q}} \leq \lambda_{\text{PFSAAWG}} \leq 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\lambda_\Delta^{-})\right)^{Q}\right)\right)^{1/Q}}$$

$$1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\mu_\Delta^{+})\right)^{Q}\right)\right)^{1/Q}} \leq \mu_{\text{PFSAAWG}} \leq 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln(1-\mu_\Delta^{-})\right)^{Q}\right)\right)^{1/Q}}$$

By Definition 4, this confirms that $\Delta^{-} \leq \text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm}) \leq \Delta^{+}$.

Theorem 12: *Shift Invariance*: Consider that $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$ characterizes a set of PFVs. If α is PFSV and $\alpha = (\Lambda_\alpha, \lambda_\alpha, \mu_\alpha)$.

$$\text{PFSAAWG}(\Delta_{11}\otimes\alpha, \Delta_{22}\otimes\alpha, \ldots, \Delta_{nm}\otimes\alpha) = \text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm})\otimes\alpha$$

Proof: **Theorem 12** can also be proved using the same method as **Theorem 5**.

Theorem 13: *Homogeneity*: Consider $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$ be an assembly of PFVs. If $r > 0$, then:

$$\text{PFSAAWG}(\Delta_{11}^{r}, \Delta_{22}^{r}, \ldots, \Delta_{nm}^{r}) = \text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm})^{r}$$

Proof: **Theorem 13** can also be proved using the same method as **Theorem 6**.

Theorem 14: *Generalized Shift Invariance*: Consider that $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$, characterizes a set of PFVs. If $\alpha = (\Lambda_\alpha, \lambda_\alpha, \mu_\alpha)$ is PSFV on k and $r > 0$, then:

$$\text{PFSAAWG}(\Delta_{11}^{r}\otimes\alpha, \Delta_{22}^{r}\otimes\alpha, \ldots, \Delta_{nm}^{r}\otimes\alpha) = \text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm})^{r}\otimes\alpha$$

Proof: **Theorem 14** can also be proved using the same method as **Theorem 7**.

Theorem 15: *Multiplicativity*: Consider there are two assemblies of PFSVs: $\Delta_{ij} = (\Lambda_{\Delta_{ij}}, \lambda_{\Delta_{ij}}, \mu_{\Delta_{ij}})$ and $\alpha_{ij} = (\Lambda_{\alpha_{ij}}, \lambda_{\alpha_{ij}}, \mu_{\alpha_{ij}})$. Then

$$\text{PFSAAWG}(\Delta_{11}\otimes\alpha_{11}, \Delta_{22}\otimes\alpha_{22}, \ldots, \Delta_{nm}\otimes\alpha_{nm}) = \text{PFSAAWG}(\Delta_{11}, \Delta_{22}, \ldots, \Delta_{nm})\otimes\text{PFSAAWG}(\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nm})$$

Proof: **Theorem 15** can also be proved using the same method as **Theorem 8**.

4 Implementation of the Suggested Method for MCDM.

Some specialists argue that **MCDM** plays a key role in narrowing down the best choice from a pool of viable options. A team of evaluators assesses each alternative, giving their **PFRV**-based input for every attribute according to predefined criteria. Their judgments are then compiled into DMs, weighted by each expert's influence. Finally, the combined data, factoring in attribute weights, is used to calculate an overall score for each option. The MCDM technique has great utility in many fields. It is extensively utilized in a variety of disciplines, such as business, economics, mathematics, and engineering.

This section presents an MCDM model that addresses DM issues by utilizing the AHP, and developed operators in combination with PF data. Let's examine a group of options represented by $\{I_1, I_2, \ldots, I_r\}$. Every alternative is a potential choice that needs to be considered. These options are evaluated based on a set of characteristics $\{M_1, M_2, \ldots, M_r\}$ where each property is associated with a certain criterion, such sustainability, cost, or performance, with weights $W_\mu \in [0,1]$, $\mu = 1, 2, \ldots, h$ so that $\sum_{\mu=1}^{h} W_\mu = 1$. The choices are evaluated by a single DM based on these characteristics. The MCDM model's goal in this framework is to identify the optimal choice among $\{Y_1, Y_2, \ldots, Y_r\}$. The selecting process's specific steps are listed below.

Table 2: The Value of Scale AHP

Scale	Definition
1	Equally Important
3	Weakly Important
5	Strongly Important
7	Very Strongly Important
9	Extremely Important
2, 4, 6, 8	Intermediate values between adjacent scales

4.1 Analytical Hierarchy Process

The derived power AOs are included into an AHP-based approach shown in this section. The Saaty (5–9) preference scale, as indicated in Table 2, is used to determine the pairwise comparison matrix of criteria in the AHP method, which we use to determine the weights of the criteria. The consistency index is then used to analyze the calculated weights' consistency.

The following describes the AHP technique's step-by-step process.

Step 1: Create the problem's hierarchical structure, including the primary criteria and the sub-criteria for assessing the options.

Step 2: Using the data in **Table 1**, create a comparison matrix and establish a pairwise comparison of the criteria.

Step 3: Create a **normalized decision matrix** $C_{\mathbf{norm}}$ by applying the expression from Eq. (5) to normalize the comparison values of the decision matrix $C_{n\times n}$.

$$e_{ij} = \frac{c_{ij}}{\sum_{j=1}^{n} c_{ij}}, \quad i,j = 1,2,\ldots,n \tag{5}$$

In other words, each normalized entry is produced by dividing each column j entry by the total number of entries in that column. Each column's total entries in the normalized decision matrix add up to 1.

Step 4: Utilizing Eq. (6), determine the **weights of the criteria** by averaging the values of each row of the normalized decision matrix.

$$\omega = \frac{\sum_{j=1}^{n} e_{ij}}{n} \tag{6}$$

Consequently, a weight vector ω that satisfies the normalcy condition is produced as a column vector in the manner described below:

Step 5: Build the matrix $C\omega$.

Step 6: Use the formula in Eq. (7) to calculate the **maximum Eigenvalue**.

$$\tau_{\max} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{(C\omega)_i}{\omega_i}\right) \tag{7}$$

Step 7: Use the following formula to determine the **consistency index**:

$$\mathrm{CI} = \frac{\tau_{\max} - n}{n-1} \tag{8}$$

A higher consistency index value indicates a greater departure from consistency, while a smaller value suggests that the DM comparative values may be consistent and that the weights that result are suitable for obtaining useful estimations. The DM comparisons are regarded as perfectly consistent when the consistency index is zero, or $\mathrm{CI} = 0$.

Step 8: By dividing the consistency index by the random index, you can find the **consistency ratio** as follows:

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}} \tag{9}$$

Where, as indicated in **Table 3**, **RI** is the random index that is defined for various values of n.

It is acceptable and the weights are consistent if the consistency ratio value is less than 0.10 ($\mathrm{CR} < 0.10$). If the consistency ratio is higher than 0.10, the comparison matrix will be inconsistent, and the AHP weights might not produce the right and significant outcomes.

Table 3: Random Index for Different Values of n

n	2	3	4	5	6	7	8	9	10
RI	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

4.2 TOPSIS method

Our **PF TOPSIS** method, which is based on the idea that the best option should be at the furthest distance from the PFNIS and the closest to the PFPIS, is designed to solve the aforementioned **MCDM** problem with PFNs.

