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Oscillatory Behaviour of Solutions of Delay Differential Equations of Fractional Order

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Abstract: In this paper, we deal with fractional delay differential equations(FDDEs). New results have been obtained on the oscillatory behaviour of solutions of FDDEs with constant as well as variable coefficients. The oscillation criteria for FDDEs with positive and negative coefficients have also been discussed. All the outcomes have been illustrated by giving suitable examples with graphical representation. The graphs are done by using MATLAB.

Keywords: Oscillation, fractional differential equations, delay differential equations.

1 Introduction

Due to its numerous applications in a variety of fields, including electromagnetic field theories, control theory, fluid flow, optics, signal processing, epidemics and infectious diseases, etc., a number of researchers have recently started working in the field of fractional differential equations.

In [1], Seemab and Rehman have studied the oscillatory and asymptotic behaviour of solutions for a class of fractional order differential equations. Chen[2] has obtained some results on oscillatory behaviour of a fractional differential equation with damping and illustrated the results with suitable examples. Liu et. al[3] have established some new oscillatory criteria for the solutions of a class of sequential differential equations of fractional order by using modified Riemann-Liouville derivative. Samko et. al[4] have discussed the fractional integral theory, fractional derivative theory and some applications related to fractional derivatives. Different types of fractional derivatives and integrals have been defined and various methods of solving fractional differential equations has been discussed by Podlubny[5]. In [6], Razaghzadeh et. al have introduced a fractional order wavelet for solving Abel integral equations as well as its generalized version.

Gyori and Ladas[7] have discussed the qualitative behaviour of solutions of delay differential equations. Ladde et. al[8] have studied the oscillatory theory of differential equations with deviating arguments. Baculikova and Dzurina[9] have studied the oscillatory behaviour of the second order neutral differential equation. In [10], Baculikova and Dzurina have established some results on oscillatory behaviour of solutions of second order neutral differential equations. Ocalan[11] has provided oscillation property of every solution of the neutral differential equation with positive and negative coefficients. Ismail et. al[12] have solved the neutral differential equation of pantograph type. In [13], Buyukkabraman has studied the oscillatory behaviour of fourth order delay differential equations. Elabassy and Saker[14] have discussed the oscillatory behaviour of solutions of delay differential equations with several positive and negative coefficients. Kon et. al[15] have discussed the oscillatory behaviour of solutions of delay differential equations. Elabassy et. al[16] have obtained some theorems on oscillatory behaviour of solutions of delay differential equations with positive and negative coefficients.

Mathematicians have recently started investigating fractional functional differential equations both quantitatively and qualitatively. In a qualitative study of differential equations, the oscillation, non-oscillation, boundedness behaviour etc.

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of the solution of the problem is studied instead of finding out the solution. Cermak and Kisela[17] have studied the oscillatory behaviour and asymptotic properties of solutions of FDDEs. Zhou et.al[18] have discussed the existence of non-oscillatory solutions for fractional neutral differential equations. Lu and Cen[19] have established some results of oscillatory behaviour of solutions of a class of the FDDEs and justified the results by giving suitable examples. Zhu and Xiang[20] have studied the oscillatory behaviour of solutions of FDDEs. Bolat[21] has discussed the oscillatory behaviour of solutions of FDDEs and illustrated the results by giving suitable examples.

Our work's main objective is to investigate the qualitative behaviour of solutions of FDDEs with constant and variable coefficients. The oscillation of delay differential equations of fractional order with constant coefficients, constant positive negative coefficients, as well as variable coefficients, has been investigated in the current study, and the observations have been adequately represented by examples. To the best of our knowledge, there is no such literature present in which the oscillatory behaviour of such types of fractional delay differential equations has been studied.

Gyori and Ladas[7], Lu and Cen[19], and Bolat[21] are those works which motivated us to do the current research.The qualitative analysis of the solutions of FDDEs is rather useful because it is not always simple to find the actual solution. As FDDEs are formed by the mathematical models of many real-world issues, understanding and analysing the behaviour of solutions of these models will be accomplished through qualitative behaviour on these types of equations.

The content of the paper is as follows. The introduction to the paper is in Section 1. Section 2 provides certain definitions and basic results that are relevant for the current work. The main result is found in Section 3. Section 4 illustrates our theoretical outcomes by giving suitable examples and graphs. Section 5 describes the conclusion of the work.

2 Definition and basic results

Some definitions and basic results that will be required in our work are provided in this section.

Definition 2.1:[5] The following definition applies to the Riemann-Liouville fractional order derivative of order α,

$$D^{\alpha}h(\rho)=\frac{1}{\Gamma(l-\alpha)}\frac{d^l}{d\rho^l}\int_0^t\frac{h(\tau)}{(\rho-\tau)^{1+\alpha-l}}d\tau \tag{1}$$

where $\alpha\in R$, $l-1<\alpha<l$, $l\in N$ and $\Gamma(z)$ is a gamma function defined by

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt,\quad R(z)>0$$

satisfying $\Gamma(z+1)=z\Gamma(z)$.

Definition 2.2:[Section 1.3, [7]] The Laplace transform(LT) of a real valued function $x:[0,\infty)\to R$ is given by ,

$$L[x(\rho)]=X(m)=\int_0^{\infty}e^{-m\rho}x(\rho)dt \tag{2}$$

If we can find a real number r_0 for which the integral in (2) converges when $Re\ m>r_0$ and diverges when $Re\ m<r_0$ for all m, then the term r_0 represents the abscissa of convergence for $X(m)$.

Lemma 2.1:[Lemma 1.3.2, [7]] (a) If $x\in C[[-\nu,\infty),R]$, $r_0<\infty$ and the LT of $x(\rho)$ is $X(m)$, then the LT of $x(\rho-\nu)$ is given by

$$L[x(\rho-\nu)]=\int_0^{\infty}e^{-m\nu}x(\rho-\nu)d\rho=e^{-m\nu}X(m)+e^{-m\nu}\int_{-\nu}^{0}e^{-m\rho}x(\rho)d\rho$$

for all m with $Re\ m>r_0$. $L[x(\rho-\nu)]$ has the same abscissa of convergence as $L[x(\rho)]$.

(b) If $x\in C^1[[0,\infty),R]$ and $r_0<\infty$, then the LT of $x'(\rho)$ is given by

$$L[x'(\rho)]=\int_0^{\infty}e^{-m\rho}x'(\rho)dt=mX(m)-x(0)$$

for all m with $Re\ m>r_0$. $L[x'(\rho)]$ has the same abscissa of convergence as $L[x(\rho)]$.

Definition 2.3:[5] The LT of the Riemann-Liouville fractional derivative can be written as

$$\int_0^\infty e^{-m\rho}\{{}_0D_t^\alpha h(\rho)\}d\rho = m^\alpha F(m) - \sum_{k=0}^{l-1} m^k{}_0D_\rho^{\alpha-k-1}h(\rho)\mid_{\rho=0} \quad (l-1\leq\alpha<l)$$

Theorem 2.1:[7] Let $x\in C[[0,\infty),R^+]$ in such away that the r_0 of the LT $X(s)$ of $x(t)$ is

finite. Then there exists a singularity at $m=r_0$ for $X(s)$, i.e., we can find a sequence

$$m_l = a_l + ib_l \quad for \;\; l = 1,2,\ldots$$

where $a_l \geq a_0$ for $l\geq 1$, $\lim_{l\to\infty} a_l = r_0$, $\lim_{l\to\infty} b_l = 0$ and $\lim_{l\to\infty}|X(m_l)| = \infty$.

Definition 2.4:[Definition 1.1.1, [7]] Consider the FDDE

$$D^\alpha x(\rho) + h(\rho, x(\rho), x(\rho-\nu_1(\rho)),\ldots,x(\rho-\nu_n(\rho))) = 0, \;\; t_0 \geq \tilde{t_0}. \tag{3}$$

A function x is said to be a solution of equation (3) on the interval I, where I is of the form $[\rho_0,T)$, $[\rho_0,T]$, or $[\rho_0,\infty)$, with $\tilde{\rho_0}\geq T$, if $x:[\rho_{-1},\rho_0]\cup I \longrightarrow R^m$ is continuous, x is continuously differentiable for $\rho\in I$ and x satisfies (3) for all $\rho\in I$.

Theorem 2.2[7] Consider the delay differential equation

$$x'(\rho) + \sum_{\tilde{i}=1}^{n} p_{\tilde{i}} x(\rho-\nu_{\tilde{i}}) = 0 \tag{4}$$

where $p_{\tilde{i}}\in R$ and $\nu_{\tilde{i}}\in R^+$ for $\tilde{i}=1,2,\ldots,n$. Then the following statements are equivalent.

(a) Every solution of equation (4) oscillates.

(b) The characteristic equation

$$\lambda + \sum_{\tilde{i}=1}^{n} p_{\tilde{i}} e^{-\lambda\tau_{\tilde{i}}} = 0 \tag{5}$$

has no real roots.

Theorem 2.3:[7] Consider the delay differential equation

$$x'(\rho) + px(\rho-\nu) = 0 \tag{6}$$

where $p,\nu\in R$. Then the following statements are equivalent.

(a) Every solution of equation (6) oscillates.

(b) $p\nu > \frac{1}{e}$

3 Main results

3.1 FDDE with constant coefficients:

Theorem 3.1.1: Let us consider a FDDE

$$D^\alpha\left[z(\rho) + D^\beta z(\rho)\right] + bz(\rho-\nu) = 0 \tag{7}$$

where $b\in R$, $\nu\in R^+$, $0<\alpha,\beta<1$. If the characteristic equation(CE) of (7) has no real roots then every solution of (7) oscillates.

Proof: Given that the CE

$$F(\lambda) = \lambda^{\alpha} + \lambda^{\alpha+\beta} + be^{-\lambda\nu} = 0$$

has no real positive root. Let on contrary $z(\rho)$ be an eventually positive solution of (7).

So $z(\rho) > 0$ for $\rho \geq -\nu(\nu > 0)$. So there exists constants M and γ such that

$$|\, z(\rho)\,| \leq Me^{\gamma\rho}, \;\; \rho \geq -\nu.$$

Thus, $L[z(\rho)] = Z(m)$ exists for Re $m > \gamma$.

Taking the LT of (7) we get

$$m^{\alpha} L\left\{z(\rho) + D^{\beta}z(\rho)\right\} - \left[D^{\alpha-1}\left(z(\rho) + D^{\beta}z(\rho)\right)\right]_{\rho=0} + be^{-m\nu}Z(m) + be^{-m\nu}\int_{-\nu}^{0} e^{-m\rho}z(\rho)d\rho = 0$$

$$\Rightarrow m^{\alpha}\left\{Z(m) + m^{\beta}Z(m) - \left[D^{\beta-1}z(\rho)\right]_{\rho=0}\right\} - \left[D^{\alpha-1}\left(z(\rho) + D^{\beta}z(\rho)\right)\right]_{\rho=0} + be^{-m\nu}Z(m) + bh(m\nu) = 0,$$

where $h(m\nu) = e^{-m\nu}\int_{-\nu}^{0} e^{-m\rho}z(\rho)d\rho$

$$\Rightarrow \left(m^{\alpha} + m^{\alpha+\beta} + be^{-m\nu}\right)Z(m) = m^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0} + \left[D^{\alpha-1}\left(z(\rho) + D^{\beta}z(\rho)\right)\right]_{\rho=0} - bh(m\nu)$$

$$\Rightarrow G(m)Z(m) = Q(m) \tag{8}$$

where $G(m) = \left(m^{\alpha} + m^{\alpha+\beta} + be^{-m\nu}\right)$ and

$Q(m) = m^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0} + \left[D^{\alpha-1}\left(z(\rho) + D^{\beta}z(\rho)\right)\right]_{\rho=0} - bh(m\nu)$.

