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A Theoretical Approach on Unitary Operators in Fuzzy Soft Settings

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Abstract: This study presents the definition of the fuzzy soft unitary operator, a particular sort of the fuzzy soft linear operators in the fuzzy soft Hilbert spaces built on the fuzzy soft inner product spaces, proposed by Faried et al. [1]. In addition, related findings like the fuzzy soft spectral theory and the connection between the fuzzy soft unitary operators and each of the fuzzy soft isometry operators and the fuzzy soft normal operators are introduced. Furthermore, the fuzzy soft unitary equivalence with its related theorems are established.

Keywords: Fuzzy set, Fuzzy soft linear operator, Fuzzy soft set, Soft set

AMS 2010 Subject Classification: 46B99; 03E72; 46S40; 47S40

Abbreviation: FS: fuzzy soft

1 Introduction

In the actual world, uncertainty manifests itself in the form of ambiguity, which leads to complexity. As a result, we constantly have a plethora of complex issues in fields such as engineering, sociology, medical science, economics, environmental science, company management, and a variety of others. We won't be able to solve those issues using traditional mathematical approaches because of the uncertainties. Zadeh [2] suggested an extension of the set theory called fuzzy sets theory to cope with uncertainty in 1965. A fuzzy set on a domain X is described by its membership (characteristic) function from X to $[0,1]$, just as a crisp set on a universal set X is characterized by its characteristic function from X to $[0,1]$. In truth, the concept of a fuzzy set is entirely non-statistical. Although the fuzzy set tool is considered as a valuable mathematical method for dealing with uncertainty, this single specific number (membership grade) contains evidence for and against element membership without showing how much of each there is. Molodtsov [3] presented the concept of the soft set theory in 1999 to manage uncertainties and deal with complex issues which cannot be handled using traditional methods in a variety of fields, including measure theory, Riemann integration, decision-making, environmental science, game theory, engineering, physics, computer science, economics, medicine, and many others. The soft set is defined as a parameterized class of subsets of the universal set that may be used to describe uncertainty by linking a set with a set of parameters. Following that, several researchers introduced innovative extended notions based on soft sets, provided examples, and investigated their properties, such as soft point [4], soft normed spaces [5], soft metric spaces [6], soft inner product spaces [7], as well as soft Hilbert spaces [8], and others. However, despite this improvement, we virtually always have inexact knowledge about the items we are considering in real-life problems and circumstances. Maji et al. [9] merged the two notions, fuzzy set, and soft set, into one novel concept called FS set, in order to enhance both. This new notion broadened the soft sets method from ordinary situations to fuzzy (more general) ones. Many authors have implemented this concept in recent years, coining topics like FS point [10], FS normed spaces [11], as well as FS metric spaces [12]. Faried et al.

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[1] presented the FS inner product spaces, as well as their characteristics and some associated findings. They also provided a definition of the FS Hilbert space, as well as investigated its features and several other related research findings [13]. They also differentiated the FS linear operators in FS Hilbert spaces [14], as well as their associated theories, such as spectral theory and verifying the FS Hilbert space's FS self-duality. And at last, they developed the FS symmetric operators [15], the FS hermitian operators [16], the FS normal operators [17] and the FS isometry operators [18], and studied their related properties. For more details and more related topics, one can refer to [19] and [20].

In this work, we build on these past researches by proposing the FS unitary operator, a specific case of FS linear operator in FS Hilbert spaces, and deriving its associated results, including FS spectral theory and the connection between the FS unitary operators and each of FS isometry operators and FS normal operators. In addition, the FS unitary equivalence with its related theorems are investigated.

2 Definitions and Preliminaries

This section provides a variety of notations, concepts, and preliminaries that will be used in the remainder of this article.

Definition 2.1.[9] Suppose that U is a universal set, E is a set of attributes, and $A \subseteq E$. An FS set over U is considered as a pair (G,A), where G is a mapping determined by $G: A \to \mathscr{F}(U)$, $mathcalF(U)$ is the class of all fuzzy subsets of U, and the fuzzy subset of U is described as a map f from U to $[0,1]$. The collection of all FS sets (G,A) over a universal set U with a certain parameter set A is indicated by $FSS(U)_A = FSS(\tilde{U})$, where $\tilde{}$ means $(\cdot,A)$ or $(\cdot)_A$.