Step 1: In PFNs, the significance of the experts is regarded as a linguistic term. Consider the PFN for the ξ-th expert's rating to be $\Lambda_\xi, \lambda_\xi, \mu_\xi$. The ξ-th expert's weight can then be determined by:

$$\varphi_\xi = \frac{\Lambda_\xi + \pi_\xi \left(\frac{\Lambda_\xi}{\Lambda_\xi+\lambda_\xi+\mu_\xi}\right)}{\sum_{\xi=1}^{n}\left(\Lambda_\xi + \pi_\xi \left(\frac{\Lambda_\xi}{\Lambda_\xi+\lambda_\xi+\mu_\xi}\right)\right)} \tag{10}$$

Fulfilling the normalized condition. In this case, $\sum_{\xi=1}^{n} \varphi_\xi = 1$ and $\pi_\xi = \sqrt{1-\Lambda_\xi-\lambda_\xi-\mu_\xi}$. **Step 2**: The matrices' data is ready for aggregation following normalization. To aggregate the attributes separately, the **PFSAAWA/PFSAAWG operator** is used.
Step 3: Every criterion is equally significant. Then no need to normalize.
Step 4: Consider J_1 and J_2 represent the benefit-type and cost-type criteria collections, respectively. One can obtain the PFPIS ($\mathbf{V}^+$) and PFNIS ($\mathbf{V}^-$) as follows:

$$V_j^+ = \begin{cases} \max_{1\le i\le n} & \text{for benefit criteria} \\ \min_{1\le i\le n} & \text{for cost criteria} \end{cases} \quad (11)$$

$$V_j^- = \begin{cases} \min_{1\le i\le n} & \text{for benefit criteria} \\ \max_{1\le i\le n} & \text{for cost criteria} \end{cases} \quad (12)$$

The distance to the PFPIS (S_j^+) and PFNIS (S_j^-) are calculated:

$$S_j^+ = \sqrt{\sum_{j=1}^{n}\left(\text{Dist}(V, V_j^+)\right)^2} \quad (13)$$

$$S_j^- = \sqrt{\sum_{j=1}^{n}\left(\text{Dist}(V, V_j^-)\right)^2} \quad (14)$$

Step 5: One can compute an alternative's **relative closeness index** with the PFPIS $\mathbf{V}^+$ as follows:

$$C_j^+ = \frac{S_j^-}{S_j^+ + S_j^-} \quad (15) \tag{11}$$

Step 6: The options are arranged in descending order of alternatives after the revised closeness index is calculated, and the option with the highest revised closeness index is the most appropriate option for the issue.

The flowchart of the developed MCDM scheme is shown in **Figure 2**.

Example 2: Green Industry Project Selection

The objective of this DM process is to determine which green industry project best fits a company's social responsibility, environmental sustainability, and economic goals. The organization is dedicated to encouraging environmentally friendly behaviors, guaranteeing financial success, and fostering technological advancement. It aims to pick a project that

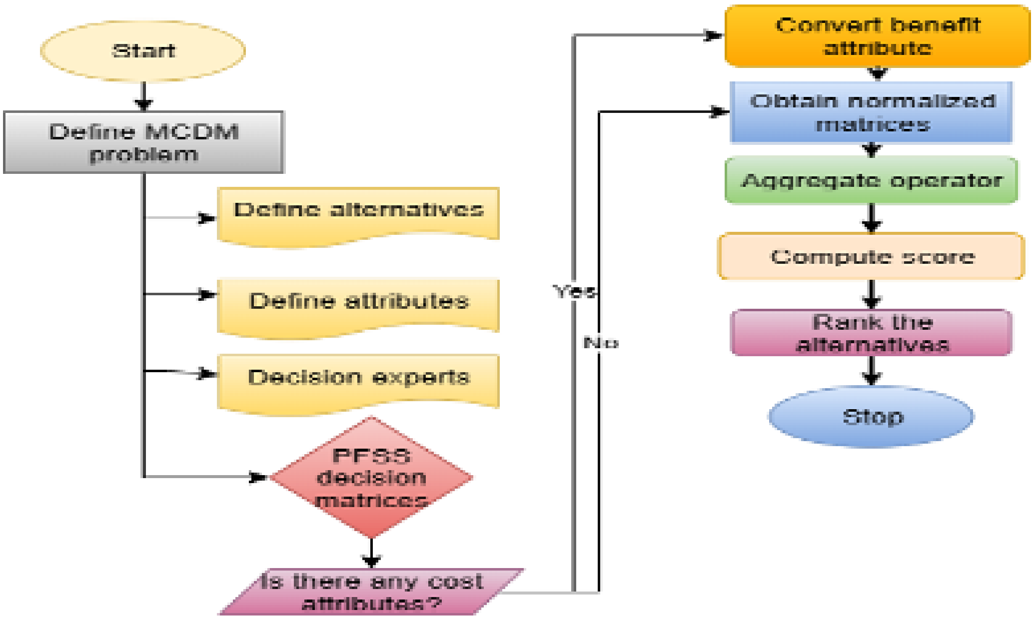

Fig. 2: The expected MCDM scheme flowchart

advances green industry practices and offers financial advantages in addition to the latter goal. The organization assesses possible projects according to important criteria like technological innovation, social benefits, environmental impact, and economic viability in order to accomplish this. The ultimate choice will affect the company's capacity to provide profitable and sustainable solutions while preserving operational effectiveness.

Assume a business that intends to finance a project related to the green industry. The organization is thinking about a number of possible projects and chooses to evaluate each one's suitability before selecting one. The business assigns three professionals with backgrounds in social policy, environmental science, and economics to evaluate the possible projects in order to guarantee a comprehensive assessment.

Criteria/Alternatives for Evaluation:

–**Economic Viability** (I_1): This metric evaluates the project's overall cost-effectiveness, taking into account the project's initial outlay, ongoing expenses, and anticipated return on investment. It has a direct effect on the company's ability to sustain its finances.

–**Environmental Impact** (I_2): The project's capacity to cut pollution, conserve resources, and cut emissions is essential to achieving the company's environmental goals.

–**Social Benefits** (I_3): These consist of the project's capacity to create jobs, support local development, and conform to environmental regulations.

–**Technological Innovation** (I_4): To sustain a competitive edge and guarantee long-term success, the project's utilization of cutting-edge, sustainable technologies and its potential for future innovation are essential.

–**Green Building Construction** (I_5): The goal of green building initiatives is to design and build structures that are resource-, energy-, and environmentally conscious at every stage of their lives. This includes utilizing energy-efficient technologies, sustainable materials, and building design techniques that lessen a structure's carbon footprint.

We assess four possible green industry projects in the following manner (using criteria M_i which are the alternatives to be evaluated):

–**Previous Performance History** (M_1): The success history of each project is assessed, as this is essential for forecasting future dependability.

- **–Projected Sustainability** (M_2): This takes into account the project's long-term viability, possible long-term environmental and social benefits, and other factors.
- **–Potential Risks and Challenges** (M_3): This section examines potential risks related to the project, including but not limited to technological obsolescence and regulatory changes.
- **–Comparative Analysis** (M_4): Compares each project's relative strengths and weaknesses to those of initiatives that are similar in the industry.
- **–Regulatory Compliance** (M_5): This feature evaluates how well the project complies with regional, governmental, and global laws and guidelines. Adherence to environmental laws and regulations is imperative in order to prevent legal complications and guarantee that the project functions within the necessary legal parameters.

These standards are used by the experts to evaluate possible projects. The assessments are given as PFSVs, which indicate the level of hesitation, MD, AD, and NMD. The following tables provide an overview of the information gathered from the experts:

4.3 Criteria Weights by the AHP Method

First, the **AHP** technique is used to determine the weights of the criteria. The pairwise comparison of criteria is built using the Saaty (5–9) preference scale as shown in **Table 4**.

Table 4: The Pairwise Comparison of Criteria

	I_1	I_2	I_3	I_4	I_5
I_1	1	3	7	5	3
I_2	0.33	1	4	6	4
I_3	0.14	0.25	1	0.2	0.14
I_4	0.20	0	5	1	1
I_5	0.33	0	7	1	1

All criteria have a benefit type; they not need to be normalized. Next, using Eq. (10) to calculate the criteria weights, the weights are given in the column weight vector **W** as follows:

$$\mathbf{W} = \{0.4275, 0.2876, 0.0393, 0.1055, 0.1401\}$$

Table 3 is used to obtain the consistency index, CI $= 0.1616$, and the random index, RI $= 1.12$ (for $n = 5$), is used to calculate the consistency ratio. The provided comparison matrix demonstrates consistent behavior, and the computed weights are suitable for DM, as the consistency ratio is 0.1443, which is less than 0.10. *(Note: The calculated CR of 0.1443 is greater than 0.10, indicating inconsistency according to the final statement, but the text states it is suitable for DM.)*

Sept 1: Determining the Expert Weights

Table 5 describes the linguistic terms used to rate the relative importance of experts and criteria and uses Eq. (10).