So from (8)

$$Z(m) = \frac{Q(m)}{G(m)}, \quad Re\ m > r_0.$$

where r_0 is defined in the Definition 2.2[7] .

Let us claim that $r_0 = -\infty$. If not then $r_0 > -\infty$. $m = r_0$ must be a singularity of $\frac{Q(m)}{G(m)}$ according to Theorem 2.1[7].However, since the real axis of this quotient does not have a singularity $r0 = -\infty$ and

$$Z(m) = \frac{Q(m)}{G(m)}\ \forall\ m \in R. \tag{9}$$

As $m \to -\infty$, we can see that (9) provides us a contradiction because $Z(m)$ and $G(m)$ are always positive while $Q(m)$ eventually turns negative.

Theorem 3.1.2: Consider the FDDE

$$D^{\alpha}\left[z(\rho) + D^{\beta}z(\rho)\right] + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}}z(\rho - \nu_{\tilde{i}}) = 0 \tag{10}$$

where $b_{\tilde{i}} \in R$, $\nu_{\tilde{i}} \in R^{+}$, $\tilde{i} = 1,...,n$, $0 < \alpha, \beta < 1$. Every solution of (10) oscillates when the

CE of (10) has no real roots .

Proof: Given that the CE

$$F(\lambda) = \lambda^{\alpha} + \lambda^{\alpha+\beta} + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}}e^{-\lambda\nu_{\tilde{i}}} = 0$$

has no real positive root. Let on contrary $z(\rho)$ be an eventually positive solution of (10).

So $z(\rho) > 0$ for $\rho \geq -\nu(\nu > 0)$. So there exists constants M and γ such that

$$| z(\rho) | \leq Me^{\gamma\rho}, \ \ \rho \geq -\nu.$$

Thus, $L[z(\rho)] = Z(m)$ exists for Re $m > \gamma$.

Taking the LT of (10) we get

$$m^{\alpha} L\left\{z(\rho)+D^{\beta}z(\rho)\right\} - \left[D^{\alpha-1}\left(z(\rho)+D^{\beta}z(\rho)\right)\right]_{\rho=0} + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} e^{-m\nu_{\tilde{i}}} Z(m) + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} e^{-m\nu_{\tilde{i}}} \int_{-\nu_{\tilde{i}}}^{0} e^{-m\rho} z(\rho) d\rho = 0$$

$$\Rightarrow m^{\alpha}\left\{Z(m)+m^{\beta}Z(m) - \left[D^{\beta-1}z(\rho)\right]_{\rho=0}\right\} - \left[D^{\alpha-1}\left(z(\rho)+D^{\beta}z(\rho)\right)\right]_{\rho=0} + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} e^{-m\nu_{\tilde{i}}} Z(m) + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} h(m\nu_{\tilde{i}}) = 0,$$

where $h(m\nu_{\tilde{i}}) = e^{-m\nu_{\tilde{i}}} \int_{-\nu_{\tilde{i}}}^{0} e^{-m\rho} z(\rho) d\rho$

$$\Rightarrow \left(m^{\alpha} + m^{\alpha+\beta} + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} e^{-m\nu_{\tilde{i}}}\right) Z(m) = m^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0}$$

$$+ \left[D^{\alpha-1}\left(z(\rho)+D^{\beta}z(\rho)\right)\right]_{\rho=0} - \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} h(m\nu_{\tilde{i}})$$

$$\Rightarrow G(m)Z(m) = Q(m) \tag{11}$$

where $G(m) = \left(m^{\alpha} + m^{\alpha+\beta} + \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} e^{-m\nu_{\tilde{i}}}\right)$ and

$Q(m) = m^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0} + \left[D^{\alpha-1}\left(z(\rho)+D^{\beta}z(\rho)\right)\right]_{\rho=0} - \sum_{\tilde{i}=1}^{n} b_{\tilde{i}} h(m\nu_{\tilde{i}})$.

So from (11)

$$Z(m) = \frac{Q(m)}{G(m)}, \quad Re\ m > r_0.$$

where r_0 is defined in the Definition 2.2[7] .

Let us claim that $r_0 = -\infty$. If not then $r_0 > -\infty$. $m = r_0$ must be a singularity of $\frac{Q(m)}{G(m)}$ according to Theorem 2.1[7]. However, since the real axis of this quotient does not have a singularity $r0 = -\infty$ and

$$Z(m) = \frac{Q(m)}{G(m)}\ \forall\ m \in R \tag{12}$$

As $m \to -\infty$, we can see that (12) provides us a contradiction because $Z(m)$ and $G(m)$ are always positive while $Q(m)$ eventually turns negative.

Theorem 3.1.3: Let us consider a FDDE

$$D^{\alpha}\left[z(\rho) - qD^{\beta}z(\rho)\right] + bz(\rho - \nu) = 0 \tag{13}$$

where $q \in R^+$, $b \in R$, $\nu \in R^+$, $0 < \alpha, \beta < 1$. If the CE of (13) has no real roots, then every

solution of (13) oscillates.

Proof: Given that the CE

$$F(\lambda) = \lambda^{\alpha} - q\lambda^{\alpha+\beta} + be^{-\lambda\nu} = 0$$

has no real positive root. Let on contrary $z(\rho)$ be an eventually positive solution of (13).

So $z(\rho) > 0$ for $\rho \geq -\nu(\nu > 0)$. So there exists constants M and γ such that

$$| z(\rho) | \leq Me^{\gamma\rho}, \ \ \rho \geq -\nu.$$

Thus, $L[z(\rho)] = Z(m)$ exists for Re $m > \gamma$.

Taking the LT of (13) we get

$$m^{\alpha} L\left\{z(\rho) - qD^{\beta}z(\rho)\right\} - \left[D^{\alpha-1}\left(z(\rho) - qD^{\beta}z(\rho)\right)\right]_{\rho=0} + be^{-m\nu}Z(m) + be^{-m\nu}\int_{-\nu}^{0} e^{-m\rho}z(\rho)d\rho = 0$$

$$\Rightarrow m^{\alpha}\left\{Z(m) - qm^{\beta}Z(m) + q\left[D^{\beta-1}z(\rho)\right]_{\rho=0}\right\} - \left[D^{\alpha-1}\left(z(\rho) - qD^{\beta}z(\rho)\right)\right]_{\rho=0} + be^{-m\nu}Z(m) + bh(m\nu) = 0,$$

where $h(m\nu) = e^{-m\nu}\int_{-\nu}^{0} e^{-m\rho}z(\rho)d\rho$

$$\Rightarrow \left(m^{\alpha} - qm^{\alpha+\beta} + be^{-m\nu}\right)Z(m) = -qm^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0} + \left[D^{\alpha-1}\left(z(\rho) - qD^{\beta}z(\rho)\right)\right]_{\rho=0} - bh(m\nu)$$

$$\Rightarrow G(m)Z(m) = Q(m) \tag{14}$$

where $G(m) = \left(m^{\alpha} - qm^{\alpha+\beta} + be^{-m\nu}\right)$ and

$Q(m) = -qm^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0} + \left[D^{\alpha-1}\left(z(\rho) - qD^{\beta}z(\rho)\right)\right]_{\rho=0} - bh(m\nu).$

So from (14)

$$Z(m) = \frac{Q(m)}{G(m)}, \quad Re\ m > r_0.$$

where r_0 is defined in the Definition 2.2[7] .

Let us claim that $r_0 = -\infty$. If not then $r_0 > -\infty$. $m = r_0$ must be a singularity of $\frac{Q(m)}{G(m)}$ according to Theorem 2.1[7]. However, since the real axis of this quotient does not have a singularity $r0 = -\infty$ and

$$Z(m) = \frac{Q(m)}{G(m)}\ \forall\ m \in R \tag{15}$$

As $m \to -\infty$, we can see that (15) provides us a contradiction because $Z(m)$ and $G(m)$ are always positive while $Q(m)$ eventually turns negative.

Theorem 3.1.4: Consider a FDDE

$$D^{\alpha}\left[z(\rho) - \sum_{\tilde{i}=1}^{s} q_{i}D^{\beta_{\tilde{i}}}z(\rho)\right] + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}z(\rho - \nu_{\tilde{j}}) = 0 \tag{16}$$

where $q_{\tilde{i}} \in R^{+}$, $\tilde{i} = 1,...,s$, $b_{\tilde{j}} \in R$, $\nu_{\tilde{j}} \in R^{+}$, $\tilde{j} = 1,...,n$, $0 < \alpha, \beta_{\tilde{i}} < 1$. If the CE of (16) has

no real roots then every solution of (16) oscillates.

Proof: Given that the CE

$$F(\lambda) = \lambda^{\alpha} - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}}\lambda^{\alpha+\beta_{\tilde{i}}} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}e^{-\lambda\nu_{\tilde{j}}} = 0$$

has no real positive root. Let on contrary $z(\rho)$ be an eventually positive solution of (10).

So $z(\rho) > 0$ for $\rho \geq -\nu(\nu > 0)$. So there exists constants M and γ such that

$$| z(\rho) | \leq Me^{\gamma\rho}, \ \ \rho \geq -\nu.$$

Thus, $L[z(\rho)] = Z(m)$ exists for Re $m > \gamma$.

