Solutions of Nonlinear Oscillators by Iteration Perturbation Method

A. M. El-Naggar and G. M. Ismail

1 Department of Mathematics, Faculty of Science, Benha University, Egypt
2 Department of Mathematics, Faculty of Science, Sohag University, 82524, Sohag, Egypt

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Abstract: In this paper, the iteration perturbation method is applied to solve nonlinear oscillations. Two examples are given to illustrate the effectiveness and convenience of this iteration procedure. Comparison with the numerical solutions is also presented, revealing that this iteration leads to accurate solutions.

Keywords: Iteration perturbation method, Nonlinear oscillators, Duffing oscillator, Van der Pol oscillator.

1 Introduction

The most common and most widely studied methods for determining analytical approximate solutions of a nonlinear oscillatory system are the perturbation methods. These methods involve the expansion of a solution to an oscillation equation in a series in a small parameter. Several researchers have studied different nonlinear problems by means of iteration procedures [1, 2, 3, 4, 5, 6, 7, 8].

The purpose of this paper is to apply the iteration procedure to determine analytical approximate solutions to the nonlinear oscillation equation. With this procedure, the analytical approximate period and the corresponding periodic solutions, valid for small as well as large amplitudes of oscillation, can be obtained. The nonlinear Duffing and Van der Pol oscillations will be taken as examples to illustrate the applicability and accuracy of the iteration procedure.

2 The iteration procedure

Consider a nonlinear conservative oscillator described as

$$\ddot{x} + f(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0,$$

where $f(x)$ is a nonlinear function and has the property

$$f(-x) = -f(x).$$

Eq. (1) can be rewritten as

$$\ddot{x} + \omega^2 x = \omega^2 x - f(x),$$

where the constant $\omega$ is a priori unknown frequency of the periodic solution $x(t)$ being sought. The original Mickens procedure is given as [1].

$$\ddot{x}_k + \omega^2 x_k = g(\omega, x_{k-1}), \quad k = 1, 2, \ldots$$

where the input of starting function is

$$x_0(t) = A \cos \omega t.$$

This iteration scheme was used to solve many nonlinear oscillating equations [9, 10, 11].

Lim et al. [3] proposed a modified iteration scheme

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = g(\omega, x_{k-1}) + g_x(\omega, x_{k-1})(x_k - x_{k-1}), \quad k = 0, 1, 2, \ldots$$

with the imputes of starting functions as

$$x_{-1}(t) = x_0(t) = A \cos \omega t.$$

where $g_x(\omega, x) = \partial g(\omega, x)/\partial x$. The modified procedure was also applied to solve many nonlinear oscillators [12, 13, 14, 15].

Chen and Liu [5] proposed a new iteration scheme, considering $\omega$ as $\omega_k$:

$$\ddot{x}_k + \omega^2 x_k = g(\omega_{k-1}, x_{k-1}), \quad k = 1, 2, \ldots$$

* Corresponding author e-mail: gamalm2010@yahoo.com

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where the right hand side of Eq (8) can be expanded in the Fourier series

\[ g(\omega_{k-1}, x_{k-1}) = \sum_{i=1}^{\varphi(k)} a_{k-1,i}(\omega_{k-1}) \cos(i\omega_{k-1}t), \]

(9)

where the coefficient \(a_{k-1,j}\) are functions of \(\omega_{k-1}\) and \(\varphi(k)\) is a positive integer. The \(k\)-th order approximation \(\omega_{k-1}\) is obtained by eliminating the so-called secular terms, i.e., letting

\[ a_{k-1,i}(\omega_{k-1}) = 0, \quad k = 1, 2, \ldots \]

(10)

Eq (10) is always a linear algebraic equation in \(\omega_{k-1}^2\).

J. H. He [7] proposed a new iteration scheme considering the following nonlinear oscillator

\[ \ddot{x} + \chi f(x, \dot{x}) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \]

(11)

We rewrite Eq (11) in the following form

\[ \ddot{x} + \chi \ddot{x}g(x, \dot{x}) = 0 \]

(12)

where \(g(x, \dot{x}) = f(x, \dot{x})/x\).

