Journal of Analysis & Number Theory
*An International Journal*

http://dx.doi.org/10.18576/jant/060102

# Error Bernoulli Polynomials and their Relation to Hermite Polynomials

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Received: 2 Jul. 2017, Revised: 10 Aug. 2017, Accepted: 15 Aug. 2017
Published online: 1 Jan. 2018

**Abstract:** In this article we introduce and investigate new families of polynomials $B_n(\frac{1}{2},x)$ called error Bernoulli polynomials through generating functions, Appell sequences and umbral calculus. We also show that these polynomials are related to the Hermite polynomials.

**Keywords:** Bernoulli numbers, Bernoulli polynomials, Hermite polynomials, Appell sequences, umbral calculus, hypergeometric Bernoulli polynomials and error Bernoulli polynomials

## 1 Introduction

It is well known that the classical Bernoulli numbers $B_k$ and the Bernoulli polynomials $B_k(x)$ are of fundamental importance in several parts of analysis and in the calculus of finite differences and have applications in various other fields such as statistics, numerical analysis, combinatorics, and so on. The Hermite polynomials are also play an important role in various fields and have interesting properties and applications. Some of the generalizations of the classical Bernoulli numbers and polynomials were investigated by Dichler [7,8], A. Hassen, H. Nguyen [2,3] and the references cited in each of these earlier works. The classical Bernoulli polynomials and their generalizations by hypergeometric Bernoulli polynomials are usually defined respectively by means of generating functions as follows:

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!} \tag{1}$$

for $|t|<2\pi$. The generating function $\frac{te^{xt}}{e^t-1}$ is related to the integral representation of the classical zeta function:

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{(s-1)}}{e^x-1}dx$$

$$\frac{t^Ne^{xt}}{N!(e^t-T_{N-1}(t))}=\sum_{k=0}^{\infty}B_k(N,x)\frac{t^k}{k!} \tag{2}$$

for $|x|<2\pi$. The generating function $\frac{t^Ne^{xt}}{N!(e^t-T_{N-1}(t))}$ is related to the integral representation of the Hypergeometric zeta functions:

$$\zeta(s)=\frac{1}{\Gamma(s+N-1)}\int_0^{\infty}\frac{x^{(s+N-2)}}{e^x-T_{N-1}(x)}dx$$

In [2] A. Hassen and H.Nguyen defined and investigated about error zeta function. They have shown that the error zeta function is connected to numbers called generalized Bernoulli numbers.

$$\zeta_{\frac{1}{2}}(s)=\frac{2}{\gamma(s-\frac{1}{2})}\int_0^{\infty}\frac{x^{2(s-1)}e^{-x^2}}{erf(x)}dx$$

The corresponding generalized Bernoulli numbers are defined as,

$$\frac{2ze^{-z^2}}{\sqrt{\pi}\,erf(z)}=\sum_{n=0}^{\infty}B_n(\frac{1}{2})\frac{z^n}{n!}$$

Motivated by this, we investigate a continuous version of

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hypergeometric Bernoulli polynomials as a generalization of hypergeometric Bernoulli polynomials by generalizing definition (2) to all real positive values of $N$. Since $e^x - T_{N-1}(x) = \frac{x^N(1F_1(1,N+1;x))}{\Gamma(N+1)}$, where $1F_1(1,N+1;x)$ is the hypergeometric series, given by:

$$1F_1(1,N+1;x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!}.$$

So that definition (2) holds for all positive real number $N$. We only focus in particular on $N = \frac{1}{2}$, where the error function $erf(x)$ makes its appearance, as described in [2]. In the present work, we define a family of polynomials $B_n(\frac{1}{2},x)$ called error Bernoulli polynomials through generating function, Appell sequences, umbral calculus and study their connection to Hermite polynomials. It is discovered that these error Bernoulli polynomials share many of the same properties found in the hypergeometric Bernoulli polynomials, via the classical Bernoulli polynomials.