Table 5: Linguistic Factors for Experts' and Criteria's Relative Importance Ratings

Linguistic term	MD (Λ)	AD (λ)	NMD (μ)
Very good (VG)	0.82	0.07	0.13
Good (G)	0.71	0.18	0.22
Median (M)	0.43	0.22	0.25
Bad (B)	0.34	0.27	0.35

Table 6: The Importance of Experts and Their Weights

Linguistic term	Weights
M	0.1982
G	0.1910
VG	0.2735
B	0.1391
M	0.1982

Table 7: Linguistic Factors for the Alternatives' Relative Importance Rating

Linguistic term	MD	AD	NMD
Extremely Good (EG)	0.96	0.0	0.0
Very Very Good (VVG)	0.95	0.01	0.02
Very Good (VG)	0.81	0.05	0.03
Good (G)	0.75	0.12	0.09
Medium Good (MG)	0.66	0.15	0.12
Medium (M)	0.56	0.17	0.18
Medium Bad (MB)	0.42	0.23	0.21
Bad (B)	0.36	0.26	0.24
Very Bad (VB)	0.24	0.31	0.32
Very Very Bad (VVB)	0.16	0.41	0.39

Table 8: Five Criteria by the Expert (Linguistic Ratings)

Alternative	Expert	M_1	M_2	M_3	M_4	M_5
I_1	$\wp^{(1)}$	EG	VVB	VVB	EG	VVG
	$\wp^{(2)}$	MB	VVB	VVG	M	EG
	$\wp^{(3)}$	MB	EG	M	EG	M
	$\wp^{(4)}$	M	VG	EG	M	MB
I_2	$\wp^{(1)}$	VB	VB	VVG	M	G
	$\wp^{(2)}$	M	MG	B	VVB	M
	$\wp^{(3)}$	MB	G	VVB	VVB	VVB
	$\wp^{(4)}$	G	G	VVB	VG	VVB
I_3	$\wp^{(1)}$	G	VG	VG	VB	MG
	$\wp^{(2)}$	VVB	G	MG	B	VVB
	$\wp^{(3)}$	VVB	VG	B	B	VG
	$\wp^{(4)}$	B	MG	B	G	VG
I_4	$\wp^{(1)}$	MB	MG	MG	VVG	MG
	$\wp^{(2)}$	VB	M	VG	VB	B
	$\wp^{(3)}$	B	G	VB	MB	G
	$\wp^{(4)}$	VVB	B	B	EG	G
I_5	$\wp^{(1)}$	B	M	MB	G	B
	$\wp^{(2)}$	G	EG	G	M	MB
	$\wp^{(3)}$	MG	VG	M	M	EG
	$\wp^{(4)}$	VB	M	G	B	MB

Table 9: Picture Fuzzy Decision Matrix of the Legal Advice (Aggregated PFVs)

Alternative	PFSAAWA			PFSAAWG		
	MD	AD	NMD	MD	AD	NMD
I_1	0.6981	0.1118	0.1021	0.4988	0.1873	0.1758
I_2	0.6034	0.1622	0.1505	0.4118	0.2353	0.2222
I_3	0.6441	0.1350	0.1167	0.4587	0.2098	0.1935
I_4	0.6004	0.1585	0.1369	0.4710	0.2052	0.1905
I_5	0.5898	0.1742	0.1610	0.3952	0.2520	0.2381

Step 2: Aggregation

The matrices' data is ready for aggregation following normalization. To aggregate the attributes separately, the **PFSAAWA/PFSAAWG** operator is used.

Step 3: Normalization Check

Every criterion may be equally significant. Then no need to normalize.

Step 4: PFPIS and PFNIS

Represent the benefit-type and cost-type criteria collections, respectively in Eq. (11-12).

Table 10: Picture Fuzzy Positive and Negative Ideal Solutions (PFPIS/PFNIS)

Ideal Solution	PFSAAWA			PFSAAWG		
	MD	AD	NMD	MD	AD	NMD
V^+	0.6981	0.1118	0.1021	0.4988	0.1873	0.1758
V^-	0.6004	0.1622	0.1505	0.4118	0.2353	0.2222

Step 5 and 6: Closeness Index and Ranking

One can compute an alternative's relative closeness index about the PFPIS Eq. (11).

Table 11: Numerical Results for PFSAAWA Operator

Alternative	$\mathbf{S_j^+}$	$\mathbf{S_j^-}$	$\mathbf{C_j^+}$	Ranking
I_1	0.0699	0.0977	0.5830	1
I_2	0.0947	0.0699	0.4249	3
I_3	0.0692	0.0516	0.4274	2
I_4	0.0987	0.0582	0.3710	4

According to **Table 11**, the ranking result is given as follows:

$$I_1 \succ I_3 \succ I_2 \succ I_4$$

Table 12: Numerical Results for PFSAAWG Operator

Alternative	$\mathbf{S_j^+}$	$\mathbf{S_j^-}$	$\mathbf{C_j^+}$	Ranking
I_1	0.0668	0.0870	0.5654	1
I_2	0.0870	0.0668	0.4346	4
I_3	0.0555	0.0550	0.4977	3
I_4	0.0518	0.0636	0.5509	2

According to **Table 12**, the ranking result is given as follows:

$$I_1 \succ I_4 \succ I_3 \succ I_2$$

Step 6 (Conclusion): According to **Table 11-12**, $\mathbf{I_1}$ (Economic Viability) is the best option because it has the maximum score value compared to all supplementary options. Additionally, the ranking outcome is revealed in **Figure 3**.

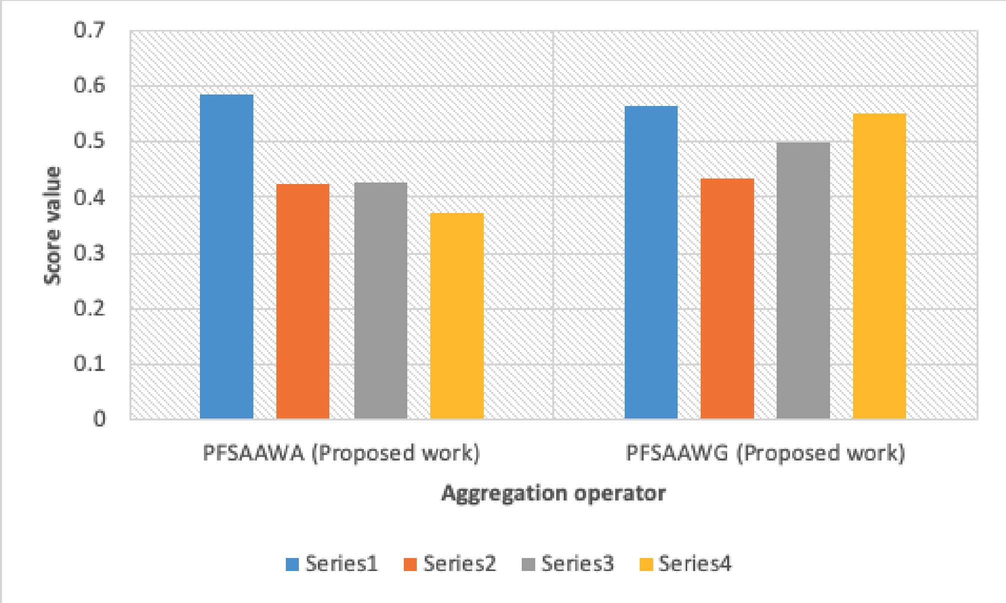

Fig. 3: Result of PFSAAWA and PFSAAWG operators

4.4 Sensitivity Analysis

An integral component of any **MCDM** problem is the weight vector. To determine the **robustness** of our ranking results and to confirm the validity of the selected DM method, we perform a **post-sensitivity analysis** in this subsection. Therefore, we alter the original weight vector per two schemes: (1) raising the weight of each attribute by 0.05, and (2) lowering the weight of each attribute by 0.05. To maintain the normalization of the modified weight vector in both cases, the weights of the remaining criteria are modified using the equation (12) proposed by Memariani et al. [62].

$$\omega_c = \left(\frac{1-\omega_c}{1-\omega_*}\right)\omega_* \tag{12}$$

Where ω_* represents the changed weight, ω_c is the resultant attribute weight, and ω_c is the attribute's associated weight.

Table 15–20 shows the computed weight values for the expected strategy. These weights form the basis for assessing the intended MCDM approach's sensitivity. Ten different sensitivity analysis scenarios were carried out in order to evaluate the DM results' robustness. **Table 15–20** lists the associated score values and the alternative rankings that result for each scenario.