Definition 2.2.([1],[10]) We have the FS set $(G,A) \in FSS(\tilde{U})$ is considered as an FS point over U, written as $(u_{f_{G(e)}},A)$ (for short, referred as $\tilde{u}_{f_{G(e)}}$), if for $e \in A$ and $u \in U$,

$$f_{G(e)}(u) = \begin{cases} \alpha \ , \ if\ u = u_0 \in U \ and\ e = e_0 \in A, \\ 0 \ , \ if\ u \in U - \{u_0\}\ or\ e \in A - \{e_0\} \end{cases},$$

where $\alpha \in (0,1]$ is the value of the membership degree. The FS point can be considered as the quadruple (u_0, e_0, G, α).

It's worth mentioning that $\mathbb{R}(A)$ and $\mathbb{C}(A)$, respectively, indicate the set of all FS real numbers and the set of all FS complex numbers. In addition, $\tilde{\theta} = (\tilde{0},\tilde{0},\tilde{0},\tilde{0})$ and $\tilde{j} = (\tilde{1},\tilde{1},\tilde{1},\tilde{1})$.

Definition 2.3.[13] Assume $(\tilde{U}, \widetilde{<\cdot,\cdot>})$ is an FS inner product space. Consequently, this space, that is FS complete in the imposed FS norm is termed as an FS Hilbert space, written as $(\tilde{H}, \widetilde{<\cdot,\cdot>})$ (briefly, $\tilde{H}$). It should be noted that every FS Hilbert space is an FS Banach space, clearly.

Theorem 2.1.[14] Given $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$, where $\tilde{H}$ is an FS Hilbert space. Then, $\tilde{T}^{\tilde{*}}$ is FS bounded and

$$\widetilde{\|\tilde{T}^{\tilde{*}}\|} \tilde{=} \widetilde{\|\tilde{T}\|}. \tag{1}$$

Theorem 2.2.[14] Let $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$, where $\tilde{H}$ is an FS Hilbert space. Then,

$$\widetilde{\|\tilde{T}^{\tilde{*}}\tilde{T}\|} \tilde{=} \widetilde{\|\tilde{T}\|}^{2} \tilde{=} \widetilde{\|\tilde{T}\tilde{T}^{\tilde{*}}\|}.$$

Definition 2.4.[14] Let $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$, where $\tilde{H}$ is an FS Hilbert space. Then, we define

$$\begin{aligned} \tilde{\rho}(\tilde{T}) \tilde{=} \{\tilde{\lambda} \tilde{\in} \mathbb{C}(A) : (\tilde{\lambda}\tilde{I} - \tilde{T})^{\tilde{-1}}\ exists\ and FS\ bounded\} \\ \tilde{=} \{\tilde{\lambda} \tilde{\in} \mathbb{C}(A) : |\tilde{\lambda}| \tilde{>} \widetilde{\|\tilde{T}\|}\}. \end{aligned}$$

We call $\tilde{\rho}(\tilde{T})$ the FS resolvent set of an FS linear operator $\tilde{T}$.

Definition 2.5.[14] $\tilde{\lambda} \tilde{\in} \mathbb{C}(A)$ is termed as an FS eigenvalue of an FS linear operator $\tilde{T}$ if $\tilde{\lambda}\tilde{I} - \tilde{T}$ is not FS injective, i.e., there exists a non-zero FS element $\tilde{v}_{f_{G(e)}} \tilde{\in} \tilde{H}$ such that $\tilde{T}\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\lambda}\tilde{v}_{f_{G(e)}}$. Moreover, $\tilde{v}_{f_{G(e)}} \tilde{\neq} \tilde{\theta}$ is called the FS eigenvector of an FS linear operator $\tilde{T}$ corresponding to $\tilde{\lambda}$. The set of all such $\tilde{\lambda}$ is called the FS point spectrum of an FS linear operator $\tilde{T}$, denoted by $\tilde{\sigma}_p(\tilde{T})$. We have

$$\tilde{\sigma}_p(\tilde{T}) \tilde{\subseteq} \tilde{\sigma}(\tilde{T}). \tag{2}$$