This paper is organized as follows. In section two we review on Appel sequences, umbral calculus and define error Bernoulli polynomials through generating function. In section three show that the error Bernoulli polynomials form an Appell sequence and define through umbral calculus. In section four we study the relation between Error Bernoulli polynomials and Hermite polynomials.

## 2 Preliminaries

In this section we define Appell sequences, umbral calculus and define the error Bernoulli polynomials through generating function.

**Definition 1.***A sequence of polynomials $P_n(x)$ is called Appell sequences if it satisfies the following conditions: $P_0(x) = 1$ and $\frac{d}{dx}P_n(x) = nP_{n-1}(x)$.*

(Appell(1832)as cited in [6])

**Definition 2.***A sequence of polynomials $P_n(x)$ is called( Lucas(1891)as cited in [6]) umbral calculus if it satisfies $P_n(x) = (P+x)^n$ where, after expanding this binomial, the exponents on $P$ are regraded to subscripts, i.e.; $P^n = P_n$ and $P_n = P_n(0)$.*

For each natural number $n$ the polynomial $B_n(\frac{1}{2},x)$ is generated by the function $w(x,z) = \frac{2ze^{xz-z^2}}{\sqrt{\pi}erf(z)}$, that is;

$$\frac{2ze^{xz-z^2}}{\sqrt{\pi}erf(z)} = \sum_{n=0}^{\infty} B_n(\frac{1}{2},x)\frac{z^n}{n!}$$

where $erf(z)$ is the error function defined by

$$erf(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt.$$

Error functions have a lot of applications in probability and differential equations. For example it satisfies the differential equation given by:

$$y'' + 2zy' = 0$$

The function defined by

$$w(x,z) = \frac{2ze^{xz-z^2}}{\sqrt{\pi}erf(z)}$$

satisfy the following differential equations:

$$w' - zw = 0$$
$$w'' - z^2 w = 0.$$

In general

$$w^{(n)} - z^n w = 0$$

for each natural number $n$, where the derivative is taken with respect to the variable $x$. It also satisfies the differential equation given by,

$$w'' + (1-z)w' - zw = 0$$

Here $B_n(\frac{1}{2},0)$ is what we call Bernoulli numbers related to the error zeta function [2]. The first few error Bernoulli polynomials are:

$$B_0(\frac{1}{2},x) = 1$$

$$B_1(\frac{1}{2},x) = x$$

$$B_2(\frac{1}{2},x) = x^2 - \frac{4}{3}$$

$$B_3(\frac{1}{2},x) = x^3 - 4x$$

$$B_4(\frac{1}{2},x) = x^4 - 8x^2 + \frac{64}{15}$$

## 3 Appell Sequence and Umbral Calculus

In this section we show that the error Bernoulli polynomials form an Appell sequence and defined through umbral calculus.

**Theorem 1.** *The error Bernoulli polynomials $B_n(\frac{1}{2},x)$ is given by the following:*

$$B_n(\frac{1}{2},x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(\frac{1}{2},0)x^k$$

*Proof.* The error Bernoulli number is given by the generating function,

$$\frac{2ze^{-z^2}}{\sqrt{\pi}erf(z)} = \sum_{n=0}^{\infty} B_n(\frac{1}{2})\frac{z^n}{n!}.$$

Therefore,

$$\frac{2ze^{xz-z^2}}{\sqrt{\pi}erf(z)} = e^{xz}\sum_{n=0}^{\infty} B_n(\frac{1}{2})\frac{z^n}{n!}$$

Which implies that,

$$\sum_{n=0}^{\infty} B_n(\frac{1}{2},x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} x^n\frac{z^n}{n!}\sum_{n=0}^{\infty} B_n(\frac{1}{2})\frac{z^n}{n!}$$

From this using product formula for series we obtain,

$$\sum_{n=0}^{\infty} B_n(\frac{1}{2},x)\frac{z^n}{n!} = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}\frac{B_{n-k}(\frac{1}{2})x^k}{k!(n-k)!})z^n$$

Therefore, we have

$$B_n(\frac{1}{2},x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}(\frac{1}{2})x^k$$