According to the data presented in **Table 10**, each scenario entails a ± 0.05 perturbation of a single attribute weight while keeping the other weights constant. The analysis shows that the alternatives' ranking is extremely stable across all ten tested scenarios, with a consistent ... *(The sentence cuts off here)*.

The validity and strength of the suggested aggregation operators are examined in this section through a thorough sensitivity analysis. The goal is to see how changing the expert and parameter weights affects the ranking results. **Table 13** lists the weights of the experts, and **Table 14** lists the weights of the parameters.

The following procedures are used to carry out the sensitivity analysis:

1. First, we modify the parameter weights (**Table 14**) while maintaining the expert weights (**Table 13**). Tables 15 and 16 present the findings.
2. Next, we modify the expert weights (**Table 13**) while maintaining the parameter weights constant (**Table 14**). Tables 17 and 18 present these findings.
3. Lastly, we jointly adjust the expert and parameter weights; Tables 19 and 20 present the outcomes.

Equation (12) is used to calculate all weight values.

Impact of Experts' Weights

The alternatives are ranked differently when the expert weights are altered, while the parameter weights stay the same. This demonstrates how expert opinions have an impact on the ranking results. Different results can result from even minor adjustments to the expert influence.

Impact of Parameter Weights

The ranking results stay the same when the expert weights are held constant while the parameter weights are altered. This suggests that when only the parameters' relative importance is changed and the expert influence is left unaltered, the rankings are more stable.

Impact of Change on Both Weights

The ranking results also change when the expert and parameter weights are altered simultaneously. This demonstrates that when both kinds of weights are changed simultaneously, the model accurately depicts changes.

Table 13: The Expert's Weights

Scenario	S_1	S_2	S_3	S_4	S_5
$S_{(1+)}$	0.2482	0.1791	0.2564	0.1304	0.1858
$S_{(1-)}$	0.1482	0.2029	0.2906	0.1478	0.2106
$S_{(2+)}$	0.2410	0.1860	0.2566	0.1305	0.1860
$S_{(2-)}$	0.1410	0.2104	0.2904	0.1477	0.2104
$S_{(3+)}$	0.3235	0.1846	0.1779	0.1295	0.1846
$S_{(3-)}$	0.2235	0.2118	0.2041	0.1487	0.2118
$S_{(4+)}$	0.1891	0.1867	0.1799	0.2576	0.1867
$S_{(4-)}$	0.0891	0.2097	0.2021	0.2894	0.2097
$S_{(5+)}$	0.2482	0.1858	0.1791	0.2564	0.1304
$S_{(5-)}$	0.1482	0.2106	0.2029	0.2906	0.1478

Table 14: The Weights Parameter

Scenario	M_1	M_2	M_3	M_4	M_5
$S_{(1+)}$	0.4775	0.2625	0.0359	0.0963	0.1279
$S_{(1-)}$	0.3775	0.3127	0.0427	0.1147	0.1523
$S_{(2+)}$	0.3376	0.3975	0.0365	0.0981	0.1303
$S_{(2-)}$	0.2376	0.4575	0.0421	0.1129	0.1499
$S_{(3+)}$	0.0893	0.4053	0.2726	0.1000	0.1328
$S_{(3-)}$	-0.0107	0.4497	0.3026	0.1110	0.1474
$S_{(4+)}$	0.1555	0.4036	0.2715	0.0371	0.1323
$S_{(4-)}$	0.0555	0.4514	0.3037	0.0415	0.1479
$S_{(5+)}$	0.1901	0.4026	0.2709	0.0370	0.0994
$S_{(5-)}$	0.0901	0.4524	0.3043	0.0416	0.1116

A comparison of the two outcomes shows that when the parameter weights are changed, the alternatives' overall ranking order shifts. This illustrates how sensitive the model is to changes in the parameters, emphasizing that the ultimate choice is based on the relative weights given to the parameters as well as expert opinions.

While **PFSAAWA** focuses the joint effect and lessens the significance of larger values, **PFSAAWG** stresses the average performance of characteristics. Therefore, differences in the final ranking are a natural result of their aggregation principles.

Table 15: The Result of the PFSAAWA Operator (Parameter Weights Modified)

Scenario	$\mathscr{P}^{(1)}$	$\mathscr{P}^{(2)}$	$\mathscr{P}^{(3)}$	$\mathscr{P}^{(4)}$	Ranking
$S_{(1+)}$	0.5859	0.4232	0.4001	0.3763	$I_1 \succ I_2 \succ I_3 \succ I_4$
$S_{(1-)}$	0.5802	0.4337	0.4708	0.3707	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(2+)}$	0.5882	0.5006	0.5662	0.4034	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(2-)}$	0.5021	0.4862	0.5774	0.3944	$I_3 \succ I_1 \succ I_2 \succ I_4$
$S_{(3+)}$	0.4022	0.5212	0.6112	0.3888	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(3-)}$	0.3622	0.5132	0.6175	0.3825	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(4+)}$	0.4196	0.5595	0.6100	0.3900	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(4-)}$	0.3678	0.5531	0.6164	0.3836	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(5+)}$	0.4254	0.5591	0.6062	0.3938	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(5-)}$	0.3682	0.5522	0.6128	0.3872	$I_3 \succ I_2 \succ I_4 \succ I_1$

Table 16: The Result of the PFSAAWG Operator (Parameter Weights Modified)

Scenario	$\mathscr{P}^{(1)}$	$\mathscr{P}^{(2)}$	$\mathscr{P}^{(3)}$	$\mathscr{P}^{(4)}$	Ranking
$S_{(1+)}$	0.5679	0.4321	0.4703	0.5262	$I_1 \succ I_4 \succ I_3 \succ I_2$
$S_{(1-)}$	0.5654	0.4346	0.5292	0.5702	$I_4 \succ I_1 \succ I_3 \succ I_2$
$S_{(2+)}$	0.5354	0.4352	0.5648	0.5848	$I_4 \succ I_3 \succ I_1 \succ I_2$
$S_{(2-)}$	0.4524	0.4253	0.5747	0.5867	$I_4 \succ I_3 \succ I_1 \succ I_2$
$S_{(3+)}$	0.3635	0.4097	0.6289	0.4201	$I_3 \succ I_4 \succ I_2 \succ I_1$
$S_{(3-)}$	0.3620	0.4206	0.6380	0.4371	$I_3 \succ I_4 \succ I_2 \succ I_1$
$S_{(4+)}$	0.3649	0.4139	0.6299	0.3959	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(4-)}$	0.3631	0.4280	0.6369	0.4021	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(5+)}$	0.3655	0.4183	0.6288	0.3989	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(5-)}$	0.3638	0.4339	0.6362	0.4052	$I_3 \succ I_2 \succ I_4 \succ I_1$

Table 17: The Result of the PFSAAWA Operator (Expert Weights Modified)

Scenario	$\mathscr{P}^{(1)}$	$\mathscr{P}^{(2)}$	$\mathscr{P}^{(3)}$	$\mathscr{P}^{(4)}$	Ranking
$S_{(1+)}$	0.5840	0.4160	0.4554	0.3657	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(1-)}$	0.5841	0.4470	0.4010	0.3837	$I_1 \succ I_2 \succ I_3 \succ I_4$
$S_{(2+)}$	0.5826	0.4174	0.4516	0.3668	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(2-)}$	0.5798	0.4502	0.3914	0.3859	$I_1 \succ I_2 \succ I_3 \succ I_4$
$S_{(3+)}$	0.6017	0.4032	0.5194	0.3594	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(3-)}$	0.6053	0.4293	0.5032	0.3740	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(4+)}$	0.5875	0.4127	0.4567	0.3800	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(4-)}$	0.5721	0.4492	0.3877	0.4078	$I_1 \succ I_2 \succ I_4 \succ I_3$
$S_{(5+)}$	0.5996	0.4004	0.5008	0.3895	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(5-)}$	0.5865	0.4135	0.4444	0.4091	$I_1 \succ I_3 \succ I_2 \succ I_4$

Table 18: The Result of the PFSAAWG Operator (Expert Weights Modified)