Taking the LT of (10) we get

$$m^{\alpha}\, L\left\{z(\rho) - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} D^{\beta_{\tilde{i}}} z(\rho)\right\} - \left[D^{\alpha-1}\left(z(\rho) - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} D^{\beta_{\tilde{i}}} z(\rho)\right)\right]_{\rho=0}$$
$$+ \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-m\nu_{\tilde{j}}} Z(m) + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-m\nu_{\tilde{j}}} \int_{-\nu_{\tilde{j}}}^{0} e^{-m\rho} z(\rho)d\rho = 0$$
$$\Rightarrow m^{\alpha}\left\{Z(m) - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} m^{\beta_{\tilde{i}}} Z(m) + \sum_{\tilde{i}=1}^{s} q_{\tilde{i}}\left[D^{\beta_{\tilde{i}}-1} z(\rho)\right]_{\rho=0}\right\} - \left[D^{\alpha-1}\left(z(\rho) - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} D^{\beta_{\tilde{i}}} z(\rho)\right)\right]_{\rho=0}$$
$$+ \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-m\nu_{\tilde{j}}} Z(m) + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} h(m\nu_{\tilde{j}}) = 0,$$

where $h(m\nu_{\tilde{j}}) = e^{-m\nu_{\tilde{j}}} \int_{-\nu_{\tilde{j}}}^{0} e^{-m\rho} z(\rho)d\rho$

$$\Rightarrow \left(m^{\alpha} - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} m^{\alpha+\beta} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-m\nu_{\tilde{j}}}\right) Z(m) = -m^{\alpha} \sum_{\tilde{i}=1}^{s} q_{\tilde{i}}\left[D^{\beta_{\tilde{i}}-1} z(\rho)\right]_{\rho=0}$$
$$+ \left[D^{\alpha-1}\left(z(\rho) - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} D^{\beta_{\tilde{i}}} z(\rho)\right)\right]_{\rho=0} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} h(m\nu_{\tilde{j}})$$
$$\Rightarrow G(m)Z(m) = Q(m) \tag{17}$$

where $G(m) = \left(m^{\alpha} - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} m^{\alpha+\beta} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-m\nu_{\tilde{j}}}\right)$ and

$Q(m) = -m^{\alpha} \sum_{\tilde{i}=1}^{s} q_{\tilde{i}}\left[D^{\beta_{\tilde{i}}-1} z(\rho)\right]_{\rho=0} + \left[D^{\alpha-1}\left(z(\rho) - \sum_{\tilde{i}=1}^{s} q_{\tilde{i}} D^{\beta_{\tilde{i}}} z(\rho)\right)\right]_{\rho=0} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} h(m\nu_{\tilde{j}})$.

So from (17)

$$Z(m) = \frac{Q(m)}{G(m)}, \quad Re\ m > r_0.$$

where r_0 is defined in the Definition 2.2[7] .

Let us claim that $r_0 = -\infty$. If not then $r_0 > -\infty$. $m = r_0$ must be a singularity of $\frac{Q(m)}{G(m)}$ according to Theorem 2.1[7]. However, since the real axis of this quotient does not have a singularity $r0 = -\infty$ and

$$Z(m) = \frac{Q(m)}{G(m)}\ \forall\ m \in R \tag{18}$$

As $m \to -\infty$, we can see that (18) provides us a contradiction because $Z(m)$ and $G(m)$ are always positive while $Q(m)$ eventually turns negative.

Let us assume the following conditions:

(A_1) $a, b, \nu, \sigma \in R^{+}$, $a \geq b$ and $\nu < \sigma$.

(A_2) $a_{\tilde{i}}, b_{\tilde{j}}, \nu_{\tilde{i}}, \sigma_{\tilde{j}} \in R^{+}$, $\tilde{i} = 1, \ldots, s$, $\tilde{j} = 1, \ldots, n$, $s > n$, $\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} > \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}$ and $\nu_{\tilde{i}} < \sigma_{\tilde{j}}$ for all $\tilde{i}, \tilde{j}$.

(A_3) $a_{\tilde{i}}, b_{\tilde{j}}, \nu_{\tilde{i}}, \sigma_{\tilde{j}} \in R^{+}$, $\tilde{i} = 1, \ldots, s$, $\tilde{j} = 1, \ldots, n$, $s > n$, $\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} > \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}$, and $\nu_{\tilde{i}} > \sigma_{\tilde{j}}$ for all $\tilde{i}, \tilde{j}$.

(A_4) $\max_{1 \leq \tilde{i} \leq s} \nu_{\tilde{i}} = \nu$, for $\tilde{i} = 1, \ldots, s$.

Theorem 3.1.5: Let (A_1) hold for the FDDE with positive and negative coefficients

$$D^{\alpha}\left[z(\rho) + D^{\beta} z(\rho)\right] + az(\rho - \nu) - bz(\rho - \sigma) = 0, \tag{19}$$

with $0 < \alpha, \beta < 1$. If the CE of (19) has no real roots then every solution of the above equation oscillates.

Proof: The CE of (19) is

$$F(\lambda) = \lambda^{\alpha} + \lambda^{\alpha+\beta} + ae^{\lambda\nu} - be^{\lambda\sigma} = 0,$$

which has no real roots. Let (19) has an eventually positive solution $z(\rho)$.

The LT of (19) is

$$m^{\alpha}\left[D^{\beta-1}z(\rho)\right] + \left[D^{\alpha-1}\left(D^{\beta}z(\rho) + z(\rho)\right)\right]_{\rho=0} + bh(m\sigma) - ah(m\nu)$$
$$= \left(m^{\alpha+\beta} + m^{\alpha} + ae^{-m\nu} - be^{-m\sigma}\right)Z(m)$$

where $h(m\sigma)$ and $h(m\nu)$ are defined in Theorem 3.1.1 .

By above expression we get,

$$G(m)Z(m) = Q(m) \tag{20}$$

where $G(m) = m^{\alpha} + m^{\alpha+\beta} + ae^{-m\nu} - be^{-m\sigma}$ and

$Q(m) = m^{\alpha}\left[D^{\beta-1}z(\rho)\right] + \left[D^{\alpha-1}\left(D^{\beta}z(\rho) + z(\rho)\right)\right]_{\rho=0} + bh(m\sigma) - ah(m\nu)$

Proceeding similarly as Theorem 3.1.1 we get the proof.

Theorem 3.1.6: Let (A_2) hold for the FDDE

$$D^{\alpha}\left[z(\rho) + D^{\beta}z(\rho)\right] + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}}z(\rho - \nu_{\tilde{i}}) - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}z(\rho - \sigma_{\tilde{j}}) = 0 \tag{21}$$

with $0 < \alpha, \beta < 1$. If the CE of (21) has no real roots, then every solution of the above equation oscillates.

Proof: The CE of (21) is

$$F(\lambda) = \lambda^{\alpha} + \lambda^{\alpha+\beta} + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}}e^{\lambda\nu_{\tilde{i}}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}e^{\lambda\sigma_{\tilde{j}}} = 0,$$

which has no real roots. Let (21) has an eventually positive solution $z(\rho)$.

The LT of (19) is

$$m^{\alpha}\left[D^{\beta-1}z(\rho)\right] + \left[D^{\alpha-1}\left(D^{\beta}z(\rho) + z(\rho)\right)\right]_{\rho=0} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}h(m\sigma_{\tilde{j}}) - \sum_{\tilde{i}=1}^{s} a_{\tilde{i}}h(m\nu_{\tilde{i}})$$
$$= \left(m^{\alpha+\beta} + m^{\alpha} + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}}e^{\lambda\nu_{\tilde{i}}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}e^{\lambda\nu_{\tilde{j}}}\right)Z(m)$$

where $h(m\sigma_{\tilde{j}})$ and $h(m\nu_{\tilde{i}})$ are defined in Theorem 3.1.2 .

By above expression we get,

$$G(m)Z(m) = Q(m) \tag{22}$$

where $G(m) = m^{\alpha+\beta} + m^{\alpha} + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}}e^{\lambda\nu_{\tilde{i}}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}e^{\lambda\nu_{\tilde{j}}}$ and

$Q(m) = m^{\alpha}\left[D^{\beta-1}z(\rho)\right] + \left[D^{\alpha-1}\left(D^{\beta}z(\rho) + z(\rho)\right)\right]_{\rho=0} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}h(m\sigma_{\tilde{j}}) - \sum_{\tilde{i}=1}^{s} a_{\tilde{i}}h(m\nu_{\tilde{i}})$

Proceeding similarily as Theorem 3.1.1 we get the proof.

Theorem 3.1.7: Let (A_1) hold for the FDDE

$$D^{\alpha}\left[z(\rho)-qD^{\beta}z(\rho)\right]+az(\rho-\nu)-bz(\rho-\sigma)=0 \tag{23}$$

where $q\in R^{+}$, $0<\alpha,\beta<1$. If the CE of (23) has no real roots, then every solution of the above equation oscillates.

Proof: The CE of (23) is

$$F(\lambda)=\lambda^{\alpha}-q\lambda^{\alpha+\beta}+ae^{-\lambda\nu}-be^{-\lambda\sigma}=0$$

Let (23) have an eventually positive solution of $z(\rho)$.

By taking LT of (23) on both sides, we get

$$-qm^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0}+\left[D^{\alpha-1}\left(z(\rho)-qD^{\beta}z(\rho)\right)\right]_{\rho=0}+bh(m\sigma)-ah(m\nu)$$
$$=\left(m^{\alpha}-qm^{\alpha+\beta}+ae^{-m\nu}-be^{-m\sigma}\right)Z(m)$$

where $h(m\sigma)$ and $h(m\nu)$ are defined in Theorem 3.1.1.

From the above equation we get,

$$G(m)Z(m)=Q(m)$$

where $G(m)=m^{\alpha}-qm^{\alpha+\beta}+ae^{-m\nu}-be^{-m\sigma}$ and

$Q(m)=-qm^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0}+\left[D^{\alpha-1}\left(z(\rho)-qD^{\beta}z(\rho)\right)\right]_{\rho=0}+bh(m\sigma)-ah(m\nu)$

By proceeding similarly as in Theorem 3.1.3 we get the proof.

Theorem 3.1.8: Let (A_2) hold for the FDDE

$$D^{\alpha}\left[z(\rho)-\sum_{\tilde{k}=1}^{p}q_{\tilde{k}}D^{\beta_{\tilde{k}}}z(\rho)\right]+\sum_{\tilde{i}=1}^{s}a_{\tilde{i}}z(\rho-\nu_{\tilde{i}})-\sum_{\tilde{j}=1}^{n}b_{\tilde{j}}z(\rho-\sigma_{\tilde{j}})=0 \tag{24}$$

where $q_{\tilde{k}}\in R^{+}$ for $\tilde{k}=1,...,p$ and $0<\alpha,\beta<1$. The solutions of (24) is oscillatory, if the CE of (24) has no real roots.

Proof: Considering the CE of (24), which is

$$F(\lambda)=\lambda^{\alpha}-\sum_{\tilde{k}=1}^{p}q_{\tilde{k}}\lambda^{\alpha+\beta}+\sum_{\tilde{i}=1}^{s}a_{\tilde{i}}e^{-\lambda\nu_{\tilde{i}}}-\sum_{\tilde{j}=1}^{n}b_{\tilde{j}}e^{-\lambda\sigma_{\tilde{j}}}=0$$

Let (24) have an eventually positive solution of $z(\rho)$.

By taking LT of (24) on both sides, we get

$$-\sum_{\tilde{k}=1}^{p}q_{\tilde{k}}m^{\alpha}\left[D^{\beta-1}z(\rho)\right]_{\rho=0}+\left[D^{\alpha-1}\left(z(\rho)-\sum_{\tilde{k}=1}^{p}q_{\tilde{k}}D^{\beta}z(\rho)\right)\right]_{\rho=0}+\sum_{\tilde{j}=1}^{n}b_{\tilde{j}}h(m\sigma_{\tilde{j}})-\sum_{\tilde{i}=1}^{s}a_{\tilde{i}}h(m\nu_{\tilde{i}})$$
$$=\left(m^{\alpha}-\sum_{\tilde{k}=1}^{p}q_{\tilde{k}}m^{\alpha+\beta}+\sum_{\tilde{i}=1}^{s}a_{\tilde{i}}e^{-m\nu_{\tilde{i}}}-\sum_{\tilde{j}=1}^{n}b_{\tilde{j}}e^{-m\sigma_{\tilde{j}}}\right)Z(m)$$

where $h(m\sigma_{\tilde{j}})$ and $h(m\nu_{\tilde{i}})$ are defined in Theorem 3.1.1.