**Theorem 2.** *The error Bernoulli polynomials $B_n(x)$ form an Appell sequence. $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$*

*Proof.* For each natural number $n$ the polynomial $B_n(\frac{1}{2},x)$ is generated by $\frac{2ze^{xz-z^2}}{\sqrt{\pi}erf(z)}$, where $erf(z)$ is the error function defined by

$$erf(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt, \qquad (N \geq 1)$$

That is;

$$\frac{2ze^{xz-z^2}}{\sqrt{\pi}erf(z)} = \sum_{n=0}^{\infty} B_n(\frac{1}{2},x)\frac{z^n}{n!}$$

Differentiating both sides with respect to the variable $x$ we get

$$\frac{2z^2e^{xz-z^2}}{\sqrt{\pi}erf(z)} = \sum_{n=0}^{\infty} B_n'(\frac{1}{2},x)\frac{z^n}{n!}, \qquad (N \geq 1)$$

Which implies that,

$$\sum_{n=0}^{\infty} B_n(\frac{1}{2},x)\frac{z^{n+1}}{n!} = \sum_{n=0}^{\infty} B_n'(\frac{1}{2},x)\frac{z^n}{n!}, \qquad (N \geq 1)$$

Equating both sides and comparing coefficients we get

$$B_0'(\frac{1}{2},x) = 0, \quad B_n'(\frac{1}{2},x) = nB_{n-1}(\frac{1}{2},x)\, for\ each\ n = 1,2,3,\ldots.$$

**Theorem 3.** *Let $B_n = B_n(0)$ be the $n^{th}$ Bernoulli number related to the error zeta function. Then $B_n(\frac{1}{2},x) = (B+x)^n$, where, after expanding this binomial, the exponents on $B$ are regraded to subscripts, i.e.; $B^n = B_n$.*

*Proof.* From the above theorem we have

$$B_n'(\frac{1}{2},x) = nB_{n-1}(\frac{1}{2},x)\, for\ each\ n = 1,2,3,\ldots.$$

If we continue to differentiate we get

$$B_n^k(\frac{1}{2},x) = k!\binom{n}{k}B_{n-k}(\frac{1}{2},x)\, for\ each\ n = 1,2,3,\ldots.$$

By the Maclaurin expansion of $B_n(\frac{1}{2},x)$, we have

$$B_n(\frac{1}{2},x) = \sum_{k=0}^{n} B_n^k(\frac{1}{2},x)\frac{x^k}{k!}.$$

Thus using the relation,

$$B_n^k(\frac{1}{2},x) = k!\binom{n}{k}B_{n-k}(\frac{1}{2},x)$$

we have the following equality relation:

$$B_n(\frac{1}{2},x) = \sum_{k=0}^{n} k!\binom{n}{k}B_{n-k}(\frac{1}{2},0)\frac{x^k}{k!}$$

Which also imply that,

$$B_n(\frac{1}{2},x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}(\frac{1}{2},0)x^k = (B+x)^n$$

## 4 Relation to Hermite Polynomials

In this section we will see the connection between the error Bernoulli polynomials and the Hermite polynomials.

Hermite polnomials are classes of orthogonal polynomials encountered in the applications, especially in mathematical physics which can be defined by the formula,

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}.$$

The Hermite polynomials (or more exactly, the Hermite polynomial multiplied by the constant factor $\frac{1}{n!}$) are the coefficients in the expansion of,

$$e^{2xz-z^2} = \sum_{n=0}^{\infty}\frac{H_n(x)z^n}{n!}.$$

The first few Hermite polynomials are:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