Definition 2.6.[14] Suppose that $\tilde{H}$ is an FS Hilbert space. If $\tilde{v}_{f_{G(e)}} \tilde{=} (\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}) \tilde{\in} \tilde{H}$. Define the FS operator $\tilde{\text{R}}$ as follows:

$$\begin{aligned} \tilde{\text{R}}\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\text{R}}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}) \\ \tilde{=} (\tilde{0}, \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}). \end{aligned} \tag{3}$$

The FS operator $\tilde{\text{R}}$ is called the FS right shift operator.

Definition 2.7.[17] Assume that $\tilde{H}$ is an FS Hilbert space. Then, we have that $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$ is called FS normal operator if

$$\tilde{T}\tilde{T}^{\tilde{*}} \tilde{=} \tilde{T}^{\tilde{*}}\tilde{T}. \tag{4}$$

Definition 2.8.[18] Let $\tilde{H}$ be an FS Hilbert space. $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$ is termed as an FS isometry operator if for each $\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} \tilde{\in} \tilde{H}$, we have:

$$\widetilde{< \tilde{T}\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{T}\tilde{v}^2_{f_{2_{G(e_2)}}} >} \tilde{=} \widetilde{< \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}} >}. \tag{5}$$

Theorem 2.3.[18] $\tilde{T} \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$ is FS isometry operator if and only if $\tilde{T}^{\tilde{*}}\tilde{T} \tilde{=} \tilde{I}$.

3 Fuzzy Soft Unitary Operators

This section introduces the FS unitary operators in FS Hilbert spaces. Furthermore, a research on associated findings involving their FS eigenvalues and FS eigenvectors and the connection between the FS unitary operators and each of FS isometry operators and FS normal operators are established.

Definition 3.1. Assume that $\tilde{H}$ is an FS Hilbert space. Then, we have $\tilde{T}\tilde{\in}\tilde{\mathbb{B}}(\tilde{H})$ is termed as an FS unitary operator if

$$\tilde{T}^{\tilde{*}} \tilde{=} \tilde{T}^{\widetilde{-1}}. \tag{6}$$

Remark 3.1. Every FS unitary operator is FS normal.

Proof. Applying (6) given in Definition (3.1) of FS unitary operator, we have:

$$\begin{aligned}\tilde{T}\tilde{T}^{\tilde{*}} &\tilde{=} \tilde{T}\tilde{T}^{\widetilde{-1}}\\ &\tilde{=} \tilde{I}\\ &\tilde{=} \tilde{T}^{\widetilde{-1}}\tilde{T}\\ &\tilde{=} \tilde{T}^{\tilde{*}}\tilde{T}.\end{aligned}$$

Definition 3.2. Every FS unitary operator is FS isometry.

Proof. Applying (6) given in Definition (3.1) of FS unitary operator, we have:

$$\begin{aligned}\tilde{T}\tilde{T}^{\tilde{*}} &\tilde{=} \tilde{T}\tilde{T}^{\widetilde{-1}}\\ &\tilde{=} \tilde{I}.\end{aligned}$$

Then, applying Theorem (2.3), implies that $\tilde{T}$ is FS isometry operator.

Example 3.1. The FS right shift operator, stated in Definition (2.6), on $\ell_2(A)$ is an FS isometry, but not FS unitary.