Scenario	$\mathscr{P}^{(1)}$	$\mathscr{P}^{(2)}$	$\mathscr{P}^{(3)}$	$\mathscr{P}^{(4)}$	Ranking
$S_{(1+)}$	0.5676	0.4324	0.5293	0.5682	$I_4 \succ I_1 \succ I_3 \succ I_2$
$S_{(1-)}$	0.5659	0.4341	0.4638	0.5253	$I_1 \succ I_4 \succ I_3 \succ I_2$
$S_{(2+)}$	0.5661	0.4339	0.5280	0.5693	$I_4 \succ I_1 \succ I_3 \succ I_2$
$S_{(2-)}$	0.5640	0.4360	0.4616	0.5274	$I_1 \succ I_4 \succ I_3 \succ I_2$
$S_{(3+)}$	0.5563	0.4286	0.5775	0.5823	$I_4 \succ I_3 \succ I_1 \succ I_2$
$S_{(3-)}$	0.5515	0.4485	0.5659	0.5613	$I_3 \succ I_4 \succ I_1 \succ I_2$
$S_{(4+)}$	0.5755	0.4245	0.5586	0.4627	$I_1 \succ I_3 \succ I_4 \succ I_2$
$S_{(4-)}$	0.5756	0.4244	0.5001	0.4158	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(5+)}$	0.5762	0.4238	0.5592	0.5318	$I_1 \succ I_3 \succ I_4 \succ I_2$
$S_{(5-)}$	0.5767	0.4233	0.5139	0.4738	$I_1 \succ I_3 \succ I_4 \succ I_2$

Table 19: The Result of the PFSAAWA Operator (Both Weights Modified)

Scenario	$\mathscr{P}^{(1)}$	$\mathscr{P}^{(2)}$	$\mathscr{P}^{(3)}$	$\mathscr{P}^{(4)}$	Ranking
$S_{(1+)}$	0.5874	0.4126	0.4211	0.3694	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(1-)}$	0.5804	0.4556	0.4350	0.3823	$I_1 \succ I_2 \succ I_3 \succ I_4$
$S_{(2+)}$	0.5742	0.4645	0.5600	0.3878	$I_1 \succ I_3 \succ I_2 \succ I_4$
$S_{(2-)}$	0.5062	0.5621	0.5682	0.4199	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(3+)}$	0.3748	0.4470	0.6252	0.3638	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(3-)}$	0.3726	0.5074	0.6161	0.3752	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(4+)}$	0.4313	0.5406	0.6103	0.3897	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(4-)}$	0.5088	0.6045	0.5917	0.4083	$I_2 \succ I_3 \succ I_1 \succ I_4$
$S_{(5+)}$	0.4077	0.4387	0.6213	0.3787	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(5-)}$	0.4548	0.5279	0.6070	0.3930	$I_3 \succ I_2 \succ I_1 \succ I_4$

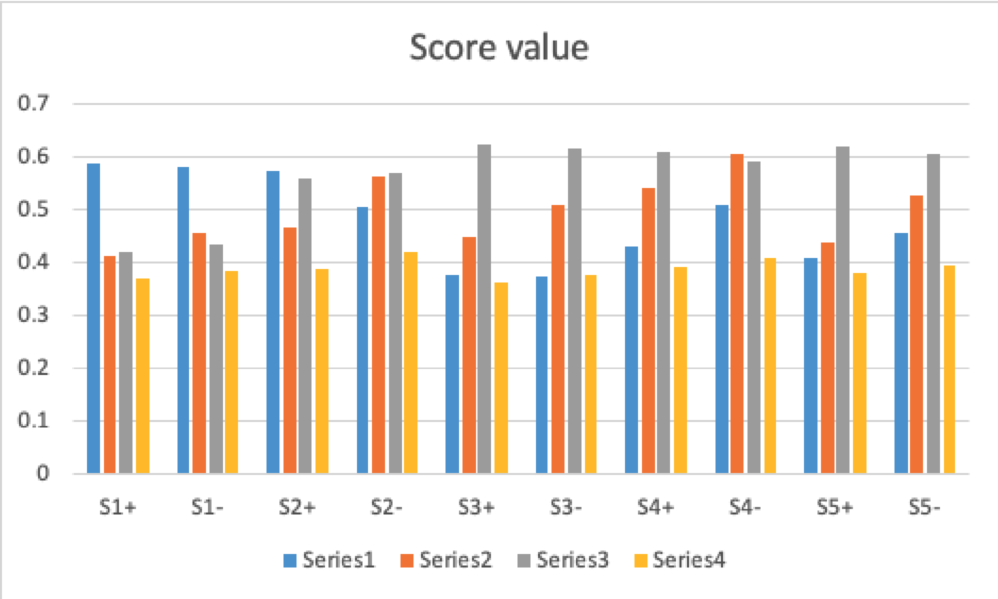

Fig. 4: The value of PFSAAWA operator

Table 20: The Result of the PFSAAWG Operator (Both Weights Modified)

Scenario	$\mathscr{P}^{(1)}$	$\mathscr{P}^{(2)}$	$\mathscr{P}^{(3)}$	$\mathscr{P}^{(4)}$	Ranking
$S_{(1+)}$	0.5691	0.4309	0.4968	0.5471	$I_1 \succ I_4 \succ I_3 \succ I_2$
$S_{(1-)}$	0.5648	0.4352	0.4893	0.5495	$I_1 \succ I_4 \succ I_3 \succ I_2$
$S_{(2+)}$	0.5163	0.4284	0.5716	0.5873	$I_4 \succ I_3 \succ I_1 \succ I_2$
$S_{(2-)}$	0.4726	0.4375	0.5625	0.5862	$I_4 \succ I_3 \succ I_1 \succ I_2$
$S_{(3+)}$	0.3742	0.4141	0.6258	0.5032	$I_3 \succ I_4 \succ I_2 \succ I_1$
$S_{(3-)}$	0.3682	0.4268	0.6318	0.4460	$I_3 \succ I_4 \succ I_2 \succ I_1$
$S_{(4+)}$	0.3756	0.4119	0.6152	0.3848	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(4-)}$	0.4433	0.5172	0.6134	0.3866	$I_3 \succ I_2 \succ I_1 \succ I_4$
$S_{(5+)}$	0.3756	0.3959	0.6244	0.3877	$I_3 \succ I_2 \succ I_4 \succ I_1$
$S_{(5-)}$	0.3805	0.4365	0.6177	0.3823	$I_3 \succ I_2 \succ I_4 \succ I_1$

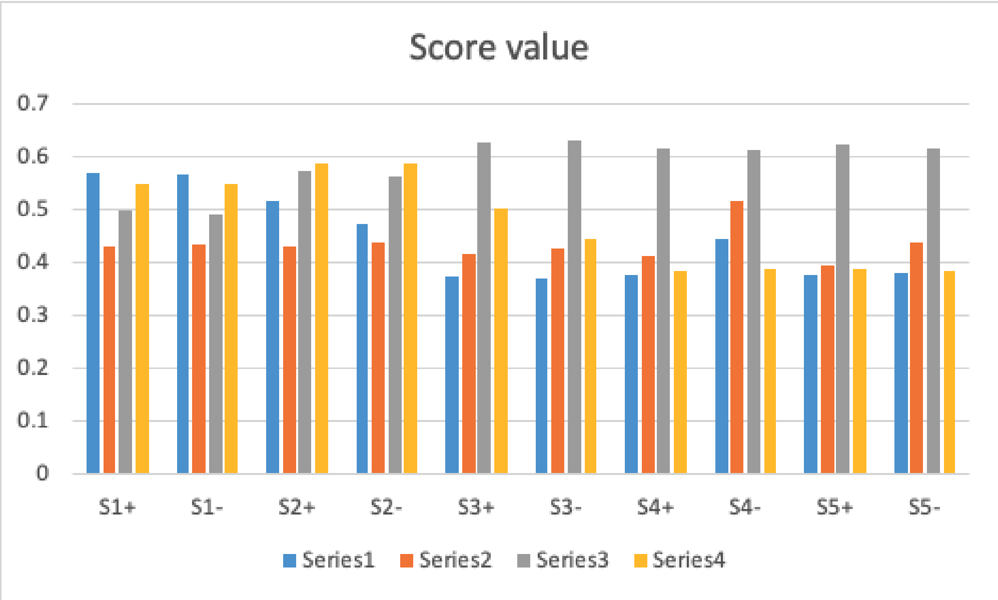

Fig. 5: The value of PFSAAWG operator

Because the two operators (average and geometric) handle information differently and because attribute or expert weights affect the aggregate, the results vary. These variations are normal and show how sensitive and adaptable the suggested approach is. However, the results are still in line with the operators' theoretical underpinnings. All things considered, this demonstrates the strength of our methodology and validates that it yields dependable and efficient outcomes for real-world applications.