In general

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^n n!(2x)^{n-2k}}{k!(n-2k)!}$$

where [t]denotes the largest integer less than or equal to $t$. The connection between error Bernoulli polynomial and Hermite polynomial arises from the function $w(x,z) = e^{xz-z^2}$ which is one of the factor of the generating function of error Bernoulli polynomials, and $w(2x,z) = e^{2xz-z^2}$ which is the generating function of the Hermite polynomials. Observe that:

$$w(x,z) = e^{xz-z^2} = \frac{\sqrt{\pi} erf(z)}{2z} \sum_{n=0}^{\infty} B_n(\frac{1}{2},x)\frac{z^n}{n!}$$

$$\frac{\sqrt{\pi} erf(z)}{2z} = \sum_{n=0}^{\infty} (-1)^n (\frac{z^{2n}}{(2n+1)n!}$$

Thus comparing with the generating function of Hermite polynomial we have the following relations:

$$e^{xz-z^2} = \sum_{n=0}^{\infty} \frac{H_n(\frac{x}{2})z^n}{n!} = \sum_{n=0}^{\infty} (-1)^n (\frac{z^{2n}}{(2n+1)n!} \sum_{n=0}^{\infty} B_n(\frac{1}{2},x)\frac{z^n}{n!}.$$

Therefore, multiplying the two series we have the following:

$$\sum_{n=0}^{\infty} \frac{H_n(\frac{x}{2})z^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{B_{n-k}}{(2k+1)k!(n-k)} z^{n+k}$$

The first few Hermite polynomials expressed in terms of the error Bernoulli polynomials as follows:

$$H_0(\frac{x}{2}) = B_0(\frac{1}{2},x)$$

$$H_1(\frac{x}{2}) = B_1(\frac{1}{2},x)$$

$$H_2(\frac{x}{2}) = B_2(\frac{1}{2},x) - \frac{2!}{3} B_0(\frac{1}{2},x)$$

$$H_3(\frac{x}{2}) = B_3(\frac{1}{2},x) - \frac{3!}{3} B_1(\frac{1}{2},x)$$

$$H_4(\frac{x}{2}) = B_4(\frac{1}{2},x) - \frac{4!}{3(2!)} B_2(\frac{1}{2},x) + \frac{4!}{5(2!)} B_0(\frac{1}{2},x)$$

We also express the error Bernoulli polynomials in terms of Hermite polynomials as follows:

$$B_0(\frac{1}{2},x) = H_0(\frac{x}{2})$$

$$B_1(\frac{1}{2},x) = H_1(\frac{x}{2})$$

$$B_2(\frac{1}{2},x) = H_2(\frac{x}{2}) + \frac{2!}{3} H_0(\frac{x}{2})$$

$$B_3(\frac{1}{2},x) = H_3(\frac{x}{2}) + \frac{3!}{3} H_1(\frac{x}{2})$$

$$B_4(\frac{1}{2},x) = H_4(\frac{x}{2}) + \frac{4!}{3(2!)} H_2(\frac{x}{2}) - \frac{4!}{3(2!)(3)} H_0(\frac{x}{2})$$

In general it can easily be seen that the Hermite and error polynomials can be given by the following relations:

$$H_{2n}(\frac{x}{2}) = \sum_{k=0}^{n} \frac{(-1)^k (2n)! B_{2n-2k}(\frac{1}{2},x)}{(2n-2k)!k!(2k+1)}$$

and

$$H_{2n+1}(\frac{x}{2}) = \sum_{k=0}^{n} \frac{(-1)^k (2n+1)! B_{2n+1-2k}(\frac{1}{2},x)}{(2n+1-2k)!k!(2k+1)}$$

It can also easily bee seen that the error Bernoulli polynomials is expressed in terms of the Hermite polynomials. Therefore, as the Hermite polynomials are orthogonal families with weight function $e^{-x^2}$ so are the error Bernoulli polynomials with the same weight function.

## 5 Conclusion

In this paper we have defined error Bernoulli polynomials, through generating functions and have shown that this definition is equivalent to the definition given through Appell sequences and umbral calculus. We have also shown the connection between these new polynomials with that of the Hermite polynomials. As the Hermite polynomials are orthogonal with weight function $e^{-x^2}$ it is easy to see from their relation that the error Bernoulli polynomials are also orthogonal with the same weight function.

## Acknowledgements

I appreciate the comments and helpful suggestion from the referee.

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