From the above equation we get,

$$G(m)Z(m)=Q(m)$$

where $G(m) = m^{\alpha} - \sum_{\tilde{k}=1}^{p} q_{\tilde{k}} m^{\alpha+\beta} + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} e^{-m\nu_{\tilde{i}}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-m\sigma_{\tilde{j}}}$ and

$Q(m) = -\sum_{\tilde{k}=1}^{p} q_{\tilde{k}} m^{\alpha} \left[D^{\beta-1} z(\rho)\right]_{\rho=0} + \left[D^{\alpha-1}\left(z(\rho) - \sum_{\tilde{k}=1}^{p} q_{\tilde{k}} D^{\beta} z(\rho)\right)\right]_{\rho=0} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} h(m\sigma_{\tilde{j}}) - \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} h(m\nu_{\tilde{i}})$

By proceeding similarly as in Theorem 3.1.4 we get the proof.

Theorem 3.1.9: Consider the FDDE

$$D^{\alpha} z(\rho) + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} z(\rho - \nu_{\tilde{i}}) - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} z(\rho - \sigma_{\tilde{j}}) = 0 \tag{25}$$

where $\nu_{\tilde{i}}, \sigma_{\tilde{j}} > 0$, $\tilde{i} = 1, \ldots, s$, $\tilde{j} = 1, \ldots, n$, $a_{\tilde{i}}, b_{\tilde{j}} \in R^{+}$, $0 < \alpha < 1$.

If all solutions of (25) oscillates then $\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} > \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}$ and $\nu_{\tilde{i}} > \sigma_{\tilde{j}}$ for all $\tilde{i}, \tilde{j}$.

Proof: The CE of (25)

$$F(\lambda) = \lambda^{\alpha} + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} e^{-\lambda \nu_{\tilde{i}}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-\lambda \sigma_{\tilde{j}}} = 0. \tag{26}$$

If every solution of (25) oscillates then (26) has no real roots. As $F(\infty) = \infty$, it follows that

$$F(0) = \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} > 0$$

$$\Rightarrow \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} > \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}$$

Furthermore, $\nu_{\tilde{i}} > \sigma_{\tilde{j}}$. Otherwise we get $\gamma_i \leq \sigma_j$, which implies that $F(-\infty) = -\infty$.

Theorem 3.1.10: Let (A_3), (A_4) hold for the FDDE (25) and $\sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}}) \leq \lambda^{\alpha-1}$ as well as $\left(\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}\right) \nu > \frac{1}{e}\left[\lambda^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}})\right]$. Then all solutions of (25) oscillate.

Proof: Let (26) has a real root λ_0. Then by using (26)

$$\begin{aligned}
\lambda_0\left(\lambda_0^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} \int_{\sigma_{\tilde{j}}}^{\nu} e^{-\lambda_0 m} dm\right) &= \lambda_0^{\alpha} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-\lambda_0 \nu} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-\lambda_0 \sigma_{\tilde{j}}} \\
&= -\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} e^{-\lambda_0 \nu_{\tilde{i}}} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-\lambda_0 \sigma_{\tilde{j}}} + \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-\lambda_0 \nu} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} e^{-\lambda_0 \sigma_{\tilde{j}}} \\
&\leq -e^{-\lambda_0 \nu}\left(\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}\right) \\
&\leq 0
\end{aligned} \tag{27}$$

For $\lambda \geq 0$

$$\begin{aligned}
\lambda^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} \int_{\sigma_{\tilde{j}}}^{\nu} e^{-\lambda m} dm &\geq \lambda^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} \int_{\sigma_{\tilde{j}}}^{\nu} ds \\
&\geq \lambda^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}}) \geq 0
\end{aligned} \tag{28}$$

So from (27) and (28) we conclude that $\lambda_0 \leq 0$. Then (26) can be written as

$$\lambda_0\left(\lambda_0^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}})\right) + \left(\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}\right) e^{-\lambda_0 \nu} < 0$$

$$\Rightarrow \lambda_0 + \frac{\left(\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}\right)}{\lambda_0^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}})} e^{-\lambda_0 \nu} < 0$$

So the equation

$$G(\lambda) = \lambda + \frac{\left(\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}\right)}{\lambda^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}})} e^{-\lambda \nu} = 0$$

has a real root in $(\lambda_0, 0)$ as $G(\lambda_0) < 0$ and $G(0) > 0$. Therefore by [[7], Theorem 2.2.4]

$$\frac{\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}}{\lambda^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}})} \nu \leq \frac{1}{e}$$

$$i.e., \left(\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}\right) \nu \leq \frac{1}{e} \left[\lambda^{\alpha-1} - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\nu - \sigma_{\tilde{j}})\right]$$

which is a contradiction. Thus every solution of (25) oscillates.

3.2 **FDDEs with variable coefficients:**

Let us assume the following conditions:

(B_1) $a(\rho), b(\rho) \in \mathrm{C}([\rho_0, \infty), R^+)$, where $0 < a \leq a(\rho)$, $0 \leq b \leq b(\rho)$.

(B_2) $a(\rho), b(\rho) \in \mathrm{C}([\rho_0, \infty), R^+)$, where $\sigma < \nu$, $0 \leq a \leq a(\rho)$, $0 \leq b(\rho) \leq b$, $a \geq b$.

(B_3) $a_{\tilde{i}}(\rho) \in \mathrm{C}([\rho_0, \infty), R^+)$, where $0 < \sum_{\tilde{i}=1}^{n} a_{\tilde{i}} < \sum_{\tilde{i}=1}^{n} a_{\tilde{i}}(\rho)$, $\nu_{\tilde{i}} \in R^+$ for $\tilde{i} = 1, ..., n$.

(B_4) $a_{\tilde{i}}(\rho), b_{\tilde{j}}(\rho) \in \mathrm{C}([\rho_0, \infty), R^+)$, $\sigma_{\tilde{j}}, \nu_i \in R^+$ where $\sigma_{\tilde{j}} < \nu_{\tilde{i}}$, $0 \leq \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} < \sum_{\tilde{i}=1}^{s} a_{\tilde{i}}(s)$,

$0 \leq \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}(\rho) < \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}$, $\sum_{\tilde{i}=1}^{s} a_{\tilde{i}} > \sum_{\tilde{j}=1}^{n} b_{\tilde{j}}$, $s > n$ for $\tilde{i} = 1, ..., s$, $\tilde{j} = 1, ..., n$.

Lemma 3.2.1: Let us consider the FDDE,

$$D^{\alpha} z(\rho) + bz(\rho) + az(\rho - \nu) = 0. \tag{29}$$

where $a, b, \nu \in R$, $0 < \alpha < 1$.

(1) If $0 < a\nu < \frac{\beta}{e^{\beta}}$ or $a < 0$, where $\frac{\beta}{e^{\beta}} = \max \frac{\rho}{e^{\rho}}$ $(0 < \rho < \infty, 1 \leq \beta \leq 2)$ then the solution of (29) is nonoscillatory.

(2) If $a\nu > \frac{\beta}{e^{\beta}}$, then the solution of (29) is oscillatory.

Proof: The CE of (29) is

$$\lambda^{\alpha} + b + ae^{-\lambda\nu} = 0.$$

(1) Let $a > 0$. From the CE, we get

$$ae^{-\lambda\nu} = -(\lambda^{\alpha} + b) \Rightarrow a\nu = \frac{-(\lambda^{\alpha} + b)\nu}{e^{-\lambda\nu}} < \frac{1}{e} < \frac{\beta}{e^{\beta}}.$$

which is a contradiction.

So $a < 0$ or $0 < a\nu < \frac{\beta}{e^{\beta}}$ and (29) has a real root. Hence the solution of (29) is nonoscillatory.

(2) Similarly as in (1), we can find that for $a\nu > \frac{w}{e^{w}}$, (29) has no real roots.

Hence the solution of (29) is oscillatory.

Theorem 3.2.1: Let (B_1) hold for the FDDE

$$D^{\alpha}z(\rho)+b(\rho)z(\rho)+a(\rho)z(\rho-\nu)=0. \tag{30}$$

where $0<\alpha<1$. If $a\nu>\frac{\beta}{e^{\beta}}$, where $\frac{\beta}{e^{\beta}}=\max\frac{\rho}{e^{\rho}}$ $(0<\rho<\infty, 1\leq\beta\leq 2)$, then every solution of (30) oscillates.

Proof: Let (30) have an eventually positive solution $z(\rho)$. As $a\leq a(\rho)$ and $b\leq b(\rho)$,

$$D^{\alpha}z(\rho)+bz(\rho)+az(\rho-\nu)\leq 0 \tag{31}$$

Let the inequality (31) has a solution $e^{\lambda\rho}$,

$$e^{\lambda\rho}(\lambda^{\alpha}+b+ae^{-\lambda\nu})\leq 0.$$

If $a\nu>\frac{\beta}{e^{\beta}}$ then $\lambda^{\alpha}+b+ae^{-\lambda\nu}\leq 0$ does not have a real root. Hence every solution (30) oscillates.

Theorem 3.2.2: Let (B_3) hold for the FDDE

$$D^{\alpha}z(\rho)+\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}(\rho)z(\rho-\nu_{\tilde{i}})=0 \tag{32}$$

where $0<\alpha<1$. If $\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}\nu_{\tilde{i}}>\frac{\beta}{e^{\beta}}$, then every solution of (32) oscillates.

Proof: Let (32) have an eventually positive solution $z(\rho)$. As $\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}<\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}(\rho)$,

$$D^{\alpha}z(\rho)+\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}z(\rho-\nu_{\tilde{i}})\leq 0. \tag{33}$$

Let the inequality (33) has a solution $e^{\lambda\rho}$,

$$e^{\lambda\rho}(\lambda^{\alpha}+\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}e^{-\lambda\nu_{\tilde{i}}})\leq 0.$$

If $\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}\nu_{\tilde{i}}>\frac{\beta}{e^{\beta}}$ then $\lambda^{\alpha}+\sum_{\tilde{i}=1}^{n}a_{\tilde{i}}e^{-\lambda\nu_{\tilde{i}}}\leq 0$ does not have a real root. Hence every solution (32) oscillates.

Theorem 3.2.3: Let (B_2) hold for the FDDE

$$D^{\alpha}z(\rho)+a(\rho)z(\rho-\nu)-b(\rho)z(\rho-\sigma)=0 \tag{34}$$

where $0<\alpha<1$. If $(a-b)\nu>\frac{\beta}{e^{\beta}}$, then every solution of (34) oscillates.

Proof: Assume that (34) has eventually positive solution. Then we get the following inequality

$$D^{\alpha}z(\rho)+az(\rho-\nu)-bz(\rho-\sigma)\leq 0 \tag{35}$$

Let the above inequality has a solution $e^{\lambda\rho}$,

$$(\lambda^{\alpha}+(a-b)e^{-\lambda}\nu)\leq 0.$$

Thus, if $(a-b)\nu>\frac{\beta}{e^{\beta}}$, then every solution of (34) oscillates.