J. H. He has constructed an iteration formula for the above equation

\[ \ddot{x}_{k+1} + \chi \ddot{x}_{k+1} + \chi \ddot{x}_{k+1} + \chi \ddot{x}_{k+1} + \chi \ddot{x}_{k+1} = 0, \quad k = 1, 2, \ldots \]

(13)

Marinca and Herisanu [8] proposed a new iteration method by combining Mickens and He’s iteration methods, considering the following nonlinear oscillator

\[ \ddot{x} + \Omega^2 x = f(x, \dot{x}, \ddot{x}) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0 \]

(14)

We rewrite Eq. (14) in the following form

\[ \ddot{x} + \Omega^2 x = x \left( \Omega^2 - \omega^2 - \frac{f(x, \dot{x}, \ddot{x})}{x} \right) := x g(x, \dot{x}, \ddot{x}). \]

(15)

where \(\Omega^2\) is a priori unknown frequency of the periodic solution \(x(t)\) being sought.

The proposed iteration scheme is

\[ x_{n+1} + \Omega^2 x_{n+1} = x_{n-1} \left[ g(x_{n-1}, \dot{x}_{n-1}, \ddot{x}_{n-1}) + g(x_{n-1}, \dot{x}_{n-1}, \ddot{x}_{n-1}) + g(x_{n-1}, \dot{x}_{n-1}, \ddot{x}_{n-1}) \right] \]

(16)

where the inputs of starting functions are [3]

\[ x_{n-1}(t) = x_0(t) = A \cos \Omega t. \]

(17)

It is further required that for each \(n\), the solution to Eq. (16) is to satisfy initial conditions

\[ x_n(0) = A, \quad \dot{x}_n(0) = 0, \quad n = 1, 2, 3, \ldots \]

(18)

Note that, for given \(x_{n-1}(t)\) and \(x_n(t)\) Eq. (16) is a second order inhomogeneous differential equation for \(x_{n-1}(t)\). Its right side can be expanded into the following Fourier series:

\[ x_{n-1} = a_1(A, \Omega, \omega) \cos \Omega t + b_1(A, \Omega, \omega) \sin \Omega t + \sum_{n=2}^{N} a_n(A, \Omega, \omega) \cos n\Omega t + \sum_{n=2}^{N} b_n(A, \Omega, \omega) \sin n\Omega t, \]

(19)

where the coefficients \(a_n(A, \Omega, \omega)\) and \(b_n(A, \Omega, \omega)\) are known functions of \(A\) and \(\Omega\), and the integer \(N\) depends upon the function \(g(x, \dot{x}, \ddot{x})\) on the right hand side of Eq. (15). In view of Eq. (19), the solution to Eq. (16) is taken to be

\[ x_{n+1} = A \cos \Omega t - \sum_{n=2}^{N} \frac{a_n(A, \Omega, \omega)}{(n^2-1)\Omega^2} (\cos n\Omega t - \cos \Omega t) - \sum_{n=2}^{N} \frac{b_n(A, \Omega, \omega)}{(n^2-1)\Omega^2} (\sin n\Omega t - \sin \Omega t), \]

(20)

where \(A\) is, tentatively, an arbitrary constant. In Eq. (20), the particular solution is chosen such that it contains no secular terms needs

\[ a_1(A, \Omega, \omega) = 0, \quad b_1(A, \Omega, \omega) = 0. \]

(21)

Eq. (21) allows the determination of the frequency \(\Omega\) as a function of \(A\) and \(\omega\). This procedure can be performed to any desired iteration step \(n\).

3 Applications

In order to illustrate the remarkable accuracy of this iteration, we compare the approximate results with numerical integration results for the following two examples.