Solution. For $\tilde{v}_{f_{G(e)}} \tilde{=} (\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots) \tilde{\in} \ell_2(A)$, the FS right shift operator, say $\tilde{U}$, is defined by:

$$\begin{aligned}\tilde{U}\tilde{v}_{f_{G(e)}} &\tilde{=} \tilde{U}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots)\\ &\tilde{=} (\tilde{0}, \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots),\end{aligned} \tag{7}$$

and the FS adjoint operator of $\tilde{U}$ (the FS left shift operator) is defined by:

$$\begin{aligned}\tilde{U}^{\tilde{*}}\tilde{v}_{f_{G(e)}} &\tilde{=} \tilde{U}^{\tilde{*}}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots)\\ &\tilde{=} (\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots).\end{aligned} \tag{8}$$

Then, using (7) and (8), we have:

$$\begin{aligned}(\tilde{U}^{\tilde{*}}\tilde{U})\tilde{v}_{f_{G(e)}} &\tilde{=} \tilde{U}^{\tilde{*}}(\tilde{U}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots))\\ &\tilde{=} \tilde{U}^{\tilde{*}}(\tilde{0}, \tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots)\\ &\tilde{=} (\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots)\\ &\tilde{=} \tilde{v}_{f_{G(e)}},\end{aligned}$$

for all $\tilde{v}_{f_{G(e)}} \tilde{\in} \ell_2(A)$. That is to say that, $\tilde{U}^{\tilde{*}}\tilde{U} \tilde{=} \tilde{I}$, and hence by Theorem (2.3), $\tilde{U}$ is FS isometry.
Now, using (7) and (8), we get:

$$\begin{aligned}(\tilde{U}\tilde{U}^{\tilde{*}})\tilde{v}_{f_{G(e)}} &\tilde{=} \tilde{U}(\tilde{U}^{\tilde{*}}(\tilde{v}^1_{f_{1_{G(e_1)}}}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots))\\ &\tilde{=} \tilde{U}(\tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots)\\ &\tilde{=} (\tilde{0}, \tilde{v}^2_{f_{2_{G(e_2)}}}, \tilde{v}^3_{f_{3_{G(e_3)}}}, \ldots)\\ &\tilde{\neq} \tilde{v}_{f_{G(e)}},\end{aligned}$$

for every $\tilde{v}_{f_{G(e)}} \tilde{\in} \ell_2(A)$. Therefore, we have $\tilde{U}\tilde{U}^{\tilde{*}} \tilde{\neq} \tilde{I}$. Consequently, we obtain that $\tilde{U}$ is not FS unitary.

Theorem 3.1. Assume that $\tilde{T}\tilde{\in}\tilde{\mathbb{B}}(\tilde{H})$ is FS unitary operator, then $\tilde{\sigma}(\tilde{T})$ lies on the FS circle $|\tilde{\lambda}| \tilde{=} \tilde{1}$.

Proof. Since $\tilde{T}$ is FS unitary, then by using (2) and (2), we have $\widetilde{\|\tilde{T}\|} \tilde{=} \widetilde{\|\tilde{T}^{\tilde{*}}\|} \tilde{=} \tilde{1}$. To prove that $\tilde{\sigma}(\tilde{T}) \tilde{\subset} \tilde{D}$, where $\tilde{D}$ is the FS unit circle $|\tilde{\lambda}| \tilde{=} \tilde{1}$. Let $\tilde{\lambda} \tilde{\notin} \tilde{D}$, then we get that $|\tilde{\lambda}| \tilde{>} \tilde{1}$, $\tilde{\lambda} \tilde{=} \tilde{0}$ or $\tilde{0} \tilde{<} |\tilde{\lambda}| \tilde{<} \tilde{1}$. Now, we study these three cases of $\tilde{\lambda}$ as follows:

1. If $|\tilde{\lambda}| \tilde{>} \tilde{1} \tilde{=} \widetilde{\|\tilde{T}\|}$, then by Definition (2.4) of FS resolvent set, we get that $\tilde{\lambda} \tilde{\in} \tilde{\rho}(\tilde{T})$.
2. If $\tilde{\lambda} \tilde{=} \tilde{0}$, we have $\tilde{0} \tilde{\in} \tilde{\rho}(\tilde{T})$, because $\tilde{T}$ is FS unitary (i.e. $\tilde{T}^{\widetilde{-1}} \tilde{=} \tilde{T}^{\tilde{*}}$ exists) and $\tilde{T}^{\widetilde{-1}} \tilde{=} (\tilde{T} - \tilde{0}\tilde{I})^{\widetilde{-1}}$.
3. If $\tilde{0} \tilde{<} |\tilde{\lambda}| \tilde{<} \tilde{1}$, then we obtain:

$$\begin{aligned}|\tilde{\lambda}^{\widetilde{-1}}| &\tilde{>} \tilde{1}\\ &\tilde{=} \widetilde{\|\tilde{T}^{\tilde{*}}\|}\\ &\tilde{=} \widetilde{\|\tilde{T}^{\widetilde{-1}}\|}.\end{aligned}$$