Theorem 3.2.4: Let (B_4) hold for the FDDE

$$D^{\alpha}z(\rho)+\sum_{\tilde{i}=1}^{s}a_{\tilde{i}}(\rho)z(\rho-\nu_{\tilde{i}})-\sum_{\tilde{j}=1}^{n}b_{\tilde{j}}(\rho)z(\rho-\sigma_{\tilde{j}})=0 \tag{36}$$

where $0 < \alpha < 1$. If $\sum_{\tilde{i}=1}^{s}(a_{\tilde{i}} - b_{\tilde{i}})\nu_{\tilde{i}} > \frac{\beta}{e^{\beta}}$, then every solution of (36) oscillates.

Proof: Assume that (36) has eventually positive solution. Then we get the following inequality

$$D^{\alpha} z(\rho) + \sum_{\tilde{i}=1}^{s} a_{\tilde{i}} z(\rho - \nu_{\tilde{i}}) - \sum_{\tilde{j}=1}^{n} b_{\tilde{j}} z(\rho - \sigma_{\tilde{j}}) \leq 0 \tag{37}$$

Let the above inequality has a solution $e^{\lambda\rho}$,

$$(\lambda^{\alpha} + \sum_{\tilde{i}=1}^{s}(a_{\tilde{i}} - b_{\tilde{i}})e^{-\lambda}\nu_{\tilde{i}}) \leq 0.$$

Thus, if $\sum_{\tilde{i}=1}^{s}(a_{\tilde{i}} - b_{\tilde{i}})\nu_{\tilde{i}} > \frac{\beta}{e^{\beta}}$, then every solution of (34) oscillates.

4 Examples

In this section, we have established some suitable examples, which illustrate our results.

Example 1: Consider the FDDE

$$D^{\frac{1}{2}}\left[z(\rho) + D^{\frac{1}{3}} z(\rho)\right] + 2cos\frac{\pi}{12} z\left(\rho - \frac{2\pi}{3}\right) = 0, \quad \rho \in \left[\frac{2\pi}{3}, \infty\right). \tag{38}$$

Here, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $b = 2cos\frac{\pi}{12}$, $\nu = \frac{2\pi}{3}$ and $0 < \frac{1}{2}, \frac{1}{3} < 1$.

The CE of (38) is

$$\lambda^{\frac{1}{2}} + \lambda^{\frac{5}{6}} + 2cos\frac{\pi}{2} e^{\frac{-2\lambda\pi}{3}} = 0,$$

which have no real roots. Figure 1(a) shows the CE of (38) which have no real roots. So the conditions of Theorem 3.1.1 are satisfied. Hence every solution of (38) oscillates. Figure 1(b) shows the solution diagram of (38).

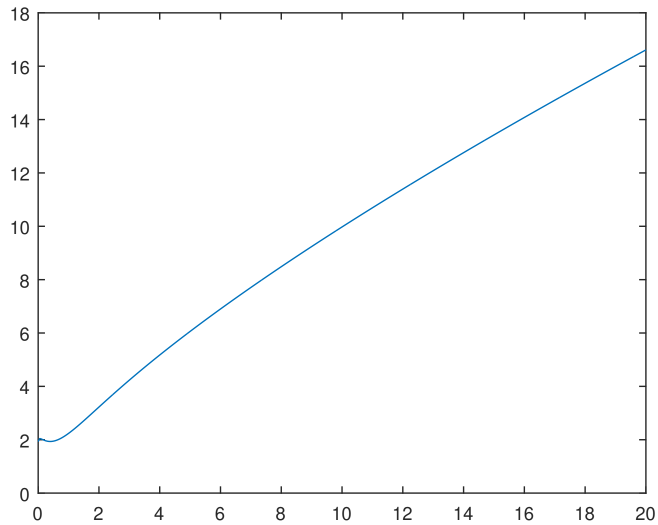

Figure 1(a): The Characteristic equation diagram for (38).

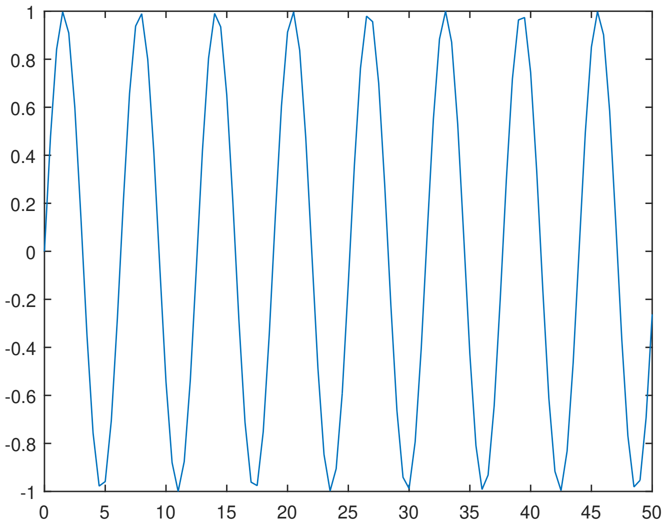

Figure 1(b): The solution diagram for (38).

Example 2: Consider the FDDE

$$D^{\frac{1}{2}}\left[z(\rho)+D^{\frac{1}{3}}z(\rho)\right]+cos\frac{\pi}{12}z(\rho-\pi)+\sqrt{3}cos\frac{\pi}{12}z\left(\rho-\frac{\pi}{2}\right)=0,\quad \rho\in[\pi,\infty). \tag{39}$$

Here $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $b_1=cos\frac{\pi}{12}$, $b_2=\sqrt{3}cos\frac{\pi}{12}$, $v_1=\pi$, $v_2=\frac{\pi}{2}$ and $0<\frac{1}{2},\frac{1}{3}<1$.

The CE of (39) is

$$\lambda^{\frac{1}{2}}+\lambda^{\frac{5}{6}}+cos\frac{\pi}{12}e^{-\lambda\pi}+\sqrt{3}cos\frac{\pi}{12}e^{\frac{-\lambda\pi}{2}}=0,$$

which have no real roots. Figure 2(a) shows the CE diagram for (39), which illustrates that (39) have no real root. All the conditions of Theorem 3.1.2 are satisfied. So every solution of (39) oscillates. Figure 2(b) shows the solution of (39).

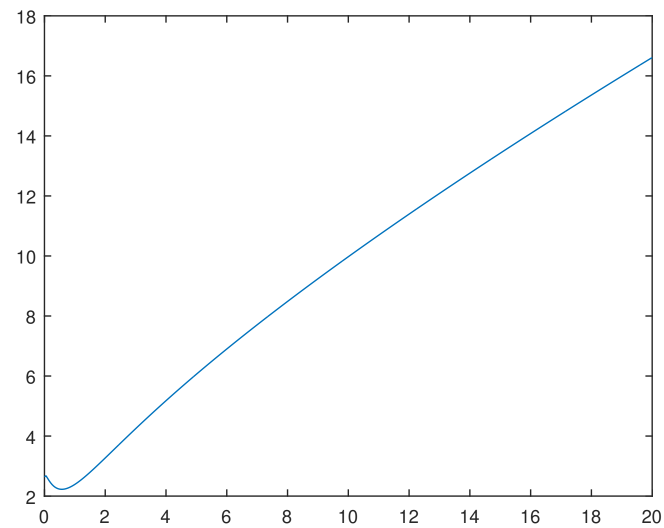

Figure 2(a): The Characteristic equation diagram for (39).

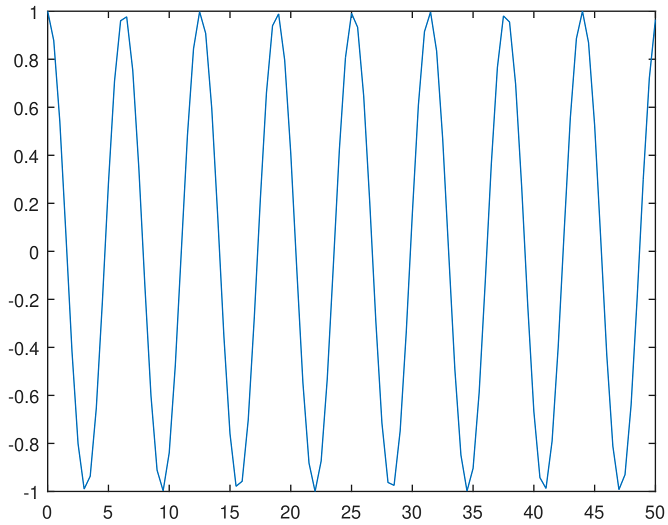

Figure 2(b): The solution diagram for (39).

Example 3: Consider the FDDE

$$D^{\frac{1}{2}}\left[z(\rho)-D^{\frac{1}{3}}z(\rho)\right]+2sin\frac{\pi}{12}z\left(\rho-\frac{7\pi}{6}\right)=0,\quad \rho\in\left[\frac{7\pi}{6},\infty\right) \tag{40}$$

Here $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $q=1$, $b=2sin\frac{\pi}{12}$, $v=\frac{7\pi}{6}$ and $0<\frac{1}{2},\frac{1}{3}<1$.

The CE of (40) is

$$\lambda^{\frac{1}{2}}-\lambda^{\frac{5}{6}}+2sin\frac{\pi}{12}e^{\frac{-7\lambda\pi}{6}}=0,$$

which have no real roots. Figure 3(a) shows the CE diagram of (40), which have no real roots. So all the conditions of Theorem 3.1.3 are satisfied.

So every solution of (40) oscillates. Figure 3(b) shows the solution of (40), which is oscillatory.

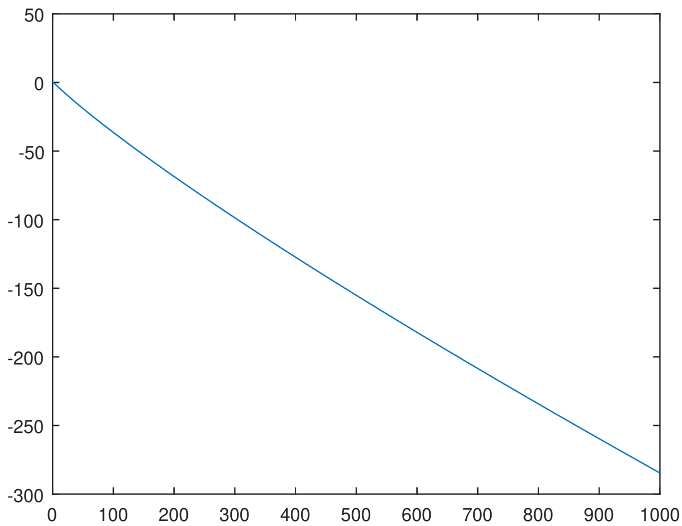

Figure 3(a): The Characteristic equation diagram for (40).