5 Comparative Studies

To evaluate the efficacy and relative advantages of the proposed MCDM method, a comprehensive comparative analysis is conducted in this section, incorporating both quantitative and qualitative dimensions against selected existing approaches. The subsequent subsections outline the comparative procedures in detail.

5.1 Quantitative Analysis

The obtained results are compared with current methods in the **PFSS** context, including geometric **Arora** [63] operator and intuitionistic fuzzy soft weighted Einstein averaging. **Ali et al.** [64] expanded the concepts of **IFSAAWA** and **IFSAAWG**. **Jan et al.** [24] introduced the AOs operator on **MAGDM** with **PFSWA** and **PFSWG**. **Table 21** shows the outcomes. This table shows that the outcomes produced by existing techniques are identical to those of a targeted strategy. In this paper, evaluation information is combined using the two AOs **PFSAAWA** and **PFSAAWG**, and the candidates are then rated using the score function.

The outcomes from the various operators defined for the PFSS framework are displayed in **Table 21**. **Table 21** makes it evident that the ranking outcomes are the same *[Note: The rankings are not all identical, but $\mathscr{P}^{(1)}$ is consistently ranked first]*. It draws attention to how crucial the suggested operator is. The most flexible operational laws, the **AATRM** and **AATCRM**, are the foundation upon which the suggested operator is built. As such, the proposed technique provides more flexible results than the existing operators. **Figure 6** displays the comparative study's geometric representation in the following manner.

Table 21: Comparative Studies

Method	Score values $\mathscr{P}^{(1)}$	$\mathscr{P}^{(2)}$	$\mathscr{P}^{(3)}$	$\mathscr{P}^{(4)}$	Ranking
PFSAAWA (Proposed)	0.5830	0.4249	0.4274	0.3710	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(2)} \succ \mathscr{P}^{(4)}$
PFSAAWG (Proposed)	0.5654	0.4346	0.4977	0.5509	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(4)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(2)}$
IFSAAWA [64]	0.6690	0.3417	0.4259	0.2631	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(2)} \succ \mathscr{P}^{(4)}$
IFSAAWG [64]	0.6559	0.4109	0.3936	0.6080	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(4)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(2)}$
IFSWEA [63]	0.2814	0.0569	0.0638	0.1543	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(4)} \succ \mathscr{P}^{(2)}$
IFSWEG [63]	0.3415	0.2265	0.3002	0.2057	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(4)} \succ \mathscr{P}^{(2)}$
PFSWA [24]	0.3941	0.1782	0.3723	0.2944	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(4)} \succ \mathscr{P}^{(2)}$
PFSWG [24]	0.4612	0.1966	0.4212	0.3412	$\mathscr{P}^{(1)} \succ \mathscr{P}^{(3)} \succ \mathscr{P}^{(4)} \succ \mathscr{P}^{(2)}$

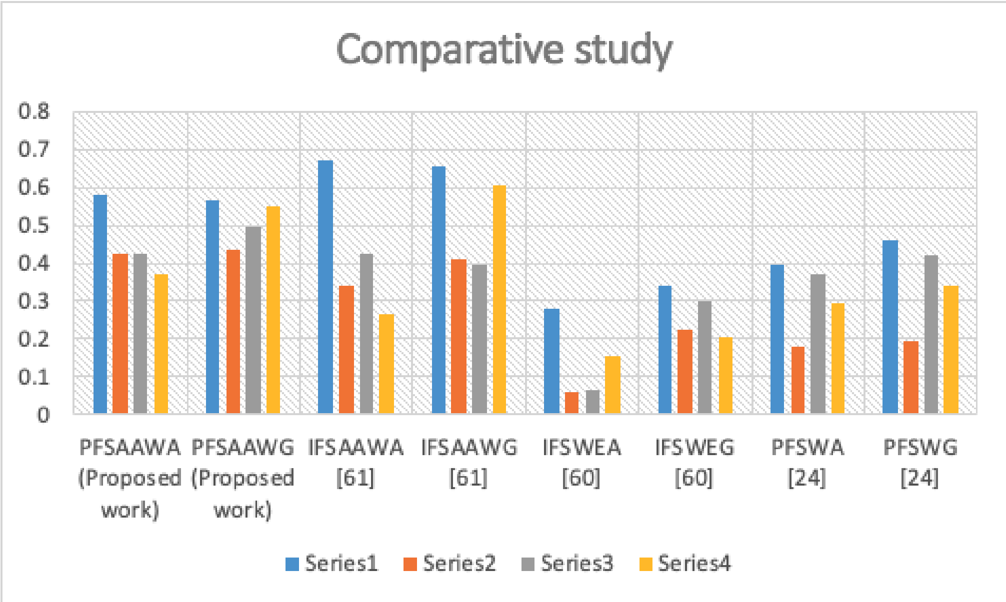

Fig. 6: The Comparison with various methods

6 Conclusion

This study addresses **DM** problems involving **PFSS** data, wherein a panel of experts expresses preferences for each alternative across multiple criteria. To resolve such problems through geometric and averaging techniques, we propose two novel operators **PFSAAWA** and **PFSAAWG** constructed on the basis of **AA operations**. The desirable properties of these operators are examined, and a DM framework grounded in their application is subsequently developed. To demonstrate the practicality and effectiveness of the proposed approach, an illustrative example is provided, followed by a comparative analysis against existing methods.

For the purposes of this study, we assume that the attributes of a given set are mutually independent; however, it is recognized that such attributes are frequently interdependent in real-world contexts. The proposed operators are formulated upon the **AATRM** and **AATCRM**, which are widely regarded as highly flexible operational laws. Consequently, these operators offer substantial versatility for the aggregation of information represented in the form of **PFSS values (PFSVs)**. This adaptability equips decision-makers with more robust and flexible tools to address complex DM scenarios.

6.1 Limitations

- –The efficacy of the **AA operators** relies on the appropriate choice of parameters, such as Q, even though they offer flexibility in modeling uncertainty under the **PFSS environment**. In practical applications, the results could become unstable or less significant if these values are not carefully selected.
- –All evaluation criteria are assumed to be **independent** in the current framework for DM. Nonetheless, certain characteristics might have an impact on one another in actual sustainable industry situations. The model's capacity to accurately represent the intricacy of expert judgments may be diminished if these relationships are not taken into consideration.
- –**Subjective bias** in weight assignment may affect the final ranking because the suggested approach relies on expert opinions.
- –The study uses **TOPSIS** and **AHP** in a particular decision-making setting, which restricts the results' applicability to other fields.

6.2 Future Research Directions

- –The flexibility and applicability of the methodology can be increased by further investigating extended FS models, such as **T-SFSS** and **complex T-SFSS (CT-SFSS)**.
- –Incorporating **Archimedean-based aggregation operators** into future work could also be a goal, as this would enable more flexible and general modeling of expert opinions in the picture bipolar fuzzy environment.
- –Furthermore, incorporating different aggregation mechanisms like the **power Maclaurin symmetric mean operator** and the **Bonferroni mean operator** could provide fresh perspectives and increase the method's range of possible applications.
- –The model can be modified and tested in domains such as **smart manufacturing**, **green logistics**, and **industrial sustainability evaluation frameworks** for wider applicability in real-world systems.
- –In order to overcome those challenges, future research will expand these techniques to incorporate other fuzzy frameworks, such as **t-spherical fuzzy hypersoft set theory**, and **complex spherical sets**. We intend to use those contemporary methods to address real-world issues in a variety of fields, such as artificial intelligence, game principles, scientific prediction, green dealer selection, and more.