Therefore, $\tilde{\lambda}^{\widetilde{-1}} \tilde{\in} \tilde{\rho}(\tilde{T}^{\tilde{*}})$. Thus, we have $\tilde{\lambda}\tilde{I} - \tilde{T} \tilde{=} \tilde{\lambda}\tilde{T}(\tilde{T}^{\tilde{*}} - \tilde{\lambda}^{\widetilde{-1}})$ and then $(\tilde{\lambda}\tilde{I} - \tilde{T})^{\widetilde{-1}} \tilde{=} \tilde{\lambda}^{\widetilde{-1}}(\tilde{T}^{\tilde{*}} - \tilde{\lambda}^{\widetilde{-1}})\widetilde{-1}\tilde{T}^{\tilde{*}}$ exists. Thus, we get that $\tilde{\lambda} \tilde{\in} \tilde{\rho}(\tilde{T})$.

Therefore, in the three cases, we have $\tilde{\lambda} \tilde{\notin} \tilde{\sigma}(\tilde{T}) \tilde{=} \tilde{\rho}(\tilde{T})^C$. Hence, $\tilde{\lambda} \tilde{\notin} \tilde{D}$ gives that $\tilde{\lambda} \tilde{\notin} \tilde{\sigma}(\tilde{T})$. That is to say that $\tilde{\lambda} \tilde{\in} \tilde{\sigma}(\tilde{T})$ gives that $\tilde{\lambda} \tilde{\in} \tilde{D}$, i.e. $\tilde{\sigma}(\tilde{T}) \tilde{\subset} \tilde{D} \tilde{=} \{\tilde{\lambda} : |\tilde{\lambda}| \tilde{=} \tilde{1}\}$.

4 Fuzzy Soft Unitary Equivalence

This section is devoted to introduce the FS equivalence with its related theorems to make the picture complete.

Definition 4.1. Let $\tilde{H}$ be an FS Hilbert space as well as $\tilde{T}_1, \tilde{T}_2 \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$. $\tilde{T}_1$ and $\tilde{T}_2$ are said to be FS unitary equivalent if

$$\tilde{T}_2 \tilde{=} \tilde{\mathscr{U}}^{\tilde{*}}\tilde{T}_1\tilde{\mathscr{U}}. \tag{9}$$

for some FS unitary operator $\tilde{\mathscr{U}}$.

Lemma 4.1. Let $\tilde{H}$ be an FS Hilbert space as well as $\tilde{T}_1, \tilde{T}_2 \tilde{\in} \tilde{\mathbb{B}}(\tilde{H})$. $\tilde{T}_1$ and $\tilde{T}_2$ are FS unitary equivalent if

$$\tilde{\mathscr{U}}\tilde{T}_2 \tilde{=} \tilde{T}_1 \tilde{\mathscr{U}}. \quad (10)$$

for some FS unitary operator $\tilde{\mathscr{U}}$.

Proof. By using (9) from Definition (4.1) of FS unitary equivalence, then $\tilde{T}_2 \tilde{=} \tilde{\mathscr{U}}^{\tilde{*}} \tilde{T}_1 \tilde{\mathscr{U}}$. But from (6) given in Definition (3.1), the following is obtained:

$$\tilde{T}_2 \tilde{=} \tilde{\mathscr{U}}^{\tilde{*}} \tilde{T}_1 \tilde{\mathscr{U}} \iff \tilde{T}_2 \tilde{=} \tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{T}_1 \tilde{\mathscr{U}}$$
$$\iff \tilde{\mathscr{U}}\tilde{T}_2 \tilde{=} \tilde{T}_1 \tilde{\mathscr{U}}.$$

Theorem 4.1. Suppose that $\tilde{T}_1$ and $\tilde{T}_2$ are FS unitary equivalent. Consequently, we have $(\tilde{T}_1 - \tilde{\lambda}\tilde{I})$ and $(\tilde{T}_2 - \tilde{\lambda}\tilde{I})$ are also FS unitary equivalent.