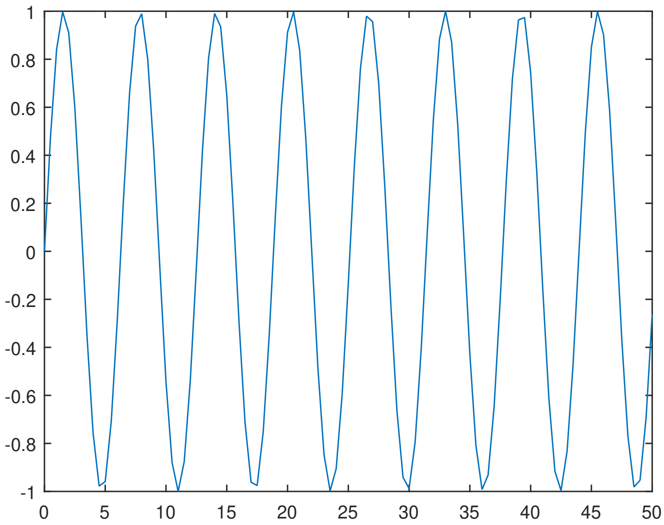

Figure 3(b): The Solution diagram for (40).

Example 4: Consider the FDDE

$$D^{\frac{1}{2}}\left[z(\rho)-D^{\frac{1}{3}}z(\rho)-D^{\frac{2}{3}}z(\rho)\right]+z\left(\rho-\frac{3\pi}{4}\right)+2cos\left(\frac{\pi}{12}\right)z\left(\rho-\frac{3\pi}{2}\right)=0,\quad \rho\in\left[\frac{3\pi}{2},\infty\right). \tag{41}$$

Here $\alpha=\frac{1}{2}$, $\beta_1=\frac{1}{3}$, $\beta_2=\frac{2}{3}$, $q_1=1$, $q_2=1$, $b_1=1$, $b_2=2cos\frac{\pi}{12}$, $\nu_1=\frac{3\pi}{4}$, $\nu_2=\frac{3\pi}{2}$ and $0<\frac{1}{2},\frac{1}{3},\frac{2}{3}<1$.

The CE of (41) is

$$\lambda^{\frac{1}{2}}-\lambda^{\frac{5}{6}}-\lambda^{\frac{7}{6}}+e^{\frac{-3\lambda\pi}{4}}+2cos\frac{\pi}{12}e^{\frac{-3\lambda\pi}{2}}=0,$$

which have no real roots. Figure 4(a) shows that the CE of (41) have no real roots. So all the conditions of Theorem 3.1.4 are satisfied.

So every solution of (41) oscillates. Figure 4(b) shows the solution of (41), which is oscillatory.

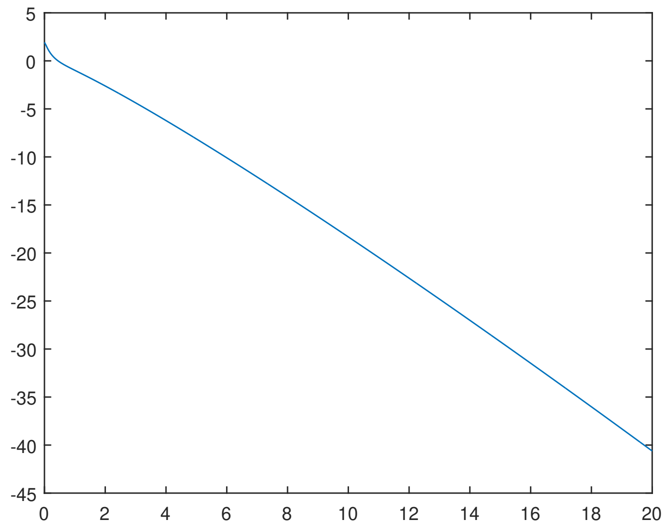

Figure 4(a): The Characteristic equation diagram for (41).

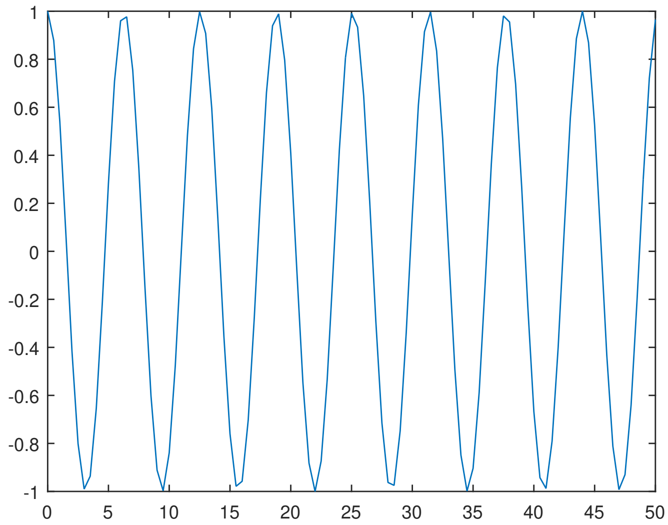

Figure 4(b): The solution diagram for (41).

Example 5: Consider the FDDE

$$D^{\frac{1}{3}}\left[z(\rho)+D^{\frac{1}{5}}z(\rho)\right]+2cos\frac{\pi}{20}cos\frac{13\pi}{60}z(\rho-\pi)-2cos\frac{\pi}{20}sin\frac{13\pi}{60}z\left(\rho-\frac{3\pi}{2}\right)=0,\ \ \rho\in\left[\frac{3\pi}{2},\infty\right) \tag{42}$$

Here $\alpha=\frac{1}{3}$, $\beta=\frac{1}{5}$, $a=2cos\frac{\pi}{20}cos\frac{13\pi}{60}$, $b=2cos\frac{\pi}{20}sin\frac{13\pi}{60}$, $\nu=\pi$, $\sigma=\frac{3\pi}{2}$ and $0<\frac{1}{3},\frac{1}{5}<1$. So $a>b$ and $\nu<\sigma$ and the CE of (42) is

$$\lambda^{\frac{1}{3}}+\lambda^{\frac{8}{15}}+2cos\frac{\pi}{20}cos\frac{13\pi}{60}e^{-\lambda\pi}-2sin\frac{\pi}{20}sin\frac{13\pi}{60}e^{\frac{-3\lambda\pi}{2}}=0,$$

which have no real roots. Figure 5(a) illustrates that the CE of (42) have no real roots. So All the conditions of Theorem 3.1.5 are satisfied. Hence every solution of (42) oscillates. Figure 5(b) shows the oscillatory solution of (42).

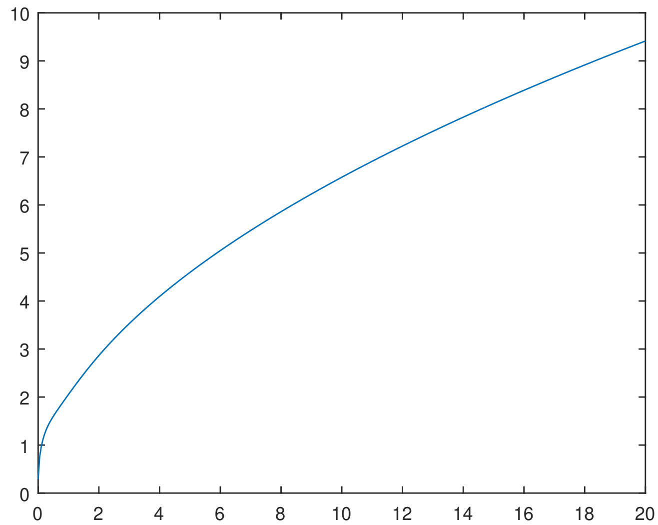

Figure 5(a): The characteristic equation diagram for (42).

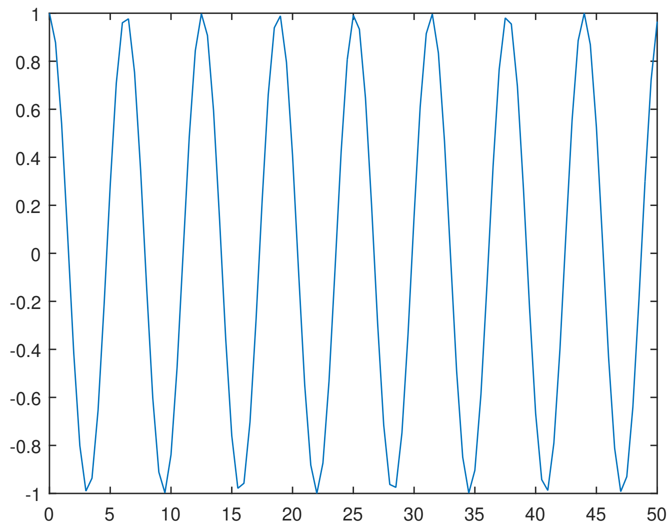

Figure 5(b): The solution diagram for (42).

Example 6: Consider the FDDE

$$D^{\frac{1}{2}}\left[z(\rho)+D^{\frac{1}{3}}z(\rho)\right]+\frac{\sqrt{3}}{2}cos\frac{\pi}{12}z\left(\rho-\frac{\pi}{2}\right)+cos\frac{\pi}{12}z(\rho-\pi)-\frac{\sqrt{3}}{2}cos\frac{\pi}{12}z\left(\rho-\frac{3\pi}{2}\right)=0,\ \rho\in\left[\frac{3\pi}{2},\infty\right) \quad (43)$$

Here $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $a_1=\frac{\sqrt{3}}{2}cos\frac{\pi}{12}$, $a_2=cos\frac{\pi}{12}$, $b_1=\frac{\sqrt{3}}{2}cos\frac{\pi}{12}$, $\nu_1=\frac{\pi}{2}$, $\nu_2=\pi$ $\sigma_1=\frac{3\pi}{2}$ and $0<\frac{1}{2},\frac{1}{3}<1$. So $a_1+a_2>b_1$, $\nu_1<\sigma_1$, $\nu_2<\sigma_1$ and the CE of (43) is

$$\lambda^{\frac{1}{2}}+\lambda^{\frac{5}{6}}+\frac{\sqrt{3}}{2}cos\frac{\pi}{12}e^{\frac{-\lambda\pi}{2}}+cos\frac{\pi}{12}e^{-\lambda\pi}-\frac{\sqrt{3}}{2}cos\frac{\pi}{12}e^{\frac{-3\lambda\pi}{2}}=0.$$

Figure 6(a) shows that the CE of (43) have no real roots. So all the conditions of Theorem 3.1.6 are satisfied. Hence every solution of the given FDDE oscillates. Figure 6(b) shows the solution for (43).