3.1 Duffing oscillator with high nonlinearity

Consider the following nonlinear Duffing equation with high nonlinearity, which models many structural systems, it is regarded as one of the most important differential equations because it appears in various physical and engineering problems such that, nonlinear optics and plasma physics [16, 17].

\[ \ddot{x} + x + \alpha x^3 + \beta x^5 + \gamma x^7 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \]

(22)

where \(x\) is displacement and, \(\alpha\), \(\beta\) and \(\gamma\) are arbitrary constants.

We rewrite Eq. (12) in the form

\[ \ddot{x}_1 + \alpha_1 x_1 = x_0(\alpha_1 x^2 - \beta_1 x^4 - \gamma_1 x^6), \]

(23)

where \(g(x, \dot{x}, \ddot{x}) = (\omega^2 - 1 - \alpha x^2 - \beta x^4 - \gamma x^6)\) and the inputs of the starting function are \(x_{-1}(t) = x_0(t) = A \cos \omega t\).
The first iteration is given by the equation
\[ \dot{x}_1 + \omega^2 x_1 = - \left( \frac{64A + 48A^3A + 40A^4A + 35A^4y^7 - 64A^2}{64} \right) \cos \omega t \]
\[ - \left( \frac{16A^3 + 20A^4A + 21A^4y^7}{64} \right) \cos 3\omega t \]
\[ - \left( \frac{4A^3 + 7A^4y^7}{64} \right) \cos 5\omega t - \left( \frac{y^7}{64} \right) \cos 7\omega t. \]
(24)

No secular terms in \( x_1 \) requires that
\[ \omega = \omega_1 = \sqrt{1 + \frac{3}{4}A^2 + \frac{5}{8}B^4 + \frac{35}{64}C^4}. \]
(25)

This equation is identical to Eq. (10) in Ref [16] and Eq. (25) in Ref [17]. Solving Eq. (24) with initial conditions (18), \( x_1 \) is obtained as
\[ x_1 = A \cos \omega t + \left( \frac{16A^3 + 20A^4A + 21A^4y^7}{5120} \right) \cos 3\omega t - \cos \omega t \]
\[ + \left( \frac{4A^3 + 7A^4y^7}{15568} \right) \cos 5\omega t - \cos \omega t \]
\[ + \left( \frac{y^7}{30720} \right) \cos 7\omega t - \cos \omega t, \]
(26)

for \( n = 1 \) into Eq. (16) with the initial functions (18) and \( x_1 \) given by Eq. (26) we obtain the following differential equation for \( x_2 \)
\[ \dot{x}_2 + \omega^2 x_2 = - \left( A + \frac{3A^4}{4} + \frac{5A^4}{8} + \frac{35A^4}{64} \right) \frac{\omega^2 A^4}{2560} + \frac{21A^4A}{2560} \cos \omega t \]
\[ - \left( \frac{184A^4}{1280} \right) \frac{\omega^2 A^4}{2560} - \left( \frac{15A^4}{6} - \frac{21A^4y^7}{64} \right) \frac{\omega^2 A^4}{2560} \cos 3\omega t + \frac{35A^4y^7}{64} \frac{\omega^2 A^4}{2560} \cos 5\omega t \]
\[ + \frac{35A^4y^7}{20480} \frac{\omega^2 A^4}{2560} + \frac{7680A^4}{327680} \frac{\omega^2 A^4}{2560} + \frac{15A^4y^7}{327680} \frac{\omega^2 A^4}{2560} + \frac{35A^4y^7}{327680} \frac{\omega^2 A^4}{2560} + \frac{7680A^4}{327680} \frac{\omega^2 A^4}{2560} \cos 7\omega t \]
\[ x \cos 7\omega t - \left( \frac{\beta A^4}{15360} + \frac{9\beta A^4}{7680} + \frac{27\beta A^4}{122880} + \frac{225\beta A^4}{327680} \right) \cos 9\omega t \]
\[ - \left( \frac{\beta A^4}{30720} + \frac{9\beta A^4}{8192} \right) \cos 11\omega t - \frac{\omega^2 A^4}{2560} \cos 13\omega t. \]
(27)