Proof. Because we have $\tilde{T}_1$ and $\tilde{T}_2$ are FS unitary equivalent, then, using (4.1) from Lemma (4.1), there is an FS unitary operator $\tilde{\mathscr{U}}$ such that

$$\tilde{\mathscr{U}}\tilde{T}_2 \tilde{=} \tilde{T}_1 \tilde{\mathscr{U}}. \quad (11)$$

But, we have:

$$\tilde{\mathscr{U}}(\tilde{T}_2 - \tilde{\lambda}\tilde{I}) \tilde{=} \tilde{\mathscr{U}}\tilde{T}_2 - \tilde{\lambda}\tilde{\mathscr{U}}. \quad (12)$$

Therefore, by substituting from (11) in (12), we get:

$$\tilde{\mathscr{U}}(\tilde{T}_2 - \tilde{\lambda}\tilde{I}) \tilde{=} \tilde{T}_1 \tilde{\mathscr{U}} - \tilde{\lambda}\tilde{\mathscr{U}}$$
$$\tilde{=} (\tilde{T}_1 - \tilde{\lambda}\tilde{I})\tilde{\mathscr{U}}.$$

Hence, $(\tilde{T}_1 - \tilde{\lambda}\tilde{I})$ and $(\tilde{T}_2 - \tilde{\lambda}\tilde{I})$ are FS unitary equivalent.

Theorem 4.2. Suppose that $\tilde{T}_1$ and $\tilde{T}_2$ are FS unitary equivalent, therefore we get

$$\tilde{\sigma}_p(\tilde{T}_1) \tilde{=} \tilde{\sigma}_p(\tilde{T}_2). \quad (13)$$

i.e., $\tilde{T}_1$ and $\tilde{T}_2$ have the same FS eigenvalues.

Proof. Because we have $\tilde{T}_1$ and $\tilde{T}_2$ are FS unitary equivalent, then, using (10) from Lemma (4.1), there is an FS unitary operator $\tilde{\mathscr{U}}$ such that $\tilde{\mathscr{U}}\tilde{T}_2 \tilde{=} \tilde{T}_1 \tilde{\mathscr{U}}$. Then,

$$\tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{\mathscr{U}} \tilde{T}_2 \tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{=} \tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{T}_1 \tilde{\mathscr{U}} \tilde{\mathscr{U}}^{\widetilde{-1}}.$$

That is to say that,

$$\tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{T}_1 \tilde{=} \tilde{T}_2 \tilde{\mathscr{U}}^{\widetilde{-1}}. \quad (14)$$

Now, let $\tilde{\lambda} \tilde{\in} \tilde{\sigma}_p(\tilde{T}_1)$, then, using Definition (2.5), there exists a non-zero FS element $\tilde{v}_{f_{G(e)}} \tilde{\in} \tilde{H}$ such that $\tilde{T}_1 \tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\lambda} \tilde{v}_{f_{G(e)}}$. Therefore, $\tilde{\mathscr{U}}^{\widetilde{-1}}(\tilde{T}_1 \tilde{v}_{f_{G(e)}}) \tilde{=} \tilde{\mathscr{U}}^{\widetilde{-1}}(\tilde{\lambda} \tilde{v}_{f_{G(e)}})$. Thus,

$$(\tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{T}_1) \tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\lambda} (\tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{v}_{f_{G(e)}}). \quad (15)$$