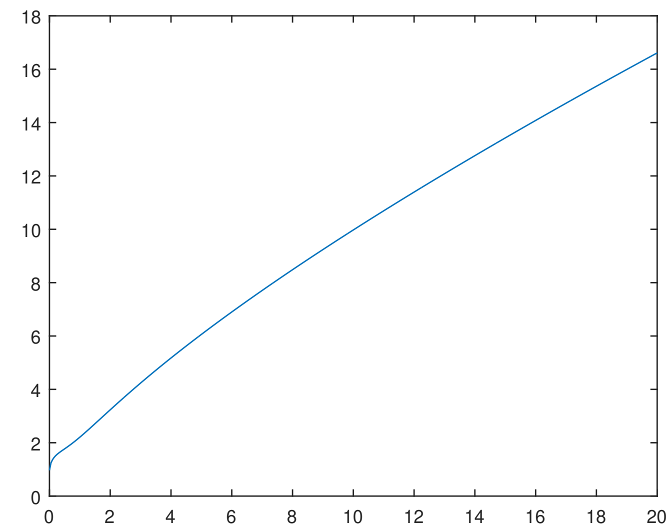

Figure 6(a): The characteristic equation diagram for (43)

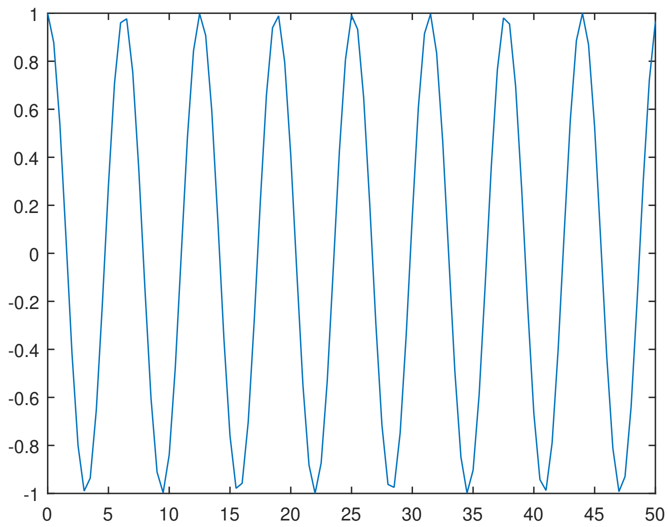

Figure 6(b): The solution diagram for (43)

Example 7: Consider the FDDE

$$D^{\frac{1}{2}}\left[z(\rho)-D^{\frac{1}{3}}z(\rho)\right]+z\left(\rho-\frac{19\pi}{12}\right)-z\left(\rho-\frac{7\pi}{4}\right)=0 \quad \rho\in\left[\frac{7\pi}{4},\infty\right). \tag{44}$$

Here $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $q=1$, $a=1$, $b=1$, $\nu=\frac{19\pi}{12}$, $\sigma=\frac{7\pi}{4}$ and $0<\frac{1}{2},\frac{1}{3}<1$. So $a=b$, $\nu<\sigma$ and the CE of (44) is

$$\lambda^{\frac{1}{2}}-\lambda^{\frac{5}{6}}+e^{\frac{-19\lambda\pi}{12}}-e^{\frac{-7\lambda\pi}{4}}=0,$$

which have no real roots. Figure 7(a) shows the CE of (44). So all the conditions of Theorem 3.1.7 are satisfied. Hence every solution of (44) oscillates. Figure 7(b) shows the solution diagram for (44), which is oscillatory in nature.

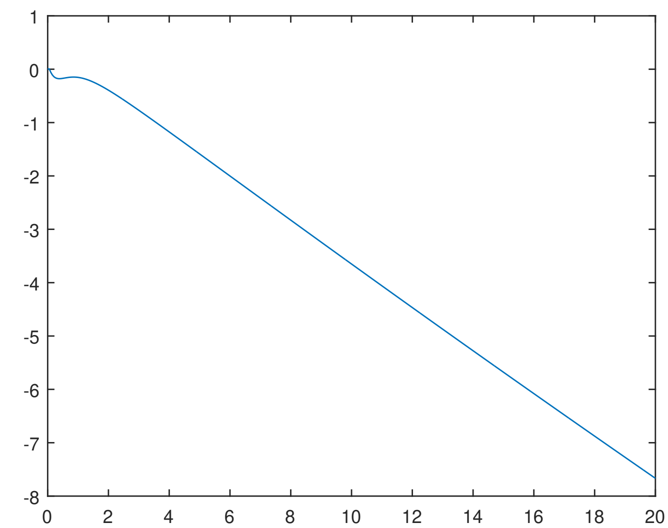

Figure 7(a): The characteristic equation diagram for (44).

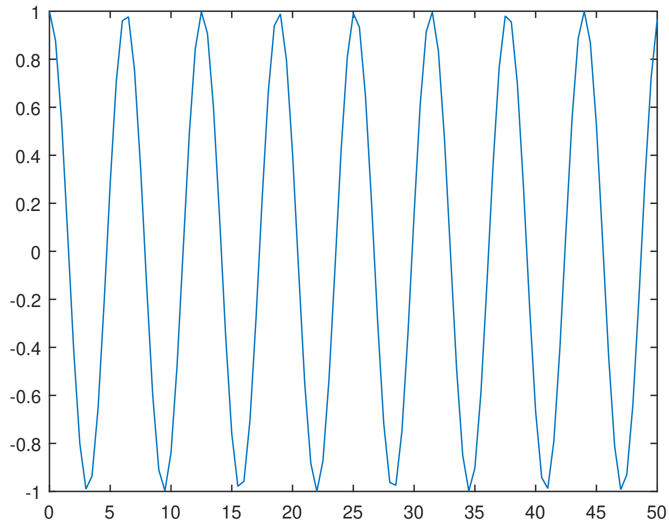

Figure 7(b): The solution diagram for (44).

Example 8: Consider the FDDE

$$D^{\frac{1}{2}}\left[z(\rho)-D^{\frac{1}{3}}z(\rho)-D^{\frac{2}{3}}z(\rho)\right]+z\left(\rho-\frac{19\pi}{12}\right)+z\left(\rho-\frac{17\pi}{12}\right)-z\left(\rho-\frac{7\pi}{4}\right)=0,\quad \rho\in\left[\frac{11\pi}{6},\infty\right) \tag{45}$$

Here $\alpha=\frac{1}{2}$, $\beta_1=\frac{1}{3}$, $\beta_2=\frac{2}{3}$ $q_1=1$, $q_2=1$, $a_1=1$, $a_2=1$, $b_1=1$, $v_1=\frac{19\pi}{12}$, $v_2=\frac{17\pi}{12}$, $\sigma_1=\frac{7\pi}{4}$ and $0<\frac{1}{2},\frac{1}{3},\frac{2}{3}<1$. So $a_1+a_2>b_1$, $v_1<\sigma_1$, $v_2<\sigma_1$ and the CE of (45) is

$$\lambda^{\frac{1}{2}}-\lambda^{\frac{5}{6}}-\lambda^{\frac{7}{6}}+e^{\frac{-19\lambda\pi}{12}}+e^{\frac{-17\lambda\pi}{12}}-e^{\frac{-7\lambda\pi}{4}}=0,$$

which have no real roots. Figure 8(a) shows the CE diagram of (45). So all the conditions of Theorem 3.1.7 are satisfied. Hence every solution of (45) oscillates. Figure 8(b) shows the solution for (45), which is oscillatory in nature.

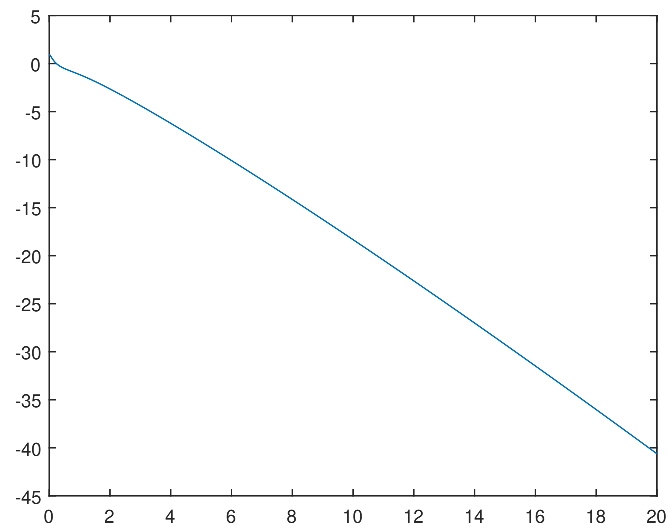

Figure 8(a): The characteristic equation diagram for (45).

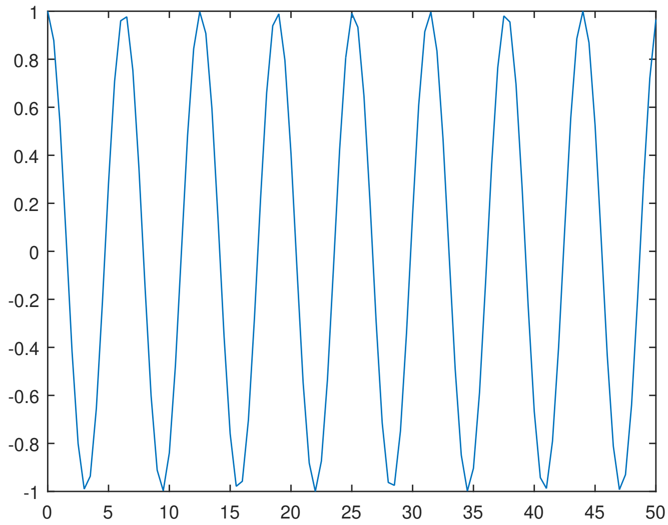

Figure 8(b): The solution diagram for (45).

Example 9: Cosider the FDDE

$$D^{\frac{1}{2}}z(\rho)+z\left(\rho-\frac{11\pi}{4}\right)+\frac{1}{2}z\left(\rho-\frac{15\pi}{4}\right)-\frac{1}{2}z\left(\rho-\frac{7\pi}{4}\right)=0,\quad \rho\in\left[\frac{15\pi}{4},\infty\right). \tag{46}$$

Here $\alpha=\frac{1}{2}$, $a_1=1$, $a_2=\frac{1}{2}$, $b_1=\frac{1}{2}$, $\gamma_1=\frac{11\pi}{4}$, $\gamma_2=\frac{15\pi}{4}$, $\sigma_1=\frac{7\pi}{4}$ and $0<\frac{1}{2}<1$. Let $\lambda=0.01$. So $a_1+a_2=1+\frac{1}{2}>1=b_1$, $\frac{11\pi}{4}>\frac{7\pi}{4}$, $\frac{15\pi}{4}>\frac{7\pi}{4}$, $\max_{1\le\tilde{i}\le 2}\nu_{\tilde{i}}=\frac{15\pi}{4}=\nu$, $\sum_{\tilde{j}=1}^{1}b_{\tilde{j}}(\nu-\sigma_{\tilde{j}})=\pi<10=\lambda^{\alpha-1}$ and $(\sum_{\tilde{i}=1}^{2}a_{\tilde{i}}-\sum_{\tilde{j}=1}^{1}b_{\tilde{j}})\nu=\frac{15\pi}{4}>2.52=\frac{1}{e}\left[\lambda^{\alpha-1}-\sum_{\tilde{j}=1}^{n}b_{\tilde{j}}(\nu-\sigma_{\tilde{j}})\right]$. Hence all the conditions of Theorem 3.1.9 and Theorem 3.1.10 satisfied. Figure 9 shows the solution of (46).