The absence of secular term gives the following equation for \( \omega^4 \)
\[ \omega^4 - \left( \frac{64 + 64A^4 - 40A^4 + 15y^7}{2560} \right) \omega^2 + \left( \frac{786432A^4 + 2046384A^4}{2560} \right) = 0. \]
(28)

Solving Eq. (28) for \( \omega \) yields
\[ \omega = \omega_2 = \sqrt{\frac{64 + 48A^4 + 40A^4 + 35y^6}{128} + \frac{\Delta_1 + \Delta_2}{64\sqrt{6}}}, \]
(29)

where
\[ \Delta_1 = 6144 + 9216A^2 + 2688A^2A^4 \]
\[ + 7680A^4 + 744A^2A^3; \]
\[ \Delta_2 = 1056A^2A^4 + 6720A^4 + 2844A^8 \]
\[ + 1240A^8 + 195\gamma^2. \]
(30)

Fig. 1 shows a comparison between the present solution obtained from formulae (29) and (30) and the numerical integration results obtained by using the Runge-Kutta method. From the results presented here and

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the results of Ref. [16], it is shown that the present results are in a good agreement with those presented in Ref. [16].

3.2 Autonomous modified Van der Pol oscillator

One of the classical equations of non-linear dynamics was formulated by Dutch physicist Van der Pol. Originally it was a model for an electrical circuit with a triode valve, and was later extensively studied as a host of a rich class of dynamical behavior, including relaxation oscillations, quasi periodicity, elementary bifurcations, and chaos [18,19]. A modified Van der Pol oscillator has been proposed to describe a self-excited body sliding on a periodic potential. This autonomous modified Van der Pol oscillator is described by the following equation [20].

\[ \dot{x} + x + \varepsilon (x^2 - 1) \dot{x} + P \sin x = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \]  

(31)

In this case we have \( g(x, \dot{x}, \ddot{x}) = \omega^2 - 1 - \frac{\varepsilon (x^2 - 1) \dot{x} - P \sin x}{x_0} \) and \( x_{-1}(t) = x_0(t) = A \cos \omega t. \) The first iteration can be written in the form

\[ \dot{x}_1 + \omega^2 x_1 = x_0 \left( \omega^2 - 1 - \frac{\varepsilon (x_0^2 - 1) \dot{x}_0 - P \sin x_0}{x_0} \right). \]  

(32)

The term \( \sin x_0 \) can be expanded in the power series

\[ \sin (A \cos \omega t) = A \cos \omega t - \frac{A^3 \cos^3 \omega t}{6} + \frac{A^5 \cos^5 \omega t}{120} + \frac{A^7 \cos^7 \omega t}{5040} + \ldots \]  

(33)

We rewrite powers \( \cos \omega t \) in Eq. (33) in terms of the cosine of multiples of \( \omega t \) with the aid of the identity [21].

\[ \cos^{2n+1} \omega t = \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k} \cos(2k+1) \omega t, \]

(34)

\[ \binom{n}{p} = \frac{n!}{p!(n-p)!}, \quad \binom{n}{0} = 1; \quad k = 1, 2, 3, \ldots, k \in N. \]

By using Eq. (34), Eq. (33) may be expressed in the form

\[ \sin (A \cos \omega t) = A \cos \omega t - \frac{A^3}{2!} (3 \cos 3 \omega t) + \frac{A^5}{4!} (5 \cos 5 \omega t + 5 \cos 3 \omega t + 10 \cos \omega t) \]

\[ - \frac{A^7}{6!} (7 \cos 7 \omega t + 7 \cos 5 \omega t + 21 \cos 3 \omega t + 35 \cos \omega t) + \frac{A^9}{8!} (9 \cos 9 \omega t + 9 \cos 7 \omega t + 36 \cos 5 \omega t + 84 \cos 3 \omega t + 126 \cos \omega t). \]  