Appendix-1: Proof of Theorem 5 (Shift Invariance)

Theorem 5: $\text{PFSAAWA}(\Delta_{11}\oplus\alpha,\Delta_{22}\oplus\alpha,\ldots,\Delta_{nm}\oplus\alpha) = \text{PFSAAWA}(\Delta_{11},\Delta_{22},\ldots,\Delta_{nm})\oplus\alpha$

Proof: First, we use the definition of $\Delta_{ij}\oplus\alpha$:

$$\Delta_{ij}\oplus\alpha = \begin{pmatrix} 1-e^{-\left((-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}+(-\ln(1-\Lambda_{\alpha}))^{Q}\right)^{1/Q}} \\ e^{-\left((-\ln\lambda_{\Delta_{ij}})^{Q}+(-\ln\lambda_{\alpha})^{Q}\right)^{1/Q}} \\ e^{-\left((-\ln\mu_{\Delta_{ij}})^{Q}+(-\ln\mu_{\alpha})^{Q}\right)^{1/Q}} \end{pmatrix}$$

Now, we calculate $\text{PFSAAWA}(\Delta_{11}\oplus\alpha,\Delta_{22}\oplus\alpha,\ldots,\Delta_{nm}\oplus\alpha)$. We use the property $-\ln(1-(1-e^{-x}))=x$ and the aggregation formula (Theorem 1).

The MD component is:

$$\begin{aligned}\Lambda_{\text{PFSAAWA}(\Delta\oplus\alpha)} &= 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln\left(1-\Lambda_{\Delta_{ij}\oplus\alpha}\right)\right)^{Q}\right)\right)^{1/Q}} \\ &= 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln\left(e^{-\left((-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}+(-\ln(1-\Lambda_{\alpha}))^{Q}\right)^{1/Q}}\right)\right)^{Q}\right)\right)^{1/Q}} \\ &= 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left((-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}+(-\ln(1-\Lambda_{\alpha}))^{Q}\right)\right)\right)^{1/Q}}\end{aligned}$$

Since $\sum_{j=1}^{m}\zeta_j = 1$ and $\sum_{i=1}^{n}\rho_i = 1$, we can separate the sum:

$$\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_\alpha))^Q = (-\ln(1-\Lambda_\alpha))^Q\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i = (-\ln(1-\Lambda_\alpha))^Q$$

Thus,

$$\Lambda_{\text{PFSAAWA}(\Delta\oplus\alpha)} = 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^Q+(-\ln(1-\Lambda_\alpha))^Q\right)^{1/Q}}$$

The NMD component is:

$$\begin{aligned}
\mu_{\text{PFSAAWA}(\Delta\oplus\alpha)} &= e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln\left(\mu_{\Delta_{ij}\oplus\alpha}\right)\right)^Q\right)\right)^{1/Q}}\\
&= e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\rho_i\left(-\ln\left(e^{-\left((-\ln\mu_{\Delta_{ij}})^Q+(-\ln\mu_\alpha)^Q\right)^{1/Q}}\right)\right)^Q\right)\right)^{1/Q}}\\
&= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left((-\ln\mu_{\Delta_{ij}})^Q+(-\ln\mu_\alpha)^Q\right)\right)^{1/Q}}\\
&= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^Q+(-\ln\mu_\alpha)^Q\right)^{1/Q}}
\end{aligned}$$

The neutral degree component $\lambda_{\text{PFSAAWA}(\Delta\oplus\alpha)}$ is derived similarly.

Now, we calculate $\text{PFSAAWA}(\Delta_{11},\ldots,\Delta_{nm})\oplus\alpha$. Let $R=\text{PFSAAWA}(\Delta_{11},\ldots,\Delta_{nm})=(\Lambda_R,\lambda_R,\mu_R)$. The MD of $R\oplus\alpha$ is:

$$\begin{aligned}
\Lambda_{R\oplus\alpha} &= 1-e^{-\left((-\ln(1-\Lambda_R))^Q+(-\ln(1-\Lambda_\alpha))^Q\right)^{1/Q}}\\
&= 1-e^{-\left(\left(-\ln\left(1-\left(1-e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^Q\right)^{1/Q}}\right)\right)\right)^Q+(-\ln(1-\Lambda_\alpha))^Q\right)^{1/Q}}\\
&= 1-e^{-\left(\left(-\ln\left(e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^Q\right)^{1/Q}}\right)\right)^Q+(-\ln(1-\Lambda_\alpha))^Q\right)^{1/Q}}\\
&= 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^Q+(-\ln(1-\Lambda_\alpha))^Q\right)^{1/Q}}
\end{aligned}$$

The NMD of $R\oplus\alpha$ is:

$$\begin{aligned}
\mu_{R\oplus\alpha} &= e^{-\left((-\ln\mu_R)^Q+(-\ln\mu_\alpha)^Q\right)^{1/Q}}\\
&= e^{-\left(\left(-\ln\left(e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^Q\right)^{1/Q}}\right)\right)^Q+(-\ln\mu_\alpha)^Q\right)^{1/Q}}\\
&= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^Q+(-\ln\mu_\alpha)^Q\right)^{1/Q}}
\end{aligned}$$

Since the expressions for $\text{PFSAAWA}(\Delta_{11}\oplus\alpha,\ldots,\Delta_{nm}\oplus\alpha)$ and $\text{PFSAAWA}(\Delta_{11},\ldots,\Delta_{nm})\oplus\alpha$ are identical for all three components, the theorem holds.

Appendix-2: Proof of Theorem 6 (Homogeneity)

Theorem 6: $\text{PFSAAWA}(\Phi\Delta_{11},\Phi\Delta_{22},\ldots,\Phi\Delta_{nm}) = \Phi\text{PFSAAWA}(\Delta_{11},\Delta_{22},\ldots,\Delta_{nm})$

Proof: First, we calculate $\text{PFSAAWA}(\Phi\Delta_{11},\Phi\Delta_{22},\ldots,\Phi\Delta_{nm})$. Using the definition of $\Phi\Delta$:

$$\Phi\Delta_{ij} = \left(1-e^{-\left(\Phi(-\ln(1-\Lambda_{\Delta_{ij}}))^Q\right)^{1/Q}}, e^{-\left(\Phi(-\ln\lambda_{\Delta_{ij}})^Q\right)^{1/Q}}, e^{-\left(\Phi(-\ln\mu_{\Delta_{ij}})^Q\right)^{1/Q}}\right)$$

The MD component of PFSAAWA$(\Phi\Delta_{ij})$ is:

$$\begin{aligned}
\Lambda_{\text{PFSAAWA}(\Phi\Delta)} &= 1 - e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left(-\ln\left(1-\Lambda_{\Phi\Delta_{ij}}\right)\right)^{Q}\right)^{1/Q}} \\
&= 1 - e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left(-\ln\left(e^{-\left(\Phi(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}}\right)\right)^{Q}\right)^{1/Q}} \\
&= 1 - e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left(\Phi(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)\right)^{1/Q}} \\
&= 1 - e^{-\left(\Phi\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}}
\end{aligned}$$

The NMD component of PFSAAWA$(\Phi\Delta_{ij})$ is:

$$\begin{aligned}
\mu_{\text{PFSAAWA}(\Phi\Delta)} &= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left(-\ln\left(\mu_{\Phi\Delta_{ij}}\right)\right)^{Q}\right)^{1/Q}} \\
&= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left(-\ln\left(e^{-\left(\Phi(-\ln\mu_{\Delta_{ij}})^{Q}\right)^{1/Q}}\right)\right)^{Q}\right)^{1/Q}} \\
&= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left(\Phi(-\ln\mu_{\Delta_{ij}})^{Q}\right)\right)^{1/Q}} \\
&= e^{-\left(\Phi\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^{Q}\right)^{1/Q}}
\end{aligned}$$

The neutral degree component is derived similarly.

Now, we calculate ΦPFSAAWA$(\Delta_{11},\ldots,\Delta_{nm})$. Let $R=$ PFSAAWA$(\Delta_{11},\ldots,\Delta_{nm})$. The MD of ΦR is:

$$\begin{aligned}
\Lambda_{\Phi R} &= 1 - e^{-\left(\Phi(-\ln(1-\Lambda_R))^{Q}\right)^{1/Q}} \\
&= 1 - e^{-\left(\Phi\left(-\ln\left(1-\left(1-e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}}\right)\right)\right)^{Q}\right)^{1/Q}} \\
&= 1 - e^{-\left(\Phi\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}}
\end{aligned}$$

The NMD of ΦR is:

$$\begin{aligned}
\mu_{\Phi R} &= e^{-\left(\Phi(-\ln\mu_R)^{Q}\right)^{1/Q}} \\
&= e^{-\left(\Phi\left(-\ln\left(e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^{Q}\right)^{1/Q}}\right)\right)^{Q}\right)^{1/Q}} \\
&= e^{-\left(\Phi\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^{Q}\right)^{1/Q}}
\end{aligned}$$

Since the expressions are identical for all three components, the theorem holds.