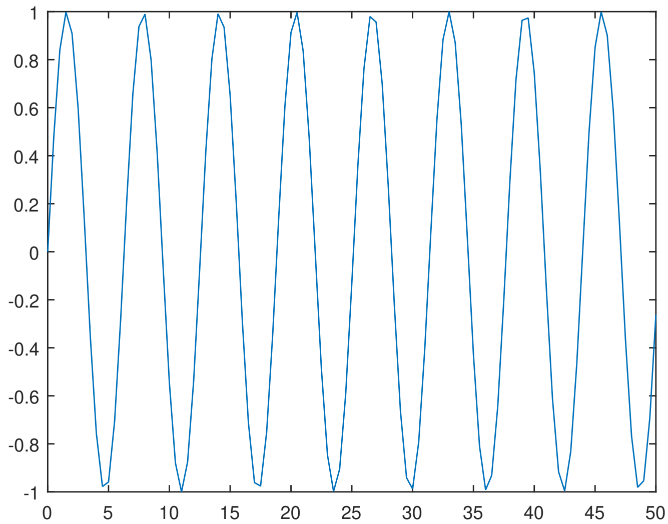

Figure 9: The solution diagram for (46).

Example 10: Consider the FDDE

$$D^{\frac{1}{3}}z(\rho)+\frac{1}{4.06\rho^{\frac{1}{3}}}z(\rho)+\left(1-\frac{5\pi}{6\rho}\right)z\left(\rho-\frac{5\pi}{6}\right)=0,\quad \rho\in[\frac{5\pi}{6},\infty). \tag{47}$$

Here $\alpha = \frac{1}{3}$, $0 < \frac{1}{3} < 1$, $a(\rho) = (1 - \frac{5\pi}{6\rho})$, $b(\rho) = \frac{1}{4.06\rho^{\frac{1}{3}}}$ and $\nu = \frac{5\pi}{6}$.

Let us choose $a = 0.5 < 1 - \frac{5\pi}{6\rho} = a(\rho)$ and $b = 0 < \frac{1}{4.06\rho^{\frac{1}{3}}}$. So $a\nu = 1.30 > 0.36 = \frac{\beta}{e^{\beta}}$ for $\beta = 1$. Hence (47) satisfies all the conditions of Theorem 3.2.1. Figure 10 shows the solution diagram for (47), which is oscillatory in nature.

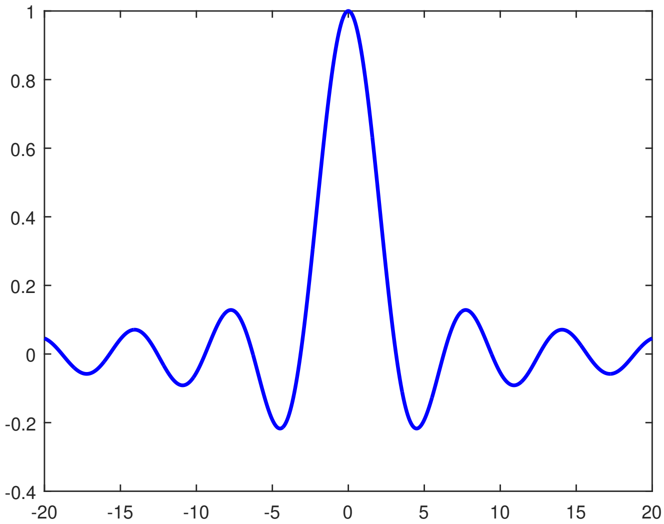

Figure 10: The solution diagram for (47).

Example 11: Consider the FDDE

$$D^{\frac{1}{3}}z(\rho) + \left(\frac{1}{\rho} + 1\right) z\left(\rho - \frac{5\pi}{6}\right) + \frac{1}{\rho} z\left(\rho - \frac{11\pi}{6}\right) = 0, \quad \rho \in \left[\frac{11\pi}{6}, \infty\right) \tag{48}$$

Here $\alpha = \frac{1}{3}$, $0 < \frac{1}{3} < 1$, $a_1(\rho) = \frac{1}{\rho} + 1$, $a_2(\rho) = \frac{1}{\rho}$, $\nu_1 = \frac{5\pi}{6}$ and $\nu_2 = \frac{11\pi}{6}$.

Let us choose $\sum_{\bar{i}=1}^{2} a_{\bar{i}} = 1 < \frac{2}{\rho} + 1 = \sum_{\bar{i}=1}^{2} a_{\bar{i}}(\rho_{\bar{i}})$. So $\sum_{\bar{i}=1}^{2} a_{\bar{i}}\nu_{\bar{i}} = 2.61 > 0.36 = \frac{\beta}{e^{\beta}}$ for $\beta = 1$. Hence (48) satisfies all the conditions of Theorem 3.2.2. Figure 11 shows the solution diagram for (48).

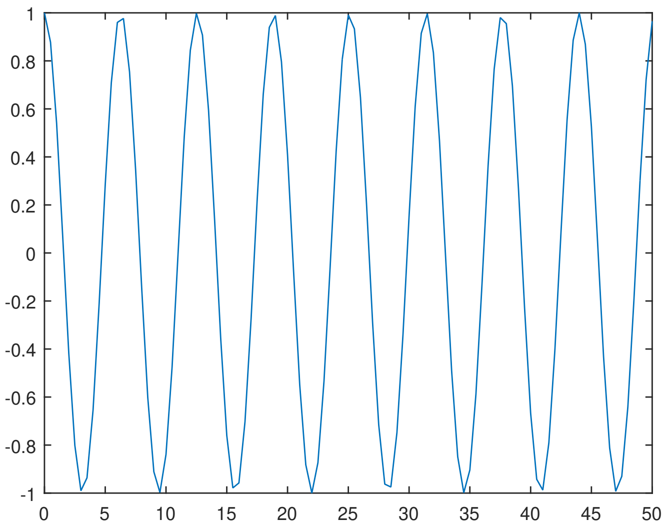

Figure 11: The solution diagram for (48)

The following counterexample illustrates Theorem 12.

Example 12: Consider the FDDE

$$D^{\frac{1}{2}}z(\rho)+\frac{e^2(\rho-2)}{2\sqrt{\pi}\rho^{\frac{3}{2}}}z(\rho-2)-\frac{e(\rho-1)}{\rho}z(\rho-1)=0, \quad \rho\in[2,\infty) \tag{49}$$

Here $\alpha=\frac{1}{2}$, $0<\frac{1}{2}<1$, $a(\rho)=\frac{e^2(\rho-2)}{2\sqrt{\pi}\rho^{\frac{3}{2}}}$, $b(\rho)=\frac{e(\rho-1)}{\rho}$, $\nu=2$ and $\sigma=1$. Let us choose $a=0$ and $b=3$. So $(a-b)\nu=-6<\frac{\beta}{e^{\beta}}$. Hence by Theorem 3.2.3, we will get a non-oscillatory solution. Figure 12 shows the solution diagram for (49) which is non-oscillatory in nature.

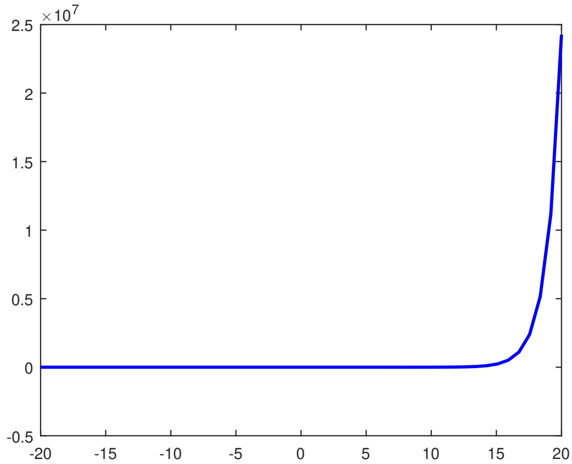

Figure 12: The solution diagram for (49).

The following counterexample illustrates Theorem 13 by contrapositive method.

Example 13: Consider the FDDE

$$D^{\frac{1}{2}}z(\rho)+\frac{(\rho-2.5)}{e^{\rho-2.5}}z(\rho-2.5)+\frac{e^2(\rho-2)}{2\sqrt{\pi}\rho^{\frac{3}{2}}}z(\rho-2)-\frac{e(\rho-1)}{\rho}z(\rho-1) \\ -\frac{\rho-0.5}{e^{\rho-0.5}}z(\rho-0.5)=0, \ \ \rho\in[2.5,\infty). \tag{50}$$

Here $\alpha=\frac{1}{2}$, $0<\frac{1}{2}<1$, $a_1(\rho)=\frac{(\rho-2.5)}{e^{\rho-2.5}}$, $a_2(\rho)=\frac{e^2(\rho-2)}{2\sqrt{\pi}\rho^{\frac{3}{2}}}$, $b_1(\rho)=\frac{e(\rho-1)}{\rho}$, $b_2(\rho)=\frac{\rho-0.5}{e^{\rho-0.5}}$, $\nu_1=2.5$, $\nu_2=2$, $\sigma_1=1$ and $\sigma_2=0.5$. Let us choose $\sum_{\tilde{i}=1}^{2}a_{\tilde{i}}=0$ and $\sum_{\tilde{j}=1}^{2}b_{\tilde{j}}=3$. So $\sum_{\tilde{i}=1}^{2}(a_{\tilde{i}}-b_{\tilde{i}})\nu_{\tilde{i}}<\frac{\beta}{e^{\beta}}$. Hence by Theorem 3.2.4, we will get a nonoscillatory solution. Figure 13 shows the solution diagram for (50), which is non-oscillatory in nature.

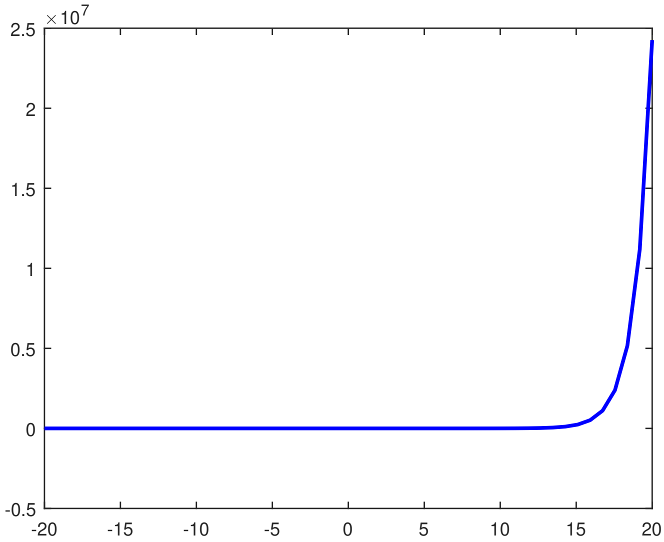

Figure 13: The solution diagram for (50)

5 Conclusion

In this paper, we have dealt with FDDEs with both constant and variable coefficients. New results on the oscillatory behaviour of solutions have been obtained in FDDEs. Graphs and suitable examples have been employed to demonstrate all the theoretical aspects. By using MATLAB, the graphs have been prepared. The novelty of the work is that, various types of FDDEs have been considered and the qualitative behaviour of their solutions have been discussed. Furthermore, we want to continue working on the study of oscillatory and non-oscillatory criteria of diffusion equations of fractional order as well as systems of fractional functional differential equations.

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