Then, by substituting from (14) in (15), we have $(\tilde{T}_2 \tilde{\mathscr{U}}^{\widetilde{-1}}) \tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\lambda} (\tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{v}_{f_{G(e)}})$. Therefore, $\tilde{T}_2 (\tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{v}_{f_{G(e)}}) \tilde{=} \tilde{\lambda} (\tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{v}_{f_{G(e)}})$. Then, $\tilde{T}_2(\tilde{w}_{h_{G(a)}}) \tilde{=} \tilde{\lambda}(\tilde{w}_{h_{G(a)}})$, where $\tilde{w}_{h_{G(a)}} \tilde{=} \tilde{\mathscr{U}}^{\widetilde{-1}} \tilde{v}_{f_{G(e)}} \tilde{\neq} \tilde{\theta}$. Hence, $\tilde{\lambda} \tilde{\in} \tilde{\sigma}_p(\tilde{T}_2)$. That is to say that,

$$\tilde{\sigma}_p(\tilde{T}_1) \tilde{\subset} \tilde{\sigma}_p(\tilde{T}_2). \quad (16)$$

On the other hand, let $\tilde{\lambda} \tilde{\in} \tilde{\sigma}_p(\tilde{T}_2)$, then, using Definition (2.5), there exists a non-zero FS element $\tilde{w}_{h_{G(a)}} \tilde{\in} \tilde{H}$ such that $\tilde{T}_2 \tilde{w}_{h_{G(a)}} \tilde{=} \tilde{\lambda} \tilde{w}_{h_{G(a)}}$. Thus, $\tilde{\mathscr{U}}(\tilde{T}_2 \tilde{w}_{h_{G(a)}}) \tilde{=} \tilde{\mathscr{U}}(\tilde{\lambda} \tilde{w}_{h_{G(a)}})$. Therefore,

$$(\tilde{\mathscr{U}}\tilde{T}_2) \tilde{w}_{h_{G(a)}} \tilde{=} \tilde{\lambda} (\tilde{\mathscr{U}} \tilde{w}_{h_{G(a)}}). \quad (17)$$

Then, by substituting from (10), stated in Lemma (4.1), in (17), we get $(\tilde{T}_1 \tilde{\mathscr{U}}) \tilde{w}_{h_{G(a)}} \tilde{=} \tilde{\lambda} (\tilde{\mathscr{U}} \tilde{w}_{h_{G(a)}})$. Therefore, $\tilde{T}_1 (\tilde{\mathscr{U}} \tilde{w}_{h_{G(a)}}) \tilde{=} \tilde{\lambda} (\tilde{\mathscr{U}} \tilde{w}_{h_{G(a)}})$. Then, $\tilde{T}_1(\tilde{v}_{f_{G(e)}}) \tilde{=} \tilde{\lambda}(\tilde{v}_{f_{G(e)}})$, where $\tilde{v}_{f_{G(e)}} \tilde{=} \tilde{\mathscr{U}} \tilde{w}_{h_{G(a)}} \tilde{\neq} \tilde{\theta}$. Hence, $\tilde{\lambda} \tilde{\in} \tilde{\sigma}_p(\tilde{T}_1)$, i.e.,

$$\tilde{\sigma}_p(\tilde{T}_2) \tilde{\subset} \tilde{\sigma}_p(\tilde{T}_1). \quad (18)$$

Hence (16) and (18) complete the proof, i.e. (13) is satisfied.

5 Conclusions and Future Work

Various researchers have introduced the fuzzy model or even the soft form of issues like normed spaces, metric spaces, as well as Hilbert spaces. Mixing fuzzy and soft sets, on the other hand, yields more comprehensive, generalized, and reliable findings. Only a few authors have looked at some of those broad extension ideas. The FS unitary operator, a specific sort of FS linear operator, has been presented in our research. Furthermore, FS spectral theory and the connection between the FS unitary operators and each of FS isometry operators and FS normal operators have been established. Finally, we investigated the FS unitary equivalence with its related theorems. The importance of the FS unitary operators is that they are the generalization of the crisp or soft ones. Although this progress, in actual-world, the problem might be so vague that the FS unitary operators fail to deal with it. As a result, we need a more general extension which can be the vague soft version of those operators. As a future work, the authors can introduce the unitary operators in the vague soft settings.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Authors' Contributions

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