Appendix-3: Proof of Theorem 7 (Generalized Shift Invariance)

Theorem 7: PFSAAWA$(\Phi\Delta_{11}\oplus\alpha,\Phi\Delta_{22}\oplus\alpha,\ldots,\Phi\Delta_{nm}\oplus\alpha)=\Phi$PFSAAWA$(\Delta_{11},\Delta_{22},\ldots,\Delta_{nm})\oplus\alpha$

Proof: From Theorem 5 and Theorem 6, we have the property $\Phi(\Delta\oplus\alpha)=\Phi\Delta\oplus\Phi\alpha$ (which is not needed here, but shows AA structure).

We use the result from Appendix-2:

$$\text{PFSAAWA}(\Phi\Delta_{11},\ldots,\Phi\Delta_{nm}) = R' = (\Lambda_{R'},\lambda_{R'},\mu_{R'})$$

where

$$\Lambda_{R'} = 1 - e^{-\left(\Phi \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i (-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}}$$

Now, we calculate ΦPFSAAWA$(\Delta_{11},\dots,\Delta_{nm}) \oplus \alpha = R' \oplus \alpha$. The MD component of $R' \oplus \alpha$ is:

$$\Lambda_{R'\oplus\alpha} = 1 - e^{-\left((-\ln(1-\Lambda_{R'}))^{Q} + (-\ln(1-\Lambda_{\alpha}))^{Q}\right)^{1/Q}}$$

Using the same steps as in the proof of Theorem 5, we get:

$$\Lambda_{R'\oplus\alpha} = 1 - e^{-\left(\Phi \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i (-\ln(1-\Lambda_{\Delta_{ij}}))^{Q} + (-\ln(1-\Lambda_{\alpha}))^{Q}\right)^{1/Q}}$$

And the NMD component is:

$$\mu_{R'\oplus\alpha} = e^{-\left(\Phi \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i (-\ln \mu_{\Delta_{ij}})^{Q} + (-\ln \mu_{\alpha})^{Q}\right)^{1/Q}}$$

Next, we calculate PFSAAWA$(\Phi\Delta_{11} \oplus \alpha, \dots, \Phi\Delta_{nm} \oplus \alpha)$. Using the AA operation for $\Phi\Delta_{ij} \oplus \alpha$:

$$\Phi\Delta_{ij} \oplus \alpha = \begin{pmatrix} 1 - e^{-\left((-\ln(1-\Lambda_{\Phi\Delta_{ij}}))^{Q} + (-\ln(1-\Lambda_{\alpha}))^{Q}\right)^{1/Q}} \\ e^{-\left((-\ln \lambda_{\Phi\Delta_{ij}})^{Q} + (-\ln \lambda_{\alpha})^{Q}\right)^{1/Q}} \\ e^{-\left((-\ln \mu_{\Phi\Delta_{ij}})^{Q} + (-\ln \mu_{\alpha})^{Q}\right)^{1/Q}} \end{pmatrix}$$

The MD of PFSAAWA$(\Phi\Delta_{ij} \oplus \alpha)$ is:

$$\begin{aligned}
\Lambda_{\text{PFSAAWA}(\Phi\Delta\oplus\alpha)} &= 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i \left(-\ln\left(1-\Lambda_{\Phi\Delta_{ij}\oplus\alpha}\right)\right)^{Q}\right)^{1/Q}} \\
&= 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i \left(-\ln\left(e^{-\left((-\ln(1-\Lambda_{\Phi\Delta_{ij}}))^{Q} + (-\ln(1-\Lambda_{\alpha}))^{Q}\right)^{1/Q}}\right)\right)^{Q}\right)^{1/Q}} \\
&= 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i \left((-\ln(1-\Lambda_{\Phi\Delta_{ij}}))^{Q} + (-\ln(1-\Lambda_{\alpha}))^{Q}\right)\right)^{1/Q}}
\end{aligned}$$

Substitute the term $(-\ln(1-\Lambda_{\Phi\Delta_{ij}}))^{Q}$:

$$(-\ln(1-\Lambda_{\Phi\Delta_{ij}}))^{Q} = \left(-\ln\left(e^{-\left(\Phi(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}\right)^{1/Q}}\right)\right)^{Q} = \Phi(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}$$

So,

$$\Lambda_{\text{PFSAAWA}(\Phi\Delta\oplus\alpha)} = 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i \left(\Phi(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q} + (-\ln(1-\Lambda_{\alpha}))^{Q}\right)\right)^{1/Q}}$$

Which, after separating the α term, is identical to $\Lambda_{R'\oplus\alpha}$. The NMD component is proven similarly.

Appendix-4: Proof of Theorem 8 (Additivity)

Theorem 8: PFSAAWA$(\Delta_{11} \oplus \alpha_{11}, \Delta_{22} \oplus \alpha_{22}, \dots, \Delta_{nm} \oplus \alpha_{nm}) =$ PFSAAWA$(\Delta_{11},\dots,\Delta_{nm}) \oplus$ PFSAAWA$(\alpha_{11},\dots,\alpha_{nm})$

Proof: First, calculate PFSAAWA$(\Delta_{11} \oplus \alpha_{11}, \dots, \Delta_{nm} \oplus \alpha_{nm})$. The MD component is:

$$\begin{aligned}
\Lambda_{\text{PFSAAWA}(\Delta\oplus\alpha)} &= 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i \left(-\ln\left(1-\Lambda_{\Delta_{ij}\oplus\alpha_{ij}}\right)\right)^{Q}\right)^{1/Q}} \\
&= 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i \left((-\ln(1-\Lambda_{\Delta_{ij}}))^{Q} + (-\ln(1-\Lambda_{\alpha_{ij}}))^{Q}\right)\right)^{1/Q}} \\
&= 1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i (-\ln(1-\Lambda_{\Delta_{ij}}))^{Q} + \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{n} \rho_i (-\ln(1-\Lambda_{\alpha_{ij}}))^{Q}\right)^{1/Q}}
\end{aligned}$$

The NMD component is:

$$\begin{aligned}\mu_{\mathrm{PFSAAWA}(\Delta\oplus\alpha)} &= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left(-\ln\left(\mu_{\Delta_{ij}\oplus\alpha_{ij}}\right)\right)^{Q}\right)^{1/Q}} \\ &= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i\left((-\ln\mu_{\Delta_{ij}})^{Q}+(-\ln\mu_{\alpha_{ij}})^{Q}\right)\right)^{1/Q}} \\ &= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^{Q}+\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\alpha_{ij}})^{Q}\right)^{1/Q}}\end{aligned}$$

Next, calculate $\mathrm{PFSAAWA}(\Delta_{11},\ldots,\Delta_{nm})\oplus\mathrm{PFSAAWA}(\alpha_{11},\ldots,\alpha_{nm}) = R_1\oplus R_2$. The MD component of $R_1\oplus R_2$ is:

$$\begin{aligned}\Lambda_{R_1\oplus R_2} &= 1-e^{-\left((-\ln(1-\Lambda_{R_1}))^{Q}+(-\ln(1-\Lambda_{R_2}))^{Q}\right)^{1/Q}} \\ &= 1-e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\Delta_{ij}}))^{Q}+\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln(1-\Lambda_{\alpha_{ij}}))^{Q}\right)^{1/Q}}\end{aligned}$$

The NMD component of $R_1\oplus R_2$ is:

$$\begin{aligned}\mu_{R_1\oplus R_2} &= e^{-\left((-\ln\mu_{R_1})^{Q}+(-\ln\mu_{R_2})^{Q}\right)^{1/Q}} \\ &= e^{-\left(\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\Delta_{ij}})^{Q}+\sum_{j=1}^{m}\zeta_j\sum_{i=1}^{n}\rho_i(-\ln\mu_{\alpha_{ij}})^{Q}\right)^{1/Q}}\end{aligned}$$

Since the expressions are identical for all three components, the theorem holds.

Regarding conflicts of interest

The writers affirm that they have none.

Data availability statement

Since no new data were generated or examined for this study, data sharing is not relevant to this article.

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Conflict of Interest

The author(s) declare no conflict of interest.
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