(35)

Substituting Eq. (35) into Eq. (32), this can be rewritten as:

\[ \dot{x}_1 + \omega^2 x_1 = \left( -A - p + \frac{A^3}{8} - \frac{A^5}{120} + \frac{A^7}{5040} + A^9 \omega^2 \right) \]

\[ \times \cos \omega t - \left( \frac{\varepsilon A^3}{3} \cos \omega t + \frac{\varepsilon A^5}{5} \cos 3 \omega t + \frac{\varepsilon A^7}{7} \cos 5 \omega t + \frac{\varepsilon A^9}{9} \cos 7 \omega t \right) \]

\[ + \frac{A^{11}}{3!} \cos 9 \omega t + \frac{A^7}{100} \left( \frac{A^3}{3} \cos 3 \omega t + \frac{A^5}{5} \cos 5 \omega t + \frac{A^7}{7} \cos 7 \omega t + \frac{A^9}{9} \cos 9 \omega t \right). \]

(36)

No secular terms in \( x_1 \) requires that

\[ A = 2, \quad \omega_1 = \sqrt{1 + \frac{A^3}{8} + \frac{A^5}{120} + \frac{A^7}{5040} + \frac{A^9}{737280}}, \]  

(37)

Solving Eq. (36) with initial conditions (18), \( x_1 \) is obtained as

\[ x_1 = A \cos \omega t + \frac{\varepsilon}{8} (3 \sin 3 \omega t - \sin 9 \omega t) \]

\[ + \frac{A^3}{12} (3 \sin 3 \omega t - \sin 9 \omega t) \]

\[ + \frac{A^5}{80} (3 \sin 3 \omega t - \sin 9 \omega t) \]

\[ + \frac{A^7}{343} (3 \sin 3 \omega t - \sin 9 \omega t) \]

\[ + \frac{A^9}{737280} (3 \sin 3 \omega t - \sin 9 \omega t). \]

(38)

Fig. 2 shows a comparison between the analytical solution obtained from formulae (37) and (38) and the numerical integration results obtained by using the Runge-Kutta method. It is seen that the solution obtained by the iteration procedure is very close to that obtained by the numerical method. One concludes that adopting the present technique to analyze the solutions of the modified Van der Pol equation, a satisfactory results are obtained for small values of parameter \( \varepsilon \).
4 Conclusion

In this paper, the iteration perturbation method has been successfully used to study the nonlinear oscillators. The examples of nonlinear oscillations have illustrated that the iteration procedure can give excellent approximate results. The first approximate frequency $\omega_1$ given in equation (25) is identical to equation (10) in Ref [16]. The examples of nonlinear oscillations have illustrated that the present method can give excellent approximate results. The second approximate frequency $\omega_2$ obtained by the second iteration gives very accurate solutions. Also, the second approximate periodic solution $x_2$ is in good agreement with the numerical integration results obtained by using a fourth order Runge-Kutta method.

References


A. M. El-Naggar
Professor of Mathematics at Benha University. Faculty of Science. His main research interests are: non-linear second order differential equations (weakly non-linear, strongly non-linear), topological study of periodic solutions of types (Harmonic, sub-harmonic of even and odd order, super-harmonic of even and odd order), analytical study of periodic solutions of different types (perturbation methods), dynamical systems (strongly non-linear or weakly non-linear), theory of elasticity (dynamical problems), theory of generalized thermo-elasticity or thermo-visco-elasticity, supervisor on about 35 thesis (M. Sc. and PhD) in the above subjects, has about 75 published papers in the previous fields.

G. M. Ismail
Lecturer of Mathematics at Sohag University, Faculty of Science. He is referee and editor of several international journals in the frame of pure and applied mathematics. His main research interests are: nonlinear differential equations, nonlinear oscillators, applied mathematics, analytical methods, perturbation methods, analysis of nonlinear differential equations.